Lorentzian homogeneous structures with indecomposable holonomy

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[Joint work with Steven Greenwood, arXiv:2404.17470]

Symmetry and shape

Celebrating the 60th birthday of Eduardo García Río

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Semi-Riemannian homogeneous spaces

- A semi-Riemannian manifold (M, g) is *homogeneous* if there is a group of isometries G that acts transitively on M.
- $M = G/H$ with $H = \{\phi \in G \mid \phi(o) = o\} \subseteq G$ the *isotropy* group at $o \in M$.
- H depends on o by conjugation in G and also on $G \subseteq \mathrm{Isom}(M, g)$.
- $H \ni \phi \longmapsto \mathrm{d}\phi|_o \in \mathbf{SO}(T_oM)$ is the isotropy representation.
- A homogeneous space G/H is *reductive* if $\mathfrak{g}$ admits an $\mathrm{Ad}(H)$-invariant decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$.

Irreducible isotropy

Some Riemannian homogeneous spaces with irreducible isotropy [Besse]:

H. Tables 203

7.106 Table 5. (Compact) non-symmetric strongly isotropy irreducible spaces
(a) *Infinite families*

$\mathfrak{g}$	$\mathfrak{k}$	Condition	$\mathfrak{g}$	$\mathfrak{k}$	Condition
$\mathfrak{su}\left(\frac{n(n-1)}{2}\right)$	$\mathfrak{su}(n)$	$5 \leqslant n$	$\mathfrak{so}\left(\frac{(n-1)(n+2)}{2}\right)$	$\mathfrak{so}(n)$	$5 \leqslant n$
$\mathfrak{su}\left(\frac{n(n+1)}{2}\right)$	$\mathfrak{su}(n)$	$3 \leqslant n$	$\mathfrak{so}((n-1)(2n+1))$	$\mathfrak{sp}(n)$	$3 \leqslant n$
$\mathfrak{su}(pq)$	$\mathfrak{su}(p) \oplus \mathfrak{su}(q)$	$2 \leqslant p \leqslant q$ $p + q \neq 4$	$\mathfrak{so}(n(2n+1))$	$\mathfrak{sp}(n)$	$2 \leqslant n$ (*)
$\mathfrak{so}(n^2-1)$	$\mathfrak{su}(n)$	$3 \leqslant n$ (*)	$\mathfrak{so}(4n)$	$\mathfrak{sp}(1) \oplus \mathfrak{sp}(n)$	$2 \leqslant n$
$\mathfrak{so}\left(\frac{n(n-1)}{2}\right)$	$\mathfrak{so}(n)$	$7 \leqslant n$ (*)	$\mathfrak{sp}(n)$	$\mathfrak{sp}(1) \oplus \mathfrak{so}(n)$	$3 \leqslant n$

Note: These spaces are constructed in 7.50 (or in 7.49 for those with a (*)).

7.107 Table 6. (Compact) non-symmetric strongly isotropy irreducible spaces
(b) *Exceptions*

$\mathfrak{g}$	$\mathfrak{k}$	Note	$\mathfrak{g}$	$\mathfrak{k}$	Note	$\mathfrak{g}$	$\mathfrak{k}$	Note
$\mathfrak{su}(16)$	$\mathfrak{so}(10)$	1	$\mathfrak{so}(133)$	E_7	3	E_6	$\mathfrak{su}(3)$	4
$\mathfrak{su}(27)$	E_6	1	$\mathfrak{so}(248)$	E_8	3	E_6	$3\mathfrak{su}(3)$	4, 5
$\mathfrak{so}(7)$	G_2	2	$\mathfrak{sp}(2)$	$\mathfrak{su}(2)$	1	E_6	G_2	4
$\mathfrak{so}(14)$	G_2	3	$\mathfrak{sp}(7)$	$\mathfrak{su}(6)$	1	E_6	$G_2 \oplus \mathfrak{su}(3)$	4
$\mathfrak{so}(16)$	$\mathfrak{so}(9)$	1	$\mathfrak{sp}(10)$	$\mathfrak{so}(12)$	1	E_7	$\mathfrak{su}(3)$	4
$\mathfrak{so}(26)$	F_4	1	$\mathfrak{sp}(16)$	$\mathfrak{sp}(3)$	1	E_7	$\mathfrak{su}(6) \oplus \mathfrak{su}(3)$	4
$\mathfrak{so}(42)$	$\mathfrak{sp}(4)$	1	$\mathfrak{sp}(28)$	E_7	1	E_7	$G_2 \oplus \mathfrak{sp}(3)$	4
$\mathfrak{so}(52)$	F_4	3	G_2	$\mathfrak{su}(3)$	4	E_7	$F_4 \oplus \mathfrak{su}(2)$	4
$\mathfrak{so}(70)$	$\mathfrak{su}(8)$	1	G_2	$\mathfrak{so}(3)$	4	E_8	$\mathfrak{su}(9)$	4
$\mathfrak{so}(78)$	E_6	3	F_4	$\mathfrak{su}(3) \oplus \mathfrak{su}(3)$	4	E_8	$F_4 \oplus G_2$	4
$\mathfrak{so}(128)$	$\mathfrak{so}(16)$	1	F_4	$G_2 \oplus \mathfrak{su}(2)$	4	E_8	$E_6 \oplus \mathfrak{su}(3)$	4

