VOLUME OF VECTOR FIELDS ON A RIEMANNIAN MANIFOLD: SOME OPEN PROBLEMS

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The volume functional

(M, g) Riemannian. On TM we consider the Sasaki metric and vector fields as maps $V: M \to (TM, g^S)$. The Volume of V is the volume of the embedded submanifold $V(M) \subset (TM, g^S)$ or also

$$\mathrm{Vol}(V) = \mathrm{vol}(M, V^* g^S)$$

It was first considered in 1986 H. Gluck and W. Ziller.

$$(V^* g^S)(X, Y) = g(L_V(X), Y)$$

$$\Updownarrow$$

$$g(X, Y) + g(\nabla_X V, \nabla_Y V)$$

$$\mathrm{Vol}(V) = \int_M \sqrt{\det(\mathrm{Id} + (\nabla V)^t \circ \nabla V)}\, \mathrm{dv}_g.$$

$$L_V = \mathrm{Id} + (\nabla V)^t (\nabla V)$$

The volume functional

The volume of V is a measure of how V fails to be parallel (i. e. $\nabla V = 0$).

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If M has finite volume

$\mathrm{Vol}(V) \geq \mathrm{vol}(M)$ with "=" if and only if V is parallel,

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Gluck and Ziller problem. *In 1986 (Comment. Math. Helv.) they proposed the following question: Which unit vector fields on odd-dimensional spheres have least volume?*

The volume functional

Gluck and Ziller, *1986*
Theorem: The unit vector fields of minimum volume on S^3 *are precisely the Hopf vector fields, and no others.*

Pedersen, *1993*
Theorem: There are smooth unit vector fields on S^n $(n>3)$ *with less volume than Hopf vector fields.*

OPEN PROBLEM

Problems about the infimum

Given a Riemannian manifold *(M,g)* with finite volume and $r > 0$ to compute

$$\mathcal{V}(M,r) = \inf\{\mathrm{Vol}(V)\ ;\ V \in \Gamma^\infty(T^r M)\}$$

and to characterise the volume-minimising vector fields.

Trivial case:

If (M,g) admits parallel vector fields ($\neq 0$) then

$$\mathcal{V}(M,r) = \mathrm{vol}(M)$$

and $\mathcal{V}(M,r) = \mathrm{vol}(V)$ *if and only if* $\nabla V = 0$.

The volume functional

Gluck and Ziller, *1986*

Theorem: The unit vector fields of minimum volume on S^3 are precisely the Hopf vector fields, and no others.

For $n = 2m+1$, $\pi : S^n \to \mathbb{C}P^m$ $\qquad \pi^{-1}(\pi(p)) = \{e^{i\theta} p\}$

$H(p) = i\, p = J(p)$ $\qquad$ standard Hopf vector field

More generally, a Hopf vector field is defined as

$$H(p) = J(p)$$

where J is any complex structure of $\mathbb{R}^{2m+2}$. They are exactly the ***unit Killing vector fields*** *of sphere S^{2m+1}*

$$\Updownarrow$$

$$\nabla V + (\nabla V)^t = 0$$

Problems about the infimum

Gluck and Ziller, *1986 (S^3, $r=1$);* Brito, *2000;* Perrone, *2008;* *— 2022*

Theorem:

For a 3-dimensional manifold M of constant curvature c>0

$$\mathcal{V}(M, r) = (1+cr^2)\,\mathrm{vol}(M).$$

Moreover, the volume-minimisers are the Killing vector fields of length r.

Problems about the infimum

— and Hurtado, *2005*

Theorem:

If $(M,g) = (S^3, g_\mu)$ *is a Berger sphere with* $\boldsymbol{\mu<1}$ *then*

$$\mathcal{V}(M, r) = (1+\mu r^2)\,\mathrm{vol}(S^3, g_\mu).$$

Moreover, $\mathrm{vol}(V) = \mathcal{V}(M, r)$ *if and only if* $V = \pm (r/\sqrt{\mu})H$, *where* H *is the Hopf vector field used to construct* g_μ.

For $\boldsymbol{\mu > 1}$,

$$\mathcal{V}(M, r) \le f(r, \mu)\,\mathrm{vol}(S^3, g_\mu) \quad \text{with } f(r, \mu) < (1+\mu r^2)$$

$$\sqrt{(1+r^2\mu)^2 + 4r^2(1+r^2\mu)\frac{1-\mu}{\mu}}$$

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Problems about the infimum

Brito, Chacón, Naveira, *2004 (r=1)*

Theorem *The volume of a vector field V of length $r > 0$ on a compact $(2m+1)$-dimensional manifold M of constant curvature $c > 0$ verifies*

$$\mathrm{Vol}(V) \geq \sum_{k=0}^{m} \frac{\binom{m}{k}^2}{\binom{2m}{2k}} r^{2k} c^k \mathrm{vol}(M)$$

and for $m \neq 1$ equality never holds.

