Marsden theorem and completeness of left-invariant semi-Riemannian metrics on Lie groups

Miguel Sánchez, Univ. Granada & IMAG

Based on A. Elshafei, AC. Ferreira, M. Sánchez, A. Zeghib: Tran. AMS (2024)

Symmetry and Shape. U. Santiago, 27/09/24

My talk is entitled

Marsden theorem and completeness of left-invariant semi-Riemannian metrics on Lie groups

Eduardo is a distinguished developer of S-R. G. in the **world**

1. Osserman manifolds in semi-Riemannian geometry (2004)
   E García-Río, DN Kupeli, R Vázquez-Lorenzo
2. Semi-Riemannian maps and their applications (2013)
   E García-Río, DN Kupeli
3. The geometry of Walker manifolds (2022)
   Peter Gilkey, Miguel Brozos-Vázquez, Eduardo García-Río, Stana Nikcević, Ramón Vázquez-Lorenzo

Further topics as:

- Lorentzian Ricci solitons,
- Null and infinitesimal isotropy in semi- Riemannian geometry,
- Lorentzian manifolds with special curvature operators,
- Curvature of indefinite almost contact manifolds...

In particular, a distinguished promoter of Lorentzian G. in **Spain**

First Int. Meet. Lorentzian Geometry, Benalmádena (2021)

A sort of Big-Bang for Spanish Lorentzian Geometry

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A sort of Big-Bang for Spanish Lorentzian Geometry ...and for a **community of very good researchers and very good friends**

Eduardo is also a distinguished promoter of the group of very good researchers and very good friends

hosting us in Santiago

Coming back, my talk is entitled

Marsden theorem and completeness of left-invariant semi-Riemannian metrics on Lie groups

and it is based on joint work with

A. Elshafei, A.C. Ferreira and A. Zeghib

(Trans AMS'24)

Main aim

Theorem (Elshafei, Ferreira, S., Zeghib '24)

Let G be a (finite-dimensional) Lie group.
If its adjoint representation has an at most linear growth,
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In particular, this includes all the known cases:

- compact (Marsden, Indiana'73)
- 2-step nilpotent (Guediri, Torino'94)
- semidirect $K \times_\rho \mathbb{R}^n$
  - K direct product of compact and abelian groups
  - $\rho(K)$ pre-compact in $GL(n, R)$

(in particular, $E(2)$ in Bromberg & Medina, SIGMA'08)

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Heuristic approach, starting at Marsden's:

compact homogeneous semi-Riemannian manifolds are complete

Planning:

1. Background and examples
2. Marsden theorem:
   - Original proof for compact homogeneous spaces
3. Clairaut metrics on Lie groups
   - A variant of Marsden's proof
4. Linear growth and geodesic completeness
5. Growth of the adjoint representation and proof of Thm.
6. *Discussion: Aff*$(\mathbb{R})$
7. *Groups of linear growth*

1. Background and examples

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  But the metric is incomplete (the party goes on!)

Geodesic completeness

- Hopf Rinow th.: basic property for Riemannian manifolds
- Subtle in the semi-Riemannian case (**no Hopf-Rinow**):

$$\text{even} \left\{ \begin{array}{c} \text{homogeneous} \\ \text{or} \\ \text{compact} \end{array} \right\} \text{Lorentz mfd possibly incomplete}$$

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Note. Completeness important for General Relativity:

- Singularity thms (as Penrose's) prove incompleteness but no curvature blow up

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$$g_0 = -2dudv \; (:= -du \otimes dv - dv \otimes du), \qquad \forall (u,v) \in \mathbb{R}^2$$

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$\rightsquigarrow$ $M = \{(u,v) \in \mathbb{L}^2 : u > 0\}$ becomes a homogeneous manifold, trivially incomplete

Example 2: highly non-Riemann. actions (Misner cylinder)

Choose $\lambda = 2$, $\phi_2(u, v) = (2u, v/2)$

$$G = \{\phi_2^k : k \in \mathbb{Z}\}$$

Let G act on (open subsets of) $\mathbb{L}^2$:

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   On $\mathbb{L}^2 \setminus \{0\}$: G acts by isometries *freely* and *discontinuously*
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3. On $M = \{(u, v) \in \mathbb{L}^2 : u > 0\}$ action *properly discontinuous*
   $\rightsquigarrow$ M/G is the Misner cylinder
   obviously incomplete
   ... and with a closed incomplete geodesic

- Misner cylinder M/G

  $M = \{(u, v) \in \mathbb{L}^2 : u > 0\}$

- Misner cylinder has an incomplete closed (lightlike) geodesic!