Notes
(1) Defined in 7.51.
(2) Defined in 7.13
(3) Defined in 7.49.
(4) Here $\mathfrak{k}$ is a maximal subalgebra of $\mathfrak{g}$ (and this is sufficient, in those cases, to characterize the embedding, see [Dyn 2]).
(5) $p\mathfrak{k}$ means $\mathfrak{k} \oplus \mathfrak{k} \oplus \cdots \oplus \mathfrak{k}$ (p times).

Lorentzian homogeneous spaces:

Theorem (Zeghib 2004)

If a Lorentzian homogeneous space (M, g) of dimension $m \geq 3$ admits an irreducible isotropy group, then it has constant sectional curvature.

Irreducible vs indecomposable

Algebraic fact [Berger '55, Di Scala & Olmos '01, de la Harpe '04]

If $H \subseteq \mathbf{O}(1,n)$ is irreducible, then $\mathbf{SO}^0(1,n) \subseteq H$.

In contrast: every compact Lie group K admits a representation $K \subset \mathbf{O}(n)$.

A better assumption in the indefinite context is *indecomposability*:

- $H \subseteq \mathbf{O}(1,n)$ is *decomposable* if $\exists$ H-invariant subspace V: $\mathbb{R}^{1,n} = V \oplus V^\perp$, and *indecomposable* otherwise, i.e. $\nexists$ non degenerate invariant subspace.
- irreducible $\implies$ indecomposable
- $\mathfrak{h} \subsetneq \mathfrak{so}(1,n+1)$ indecomposable, then $V \cap V^\perp = \mathbb{R}e_-$ is invariant, so that

$$\mathfrak{h} \subseteq \mathfrak{stab}_{\mathfrak{so}(1,n+1)}(\mathbb{R}e_-) = \left\{ \begin{pmatrix} a & u^t & 0 \\ 0 & B & -u \\ 0 & 0 & -a \end{pmatrix} \middle| \begin{array}{l} a \in \mathbb{R}, \\ u \in \mathbb{R}^n, \\ B \in \mathfrak{so}(n) \end{array} \right\} = \underbrace{(\mathbb{R} \oplus \mathfrak{so}(n))}_{=\mathfrak{co}(n)} \ltimes \mathbb{R}^n,$$

with $\mathrm{pr}_{\mathbb{R}^n}(\mathfrak{h}) = \mathbb{R}^n$.

Are there any homogeneous Lorentzian manifolds with indecomposable, non irreducible isotropy?

Plane waves

A Lorentzian mfd (M, g) is a *plane wave* if it has a null[1] vector field ξ:

$$\nabla\xi = 0, \qquad \underbrace{R(X,Y) = 0,}_{\iff R \in \Lambda^2 \otimes \mathbb{R}^n} \quad \nabla_X R = 0 \quad \forall X, Y \in \xi^\perp.$$

There are coordinates $(t, x^1, \ldots, x^n, v) = (t, \mathbf{x}, v)$:

$$g_Q \;=\; 2\,\mathrm{d}v\,\mathrm{d}t + \mathrm{d}\mathbf{x}^\top \mathrm{d}\mathbf{x} + \mathbf{x}^\top Q(t)\,\mathbf{x}\,\mathrm{d}t^2$$

If $M = \mathbb{R}^{n+2}$ and g_Q on M, then (M, g_Q)