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and for $m \neq 1$ equality never holds.

$$c(m;r) = \sum_{k=0}^{m} \frac{\binom{m}{k}^2}{\binom{2m}{2k}} r^{2k}.$$

M a compact (2m+1)-dimensional manifold of constant curvature 1.

- *If* $m=1$, $\mathcal{V}(M, r) = \mathrm{Vol}(H) = (1+r^2)\,\mathrm{vol}(M) = c(1; r)\,\mathrm{vol}(M)$

- *If* $m > 1$,

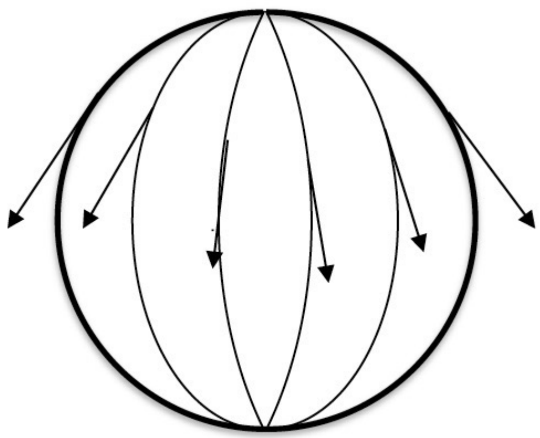
  $c(m; r)\,\mathrm{vol}(M) \leq \mathcal{V}(M, r) \leq \mathrm{Vol}(H) = (1+r^2)^m\,\mathrm{vol}(M)$

Problems about the infimum

- *For all n, radial vector fields of length r are defined on S^n minus two antipodal points.*
- *If $n=2m+1$, $\mathrm{Vol}(R) = c(m;r)\,\mathrm{vol}(S^{2m+1})$*

$$c(m;r) = \sum_{k=0}^{m} \frac{\binom{m}{k}^2}{\binom{2m}{2k}} r^{2k}.$$

If $M = S^{2m+1}$ this value is the volume of the radial vector fields which are defined in the sphere minus two points.



Problems about the infimum

- *For all n, radial vector fields of length r are defined on S^n minus two antipodal points.*
- *If $n=2m+1$,* $\mathrm{Vol}(R) = c(m;r)\,\mathrm{vol}(S^{2m+1})$

$$c(m;r) = \sum_{k=0}^{m} \frac{\binom{m}{k}^2}{\binom{2m}{2k}} r^{2k}.$$

Is $\mathcal{V}(S^{2m+1}, r) = c(m;r)\,\mathrm{vol}(S^{2m+1})$?

$$\mathcal{V}(M,r) = \inf\{\mathrm{Vol}(V)\ ;\ V \in \Gamma^\infty(T^r M)\}$$

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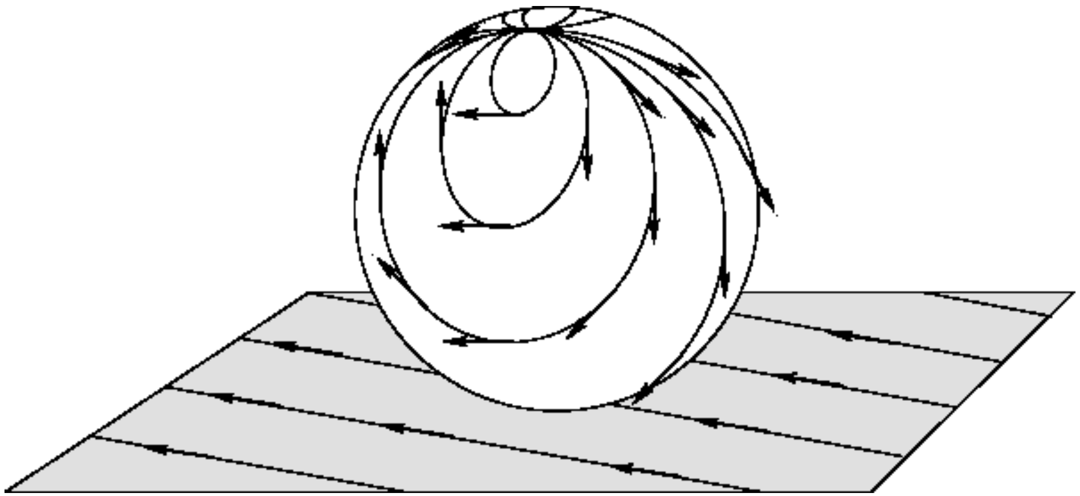
Problems about the infimum

Parallel Transport vector fields of length *r*.
For any orthonormal 2- frame $\{a , b\}$ of R^{n+1} and $r>0$, we consider the vector field $rP^{a,b}$ obtained by parallel transport of rb along radial geodesics from a, which is defined on $S^n - \{-a\}$.