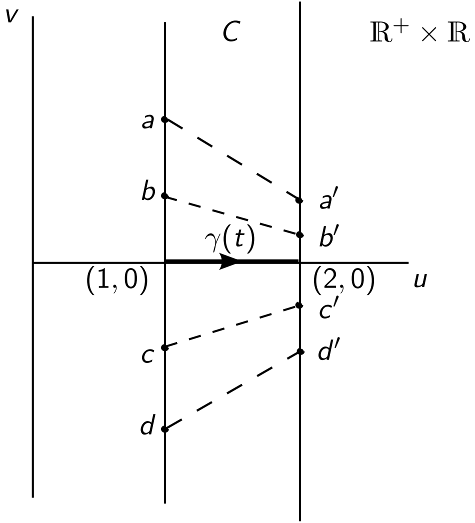
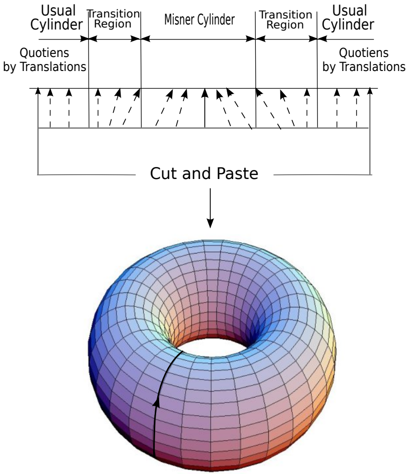


Figure:

Example 4: Incomplete Lorentzian tori (intuitive)

- Misner cylinder shows that an incomplete geodesic may remain in a compact region.
- Intuitively, it's easy to go from the cylinder to a torus!



Example 4: Incomplete Lorentzian tori (explicit)

$(\mathbb{R}^2, g)$ in usual coordinates

$$g = 2dxdy - 2\tau(x)dy^2,$$

where $\tau : \mathbb{R} \to \mathbb{R}$ satisfies:

(1) 1-periodic
$\rightsquigarrow$ The metric g is inducible in the quotient torus $\mathbb{R}^2/\mathbb{Z}^2$

(2) $\tau(0) = 0$.
$\rightsquigarrow$ The coordinate axis y is the image of a lightlike geodesic

(3) $\tau'(0) \neq 0$.
$\rightsquigarrow$ Such a lightlike geodesic is incomplete:

- Christoffel symbol: $\Gamma^y_{yy}(x, y) = \frac{1}{2} g^{yx} (2\partial_y g_{xy} - 2\partial_x g_{yy}) = \tau'(x)$,
- Equation for the component $y(t)$: $y''(t) + \tau'(0) y'(t)^2 = 0$.
- Incomplete y-solution: $y(t) = \ln(t)/\tau'(0)$

Example 4: Incomplete Lorentzian tori: Killing family

Notes:

1. Clifton-Pohl's torus : $\mathbb{R}^2/\mathbb{Z}^2$

$$g = \pi^2 \cos 2\pi x (2dxdy) - \pi^2 \sin 2\pi x (dx^2 - dy^2)$$

   1st example compact+incomplete

2. Lorentzian tori with a Killing field K ($K = \partial_y$ above)
   incomplete $\Leftrightarrow$ $g(K,K)$ non-constant sign
   $\Leftrightarrow$ space, time & lightlike incomplete geod.
   - S., Trans AMS '97 (systematic study)
   - Many subtler properties: Mehidi, Math Z'22, Geom. Ded.'23

2. Marsden theorem

Theorem (Marsden'73)

A compact homogeneous semi-Riemannian manifold is complete.