- $\mathbf{Hei}_{2n+1}$ acts transitively on each leaf of $\xi^\perp$
- (M, g_Q) is homogeneous $\iff \exists$ KVF $\pitchfork \xi^\perp \overset{[\text{Blau\&O'Loughlin'03}]}{\iff} \exists F \in \mathfrak{so}(n)$:

$$Q(t) = \mathrm{e}^{tF} Q_0 \mathrm{e}^{-tF}, \quad \text{or} \quad Q(t) = \tfrac{1}{t^2}\mathrm{e}^{\log tF} Q_0 \mathrm{e}^{-\log tF} \text{ on } \{t > 0\}.$$

- with isotropy $\mathfrak{h} = \mathfrak{z}_{\mathfrak{so}(n)}(Q_0, F) \ltimes \mathbb{R}^n$, i.e. $\mathfrak{h}$ indecomposable.

[1] null := light-like (isotropic) and $\neq 0$

Dichotomy for Lorentzian symmetric spaces

$G = \langle \phi_p \phi_q \mid p, q \in M \rangle$ transvection group; the isotropy group H in G is equal to the holonomy group of (M, g).

Theorem (Cahen & Wallach, '70)

An indecomposable Lorentzian symmetric space either has constant sectional curvature or is universally covered by a Cahen–Wallach space *(a plane wave with* $M = \mathbb{R}^{n+2}$ *and* Q *constant and* $\det(Q) \neq 0$*).*

Other rigidity results:

- If (M, g) is a locally homogeneous pp-wave ($\nabla \xi = 0$ and $R(X, Y) = 0 \ \forall X, Y \in \xi^{\perp}$), of dim ≥ 4, then it is a plane wave [Globke & L '16]
- If (M, g) has a transitive group of essential conformal transformations, then g is conformally equivalent to a homogeneous plane wave [Alekssevsky & Galaev '24]

Reductive Lorentzian homogeneous spaces with indecomposable isotropy

All known (to us) examples of dimension ≥ 4 are plane waves.

Conjecture

A reductive Lorentzian homogeneous space G/H of dimension $m \geq 4$ with indecomposable isotropy $H \not\supset \mathbf{SO}^0(1, m-1)$ is a plane wave.

An *Ambrose–Singer connection* on (M, g) is connection $\widetilde{\nabla}$ with

$$\widetilde{\nabla} g = 0, \quad \widetilde{\nabla}\widetilde{R} = 0, \quad \widetilde{\nabla} T = 0, \qquad \widetilde{R} = \text{curvature},\ T = \text{torsion of } \widetilde{\nabla}.$$

There is a close relation between local homogeneity and the existence of an AS connection, and between the isotropy and the holonomy of $\widetilde{\nabla}$ ...

Theorem (Greenwood & L '24)

If $(M^{m\geq 4}, g)$ is a Lorentzian manifold that admits an AS-connection with indecomposable, non-irreducible holonomy, then the universal cover of (M, g) is a locally homogeneous plane-wave.

AS-connections, homogeneous structures and homogeneous spaces

A *homogeneous structure* is a section S of $T^*M \otimes \mathfrak{so}(TM, g)$ such that

$$\nabla_X S = S(X) \cdot S, \quad \nabla_X R = S(X) \cdot R, \quad \nabla = \text{Levi-Civita connection.}$$

$$\text{homog structures} \longleftrightarrow \text{AS connections}, \quad S \longleftrightarrow \widetilde{\nabla}_X Y = \nabla_X Y - S(X, Y).$$

- G/H reductive homogeneous $\implies$ AS connection: the canonical connection is an AS connection (defined by $(\widetilde{\nabla}_X Y)_o = 0$ for all $X, Y \in \mathfrak{m}$, torsion $-[X, Y]_{\mathfrak{m}}$, curvature $= -[X, Y]_{\mathfrak{h}}$).
- Isotropy & holonomy: If $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, then $\widetilde{\mathfrak{g}} := \mathrm{pr}_{\mathfrak{h}}([\mathfrak{m}, \mathfrak{m}]) \oplus \mathfrak{m}$ is an ideal in $\mathfrak{g}$ and $G/H = \widetilde{G}/\widetilde{H}$ with $\widetilde{\mathfrak{h}} := \mathrm{pr}_{\mathfrak{h}}([\mathfrak{m}, \mathfrak{m}]) = \mathfrak{hol}(\widetilde{\nabla})$ that is equal to the holonomy algebra of the canonical connection.
- Converse [Ambrose & Singer '58, Tricerri & Vanhecke '83, Gadea & Oubiña '92]:
  If (M, g) is complete, simply-connected and with AS connection, then (M, g) is reductive homogeneous.
- (M, g) is locally reductive homogeneous $\iff$ $\exists$ AS connection [e.g. Castrillón-López & Calvaruso '19]