E. Llinares, 2004

Problems about the infimum

Pedersen, *1993*

Theorem: *On the sphere S^{2m+1}*

- *There is a curve of smooth unit vector fields P_t such that*

  $$\mathrm{Vol}\,(P_t) \to \mathrm{Vol}\,(P) = c^*(m;1)$$

- *If $m>1$* $\mathrm{Vol}(P) < \mathrm{Vol}(H)$. *Therefore, there are smooth unit vector fields with less volume than Hopf vector fields.*

If $m>1$ then

$$c(m;\,1)\,\mathrm{vol}(S^{2m+1}) \leq \mathcal{V}(S^{2m+1},\,1) \leq c^*(m;\,1)\,\mathrm{vol}(S^{2m+1})$$

$$c(m;1) = 4^m \binom{2m}{m}^{-1}$$

$$c^*(m;1) = 4^{2m} \binom{4m}{2m}^{-1}$$

$$c(m;r) = \sum_{k=0}^{m} \frac{\binom{m}{k}^2}{\binom{2m}{2k}} r^{2k}.$$

Problems about the infimum

Borrelli and —, *2006; —, 2023*

Proposition: *On the sphere* S^{2m+1}

- *There is a curve of smooth vector fields* P_t *of length r such that*

$$\mathrm{Vol}\,(P_t) \to \mathrm{Vol}\,(rP) = c^*(m;r)\ \mathrm{vol}(S^{2m+1})$$

- *If* $m > 1$, $\mathcal{V}(S^{2m+1}, r) \leq \min\{c^*(m;r), (1+r^2)^m\}\mathrm{vol}(S^{2m+1})$.

- *If* $c^*(m;r) < (1+r^2)^m$ *then* $\mathrm{Vol}(rP) < \mathrm{Vol}(rH)$ *and there are smooth vector fields with less volume than Hopf vector fields.*

- *For all* $m>1$ *there is a value* $r_1(m)$ *such that for* $r > r_1(m)$

$$\mathrm{Vol}(rH) < \mathrm{Vol}(rP)$$

Problems about the infimum

— and Llinares, *2001,* Borrelli and —, *2006*

- *Hopf vector fields of length r on S^{2m+1} are critical points of the volume functional. They are stable if and only if*

$$r \geq r_0(m) = (2m-3)^{1/2}$$

- *If $1 \leq r_0(m) < \mathbf{r}$ and $r_1(m) < \mathbf{r}$, Hopf vector fields rH are stable critical points and there are no examples of smooth vector fields of length r with less volume than the Hopf vector fields.*

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Problems about the infimum

Borrelli and Zoubir, *2010*

Let M be any space form of positive curvature different from the sphere. The Hopf vector fields of length r are ***stable for all*** $\boldsymbol{r > 0}$.

—, *2023*

In such a manifold, for each unit smooth vector field V, which is not a Hopf vector field, there is $r_V > 0$ *such that for all* $r < r_V$

$$\mathrm{Vol}(rH) < \mathrm{Vol}(rV)$$

Moreover, there are no examples of smooth vector fields of length r with less volume than the Hopf vector fields.

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Moreover, there are no ~~examples of~~ smooth vector fields of length *r* with less volume than the Hopf vector fields.

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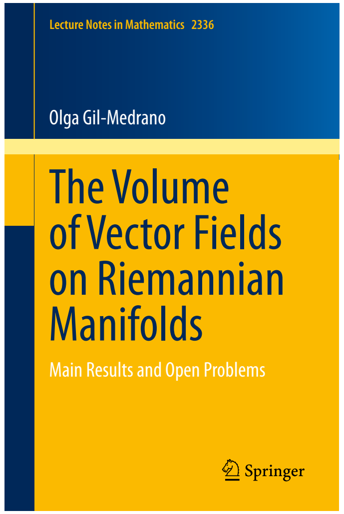
Conjecture



Some useful links at

https://www.uv.es/gilo/Pagina_web2/Book.html

https://doi.org/10.1007/978- 3- 031- 36857- 8

Thanks for your attention!

Happy Birthday Eduardo!