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A spectral theory for simply periodic solutions of the sinh-Gordon equation

Abstract. In the talk, I describe a spectral theory for solutions $u : N \subset \mathbb{C} \to \mathbb{C}$ of the sinh-Gordon equation

$$\Delta u + \sinh(u) = 0$$

which are simply periodic in the sense that their domain N is a horizontal strip in $\mathbb{C}$ and

$$u(z+1) = u(z) \qquad \text{holds for all } z \in N\,.$$

Solutions $u : N \to \mathbb{R}$ of the sinh-Gordon equation (periodic or not) are of interest in particular, because they give rise to minimal surfaces in S^3 (or to constant mean curvature surfaces in $\mathbb{R}^3$). The case where u is doubly periodic (i.e. $N = \mathbb{C}$ and u has two $\mathbb{R}$-linearly independent periods) corresponds to minimal tori in S^3; this case has been studied extensively and a complete classification has been given by Pinkall/Sterling (1989) and independently by Hitchin (1990). In contrast thereto, simply periodic solutions of the sinh-Gordon equation give rise to a far larger class of minimal surfaces in S^3, for example the Lawson surfaces (which are compact, immersed minimal surfaces in S^3 of genus $g \geq 2$) are obtained in this way.

I will describe how one can associate to a simply periodic solution u of the sinh-Gordon equation a set of spectral data (Σ, D). Here Σ is the spectral curve associated to u, which is a non-compact, hyperelliptic Riemann surface over $\lambda \in \mathbb{C}^*$. D is the spectral divisor on Σ, corresponding to a holomorphic line bundle on Σ.

The fundamental difference between the present situation, where u is simply periodic, and the situation investigated by Hitchin, with u doubly periodic, is that in the doubly periodic case, the spectral curve Σ is of finite genus, and therefore can be compactified, whereas in the present simply periodic case, Σ is of infinite genus, and its branch points accumulate near $\lambda = 0$ and near $\lambda = \infty$.

The *direct problem* for a given simply periodic solution u is to construct the corresponding spectral data (Σ, D), and to discuss their behavior. In this context, I will in particular characterise the asymptotic behavior of the spectral divisor D near the two singularities $\lambda = 0$ and $\lambda = \infty$ of Σ. It turns out that the divisor D asymptotically approximates for $\lambda \to 0$ and for $\lambda \to \infty$ the spectral divisor D^0 of the "vacuum solution" $u \equiv 0$ of a certain order.

Finally, the *inverse problem* will be discussed. This concerns the reconstruction of the solution u from its spectral data (Σ, D).