Infinitesimal model for for homogeneous spaces

- Let $\widetilde{\nabla}$ be an AS connection with $\widetilde{R}$ and T.
- At $o \in M$, set $\mathfrak{m} = T_oM$, $\widetilde{\mathfrak{h}} := \mathfrak{hol}_o(\widetilde{\nabla}) = \mathrm{span}\{\widetilde{R}|_o(X, Y) \mid X, Y \in \mathfrak{m}\}$,
- Lie bracket on $\widetilde{\mathfrak{g}} := \widetilde{\mathfrak{h}} \oplus \mathfrak{m}$ by extending the Lie bracket of $\widetilde{\mathfrak{h}} \subseteq \mathfrak{so}(\mathfrak{m})$ by

$$[H, X] := H(X), \qquad [X, Y] := -\widetilde{R}|_o(X, Y) - T|_o(X, Y),$$

- $\exists$ unique simply connected Lie group $\widetilde{G}$ with Lie algebra $\widetilde{\mathfrak{g}}$ and unique connected subgroup $\widetilde{H}$ with Lie algebra $\widetilde{\mathfrak{h}}$. If $\widetilde{H}$ is closed in $\widetilde{G}$, the homogeneous space $\widetilde{G}/\widetilde{H}$ is locally isometric to (M, g).
  $(\mathfrak{m}, \widetilde{R}|_o, T|_o)$ is an *infinitesimal model* of the locally homogeneous space $M = \widetilde{G}/\widetilde{H}$.

Version of the Theorem

A Lorentzian reductive locally homogeneous space of dimension $m \geq 4$ is a plane wave if it admits an infinitesimal model that has indecomposable isotropy $\mathfrak{h} \neq \mathfrak{so}(1, m-1)$.

Previous results

Torsion T is a section of $\Lambda^2 \otimes \Lambda^1$, where $\Lambda^k := \Lambda^k T^* M$, whose fibres split into three irreducible $\mathfrak{so}(1, m-1)$-modules,

$$\begin{array}{ccccccc} \Lambda^2 \otimes \Lambda^1 & \simeq & \Lambda^3 & \oplus & \ker(\mathrm{pr}_{\Lambda^3}) \cap \ker(\mathrm{trace}) & \oplus & \Lambda^1 \\ & & \uparrow & & \uparrow & & \uparrow \\ \text{torsion} & & \text{skew} & & \text{twistorial} & & \text{vectorial} \end{array} .$$

Since $\widetilde{\nabla} T = 0$, the algebraic type of T does not change.

- T vectorial: if $\mathrm{tr}(T)$ null, then singular homogeneous plane wave [Montesinos Amilibia '01], otherwise constant sectional curvature [Gadea & Oubiña '97].
- T is twistor-free with null vectorial part $\implies$ singular homogeneous plane wave [Meessen '06].
- T skew and $\mathfrak{hol}(\widetilde{\nabla})$ is indecomposable $\implies$ regular homogeneous plane wave [Ernst & Galaev '22].

So far, no results for twistorial torsion.

The following results [Greenwood & L '24] do not make any assumption on algebraic type.

Lorentzian Ambrose–Singer connections

Let (M^{n+2}, g) be Lorentzian manifold with Levi-Civita connection ∇, and $\widetilde{\nabla}_X Y = \nabla_X Y - S(X, Y)$ an Ambrose–Singer connection, with S a section of $T^*M \otimes \mathfrak{so}(TM)$. We have

$$\widetilde{\nabla} g = 0, \qquad \widetilde{\nabla} S = 0 \quad (\iff \widetilde{\nabla} T = 0), \qquad \widetilde{\nabla} \widetilde{R} = 0$$

Lorentzian *parallel torsion* *loc symmetric*

- Let $\mathfrak{h} \subset \mathfrak{g} := \mathfrak{so}(1, n+1)$ be the holonomy algebra of $\widetilde{\nabla}$.
- $\widetilde{\nabla} S = 0$ and $\widetilde{\nabla}\widetilde{R} = 0 \implies \mathfrak{h} \cdot S = 0$ and $\mathfrak{h} \cdot R = 0$, i.e. S and R lie in the maximal trivial $\mathfrak{h}$-submodule in $\Lambda^1 \otimes \mathfrak{g}$ and $\Lambda^2 \otimes \mathfrak{g}$.
- If $\mathfrak{h} \neq \mathfrak{g}$ is indecomposable, we will show that the above implies that $\mathfrak{h} = \mathbb{R}^n$, and moreover that the holonomy of the Levi-Civita connection is also in $\mathbb{R}^n$.

Locally symmetric Lorentzian connections, $\widetilde{\nabla} g = 0$ and $\widetilde{\nabla}\widetilde{R} = 0$

Let (M^{n+2}, g) Lorentzian mfd, $\widetilde{\nabla}$ Lorentzian connection, i.e. $\widetilde{\nabla} g = 0$, with indecomposable holonomy $\mathfrak{h} \subsetneq \mathfrak{so}(1, n+1)$, i.e. $\mathfrak{h} \subseteq \mathfrak{stab}_{\mathfrak{g}}(\mathbb{R}e_-)$.

- $\widetilde{\nabla}$ admits a parallel null line bundle $\mathcal{L} \subset \mathcal{L}^\perp \subset TM$.
- If (M, g) is time-oriented, $\exists$ recurrent $\xi \in \Gamma(\mathcal{L})$: $\nabla \xi = \theta \otimes \xi$, for $\theta \in \Gamma(\Lambda^1)$.

$$\widetilde{R}(X, Y)\xi = \mathrm{d}\theta(X, Y)\xi, \qquad (\widetilde{\nabla}_X \widetilde{R})(Y, Z)\xi = (\nabla_X \mathrm{d}\theta)(Y, Z)\xi.$$

- If M is simply connected, then ξ can be rescaled to a parallel null vector field $\iff \mathrm{d}\theta = 0$.
- If $\widetilde{\nabla}$ is locally symmetric, i.e. $\widetilde{\nabla}\widetilde{R} = 0$, then $\widetilde{\nabla}\mathrm{d}\theta = 0$.
  In particular, $[\mathfrak{h}, \mathrm{d}\theta] = 0$, i.e. $\mathrm{d}\theta \in \mathfrak{z}_{\mathfrak{g}}(\mathfrak{h}) \overset{\text{a bit of algebra on the next slide}}{\Longrightarrow}$

Proposition

Let $\widetilde{\nabla}$ be a locally symmetric Lorentzian connection on $(M^{m \geq 3}, g)$ with indecomposable, non-irreducible holonomy algebra. Then, on the universal cover of M, $\widetilde{\nabla}$ admits a parallel null vector field.

Algebra 1: indecomposable subalgebras in $\mathfrak{g} := \mathfrak{so}(1, n+1)$

- In $V := \mathbb{R}^{1,n+1}$ consider two null vectors $e_\pm$ with $\langle e_-, e_+\rangle = 1$, so that
$$V = V_- \overset{\perp}{\oplus} V_0 \overset{\perp}{\oplus} V_+, \quad \text{where } V_\pm := \mathbb{R}\cdot e_\pm \text{ and } V_0 = \mathbb{R}^n.$$
- Let $\mathfrak{g}_0 \simeq \mathfrak{co}(n) \simeq \mathbb{R} \oplus \mathfrak{so}(n)$ be the subalgebra of $\mathfrak{g}$ that preserves so that
$$\mathfrak{g} = \left\{ \begin{pmatrix} a & v^\top & 0 \\ u & A & -v \\ 0 & -u^\top & -a \end{pmatrix} \middle| \begin{array}{l} a \in \mathbb{R}, A \in \mathfrak{so}(n) \\ v \in \mathbb{R}^n, \\ u \in \mathbb{R}^n \end{array} \right\} = \underbrace{\mathfrak{g}_- \oplus \mathfrak{co}(n)}_{=\mathfrak{p}:=\mathfrak{stab}_\mathfrak{g}(V_-)} \oplus \mathfrak{g}_+$$
with $[\mathfrak{g}_i, \mathfrak{g}_j] = \mathfrak{g}_{i+j}$ and $\mathfrak{g}_i(V_j) = V_{i+j} \mod 2$, with
$$\pi_\pm : \mathfrak{g} \to \mathfrak{g}_\pm, \qquad \pi_0 = \pi_\mathbb{R} + \pi_{\mathfrak{so}(n)} : \mathfrak{g} \to \mathfrak{g}_0 = \mathfrak{co}(n) = \mathbb{R} \oplus \mathfrak{so}(n).$$
- $\mathfrak{h} \subsetneq \mathfrak{g}$ indecomposable $\overset{\text{mod conjug}}{\iff}$ $\mathfrak{h} \subset \mathfrak{p}$ and $\pi_-(\mathfrak{h}) = \mathfrak{g}_-$. Set $\mathfrak{h}_0 := \pi_0(\mathfrak{h})$.

Lemma

$$\mathfrak{z}_\mathfrak{g}(\mathfrak{h}) = \begin{cases} \{\overline{z} \in \mathfrak{g}_- \mid \mathfrak{h}_0 z = 0\}, & \text{if } \pi_\mathbb{R}(\mathfrak{h}) = 0, \text{ i.e. if } \mathfrak{h} \subset \mathfrak{so}(n) \ltimes \mathbb{R}^n \\ 0, & \text{otherwise.} \end{cases}$$

This implies: either $\mathrm{d}\theta = 0$ or $\mathfrak{h} \in \mathfrak{so}(n) \ltimes \mathbb{R}^n$, so in both cases the recurrent vector field rescales to a parallel one.

Algebra 2: the form of S — trivial $\mathfrak{h}$-modules in $V^* \otimes \mathfrak{g}$

- We denote $V^0 := V_0^*$ and $V^\pm := V_\pm^* = \mathbb{R}e^\pm$ with $e^\pm := \langle e_\mp, . \rangle$,
- for $X \in \mathfrak{g}$ denote $X_a := \pi_a(X)$ for $a = -, 0, +$, and the same for subsets.

Theorem

Let $\mathfrak{h} \subseteq \mathfrak{so}(n) \ltimes \mathfrak{g}_-$ indecomposable with $n \geq 2$. If $W \subseteq V^* \otimes \mathfrak{g}$ is a trivial $\mathfrak{h}$-module, then

$$W \subseteq (V^0 \otimes \mathfrak{g}_-) \oplus (V^+ \otimes \mathfrak{p})$$

and every $S \in W$ is determined by $S(e_+)$ as

$$S(x)e_+ = -(S(e_+))_0 \cdot x, \text{ for all } x \in V_0, \quad \text{and} \quad [\mathfrak{h}_0, (S(e_+))_0] = \{0\}.$$

In particular, $S(e_-) = 0$ and $S(X, e_-) \in V_-$.

Algebra 3: trivial $\mathfrak{h}$-modules in the torsion module $\Lambda^2 V^* \otimes V$

Isomorphism $V^* \otimes \mathfrak{g} \ni S_{ab}{}^c \longmapsto T_{ab}{}^c := S_{[ab]}{}^c \in \Lambda^2 \otimes V$

Corollary

If W is a trivial $\mathfrak{h}$-module in $\Lambda^2 V^* \otimes V$, then

$$W \subseteq \Big((V \wedge V^+) \oplus (V^0 \wedge V^0)\Big) \otimes V_- \;\oplus\; (V^0 \wedge V^+) \otimes V_0.$$

Moreover, if for $T \in W$, we define $b \in \mathbb{R}$ and $\omega \in \Lambda^2 V_0$ by

$$T(e_+, \mathrm{e}_-) = b\, e_-, \qquad \omega(x, y) := \langle T(x, y), \mathrm{e}_+ \rangle,$$

then $\langle T(e_+, x), y\rangle = b\, x^\top y + \omega(x, y)$.

- T is totally skew $\iff$ $S(e_+) \in \mathfrak{so}(n)$ $\iff$ $T(e_+) \in \mathfrak{so}(n)$
- T is twistorial $\iff$ $S(e_+) \in \mathfrak{g}_-$, $\iff$ $T(e_+) \in V^0 \otimes V_-$, and
- T is vectorial $\iff$ $S(e_+) \in \mathbb{R}$ $\iff$ $T(e_+) \in V^- \otimes V_-$.

Assume that $\widetilde{\nabla}_X Y = \nabla_X Y - S(X, Y)$ admits a parallel null vector field ξ and has parallel torsion.

- Theorem A2 $\implies S(.,\xi) = \alpha\xi^\flat \otimes \xi$, for constant α.
- ξ is recurrent for ∇ with $\theta = \alpha\,\xi^\flat$, so

$$\mathrm{d}\theta(X, Y) = \alpha\,\mathrm{d}\xi^\flat(X, Y) = -2g(\xi, \underbrace{T(X, Y)}_{\in\xi^\perp \text{ by A2}}) = 0.$$

Proposition

Let $(M^{m\geq 4}, g)$ be a Lorentzian, ∇ = Levi-Civita connection, $\widetilde{\nabla}$ = Lorentzian connection with parallel torsion. If $\widetilde{\nabla}$ admits a parallel null vector field ξ and $\mathfrak{hol}(\widetilde{\nabla})$ indecomposable, then also ∇ admits a parallel null vector field and $\mathrm{pr}_{\mathfrak{so}(n)}(\mathfrak{hol}(\nabla)) = \mathrm{pr}_{\mathfrak{so}(n)}(\mathfrak{hol}(\widetilde{\nabla}))$

This was known only for parallel totally skew torsion [Ernst & Galaev '22]

Algebra 4: algebraic curvature tensors with torsion

Let $T \in \Lambda^2 V^* \otimes V$, and $\mathfrak{h} \subset \mathfrak{so}(V)$.

$$\mathcal{R}(V, \mathfrak{h}, T) := \left\{ R \in \Lambda^2 V^* \otimes \mathfrak{h} \,\middle|\, \underset{u,v,w}{\mathfrak{S}} (R(u,v)w + T(T(u,v),w)) = 0 \ \forall u,v,w \in V \right\},$$

Theorem

Let $\mathfrak{h} \subseteq \mathfrak{so}(n) \ltimes \mathfrak{g}_-$ indecomposable, with $n \geq 1$, and $T \in \Lambda^2 V^* \otimes V$ such that $\mathfrak{h} \cdot T = 0$ (i.e. as in Corollary A3). Then $\mathcal{R}(V, \mathfrak{h}, T)$ injects into

$$\mathcal{R}(V_0, \mathfrak{h}_0) \oplus \mathcal{P}(V_0, \mathfrak{h}_0) \oplus V_0 \otimes V_0,$$

where $\mathcal{P}(V_0, \mathfrak{h}_0) := \left\{ P \in V^0 \otimes \mathfrak{h}_0 \mid \underset{x,y,z}{\mathfrak{S}} (P(x,y))^\top z = 0 \ \forall x,y,z \in V_0 \right\}.$

Corollary

Under the above assumptions, if $R \in \mathcal{R}(V, \mathfrak{h}, T)$ such that $\mathfrak{h} \cdot R = 0$, then $R \in \Lambda^2 V^* \otimes \mathfrak{g}_-$.

Proposition on Lorentzian symmetric connections and Theorem A4 $\Longrightarrow$

Theorem

Let $(M^{m\geq 3}, g)$ be Lorentzian manifold with Ambrose–Singer connection $\widetilde{\nabla}$ with indecomposable, non-irreducible holonomy. Then, on the universal cover, $\widetilde{\nabla}$ admits a parallel null vector field ξ, and $\widetilde{R}(X, Y) = 0$ for all $X, Y \in \xi^{\perp}$.

In particular, this implies the Cahen–Wallach result: *a locally symmetric Lorentzian manifold with indecomposable, non-irreducible holonomy is locally isometric to a Cahen–Wallach space.*

Levi-Civita vs. Ambrose–Singer: proof of the main result

Let (M, g) be a Lorentzian manifold of dim ≥ 4 and $\widetilde{\nabla}$ an Ambrose-Singer connection with indecompsable, non-irreducible holonomy.

- By the previous theorem, $\widetilde{\nabla}$ admits a parallel null vector field and $\widetilde{\nabla}^{\mathcal{S}}$ and $\widetilde{R}(X, Y) = 0$ for all $X, Y \in \xi^{\perp}$.
- By the proposition for connections with parallel torsion, also ∇ admits a parallel null vector field and $R(X, Y) = 0$ for all $X, Y \in \xi^{\perp}$.
  This already implies that (M, g) is a pp-wave, i.e. $R \in V^{+} \wedge V^{0} \otimes \mathfrak{g}_{-}$
- $0 = \widetilde{\nabla}_X R = \nabla_X R - S(X) \cdot R$, and by Theorem A2, $S(X) \in \mathfrak{g}_{-}$ for all $X \in \xi^{\perp}|_p$.
  Hence, for all $X, Y \in \xi^{\perp}|_p$ and $V \in T_pM$

$$(S(X) \cdot R)(Y, V) = \underbrace{[S(X), R(Y, V)]}_{=0,\ \mathfrak{g}_{-}\ \text{abelian}} - R(\underbrace{S(X, Y)}_{\in \mathbb{R}\cdot\xi}, V) - R(Y, \underbrace{S(X, V)}_{\in V_0}) = 0.$$

  This shows that $S(X) \cdot R = 0$ and hence $\nabla_X R$ for all $X \in \xi^{\perp}$.
  I.e. (M, g) is a plane wave.

The case of dimension 3

Theorem A2 fails in dimension $3 = n + 2$:

Let $V = \mathrm{span}(e_-, e_1, e_+)$ and $\mathfrak{h} = \mathbb{R} \cdot X$, where $X := \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{so}(1,2)$.

$S \in V^* \otimes \mathfrak{so}(1,2)$ such that $\mathfrak{h} \cdot S = 0 \iff$

$$S(e_-) = aX, \quad S(e_1) = \begin{pmatrix} a & -b & 0 \\ 0 & 0 & b \\ 0 & 0 & -a \end{pmatrix}, \quad S(e_+) = \begin{pmatrix} -b & -c & 0 \\ -a & 0 & c \\ 0 & a & b \end{pmatrix},$$

a, b, c in $\mathbb{R}$, i.e. the maximal trivial $\mathfrak{h}$-submodule of $V^* \otimes \mathfrak{g}$ is of dim 3 and not in $V^* \otimes \mathfrak{p}$ (as in Theorem A2).

Proposition

If a simply-connected Lorentzian manifold (M^3, g) admits an Ambrose–Singer connection with indecomposable, non-irreducible holonomy, then

- *(M^3, g) is a plane-wave, or*
- *a left-invariant metric on $\widetilde{\mathbf{SL}}(2, \mathbb{R})$ with holonomy algebra $\mathfrak{so}(1,2)$ and negative scalar curvature.*

An example

Consider $\mathfrak{m} := \mathbb{R}^3 = \mathrm{span}(e_-, \mathrm{e}_1, \mathrm{e}_+)$ with Lie bracket defined by

$$[e_-, e_1] = -2ae_- \qquad [e_-, e_+] = 2ae_1, \qquad [e_1, e_+] = -2ae_+ + \frac{(2ac+1)}{2a} e_-,$$

- $\mathfrak{m} = [\mathfrak{m}, \mathfrak{m}]$ and the Killing form is indefinite $\implies$ isomorphic to $\mathfrak{sl}(2, \mathbb{R})$.
- Let $\langle ., . \rangle$ be the Minkowski inner product on $\mathfrak{m}$ with (e_-, e_1, e_+) as before. $\rightsquigarrow$ left-invariant metric g on the 1-connected Lie group $M \simeq \widetilde{\mathbf{SL}}(2, \mathbb{R})$.
- $\nabla_{e_-} e_- = 0, \qquad \nabla_{e_1} e_- = ae_-, \qquad \nabla_{e_+} e_- = -ae_1$, so no parallel null vf!
- Left invariant homogeneous structure $S \in \mathfrak{m}^* \otimes \mathfrak{so}(\mathfrak{m})$ from the previous slide, the left invariant null vector field corresponding to e_- is parallel for $\widetilde{\nabla}$, since

$$S(e_-, e_-) = 0, \qquad S(e_1, e_-) = ae_-, \qquad S(e_+, e_-) = -ae_1.$$

- $\widetilde{\nabla}$ is not flat, so $\mathfrak{hol}(\widetilde{\nabla}) = \mathbb{R} \cdot X$ indecomposable.

$\rightsquigarrow$ Dimension restriction $m \geq 4$ in the main theorem is sharp.

$\rightsquigarrow$ Counterexample to Conjecture in dimension $m = 3$: $\mathfrak{h} := \mathbb{R} \cdot X$ and

$$\mathfrak{g} \; = \; \mathbb{R} \cdot X \ltimes \mathfrak{m} \; \simeq \; \mathbb{R} \ltimes \mathfrak{sl}(2, \mathbb{R}).$$

$M = G/H$ is not a plane wave, and has const sect curvature $\iff 1 + 2ac = 0$.

Moitas grazas!