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Proof. Let $\gamma : [0, b) \to M$, $b < \infty$ be a geodesic:

1. General: γ' in compact $C_\gamma \subset TM \Rightarrow \gamma$ extends smoothly to b
2. Under our hyp.: such a C_γ exists (and will contain $\gamma'([0, \infty))$

1. γ' in a compact subset $C_\gamma \subset TM \Rightarrow \gamma$ extensible to b

Lemma (Step 1)

For any affine conn. ∇ on M:

- $\{\gamma'(t_m)\}_m$ *converges in* TM *for some* $\{t_m\} \nearrow b$
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Proof. Consequence of $\rho := \gamma'$ is an integral curve of the geodesic vector field $\mathcal{G}$ on TM.
(existence and uniqueness of its local flow through the limit) $\square$

2. $\exists C_\gamma$ compact containing γ'

Lemma (Step 2)

Let $K_1, \dots K_m$ be ($m \geq$dim M) a base of Killing algebra and

$$\begin{aligned} c_i &:= & g(\gamma'(0), K_i), & \quad i = 1, \dots m \\ C_\gamma &:= & \{v \in TM : c_i = g(v, K_i), & \quad i = 1, \dots m\} \end{aligned}$$

(a) C_γ *contains* $\gamma'(t), \forall t \in [0, b)$

(b) C_γ *is compact*

Proof. (a) For any geodesic γ and Killing K:

$$g(\gamma', K) \qquad \text{is constant}$$

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Proof. (b) Steps:

1. C_γ is closed (trivial)
2. $\Pi : TM \to M$ restricted to C_γ injective (c_i's overdetermine v)

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At each $p \in \partial(\Pi(C_\gamma))$, choose n Killing independent at p, a normal neigh. and use overdetermination of geod. through p

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   At each $p \in \partial(\Pi(C_\gamma))$, choose n Killing independent at p, a normal neigh. and use overdetermination of geod. through p
4. $\Pi|_{C_\gamma}$ is continuously invertible (as v varies cont. with c_i's) □

Notes. Going further

- Weaken: homogeneous $\longrightarrow$ conformally-homogeneus
    - $\gamma'([0, b)))$ is proven to lie in a compact subset of TM
    - ...but, as a difference with Marsden's, $\gamma'([0, \infty))$, **possibly not**

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- Semi-Riemannian g of index s:
  - s pointwise indep. timelike ($g(K,K) < 0$) conf. Killing v.f. $\Rightarrow$ completeness
  - Extensible to **non-compact M** case under some assumptions

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- **Corollary** [precedent of ours] for non-compact Lorentz M:
  - $\exists$ timelike Killing K with $|g(K,K)| \geq \epsilon > 0$
  - It is complete the ("Wick-rotated") Riemann $g_R := g - 2(K^\flat \otimes K^\flat)/g(K,K)$

$\Rightarrow$ complete g

Notes. Going even beyond

- Compact Lorentz with K lightlike
  - K Killing $\not\Rightarrow$ complete, Hanounah, Mehidi, arxiv: 2403.15722
  - K Parallel $\Rightarrow$ complete, Mehidi, Zeghib, arxiv: 2205.07243
    - Applicable even **weakening compactnes**
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- For (M, ∇) compact affine (possibly non-symmetric) Precompact holonomy $\Rightarrow$ completeness (Aké, S., JMAA '16)

- G admits a natural uniformity
  Base of entourages: $\{V_U : U$ is a neighbourhood of $1\}$ where $V_U := \{(p, q) \in G \times G : \ q^{-1}.p \in U\}$.
  $\rightsquigarrow$ Cauchy filters, completeness

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  $\rightsquigarrow$ Cauchy filters, completeness
- All the left invariant **Riemanian** metrics g_R, g'_R are:
  - Complete (homogeneous positive def. spaces)
  - Bilipschitz bounded: $c\, g_R \leq g_R{}' \leq g_R/c$. $(c \in \mathbb{R})$

  $\rightsquigarrow$ induce the natural uniformity on G

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- Right invariant: $Y_i(p) = e_i.p$
  - Y_i Killing for g
- Clairaut forms and coframe: $\omega^i := g(Y_i, \cdot)$
  - For any geodesic γ: $\omega^i(\gamma') \equiv c \in \mathbb{R}$
  - Transformation law ($\dagger \equiv g$-adjoint operator):

$$\begin{aligned} \omega^i_p(p.u) &= g_p(Y_i(p), p.u) = g_p(e_i.p, p.u) = g_1(\mathrm{Ad}_{p^{-1}}(e_i), u) \\ &= g_1(e_i, ((\mathrm{Ad}_p)^{-1})^\dagger)(u)) = \omega^i_1((\mathrm{Ad}^\dagger_p)^{-1}(u)). \end{aligned}$$

no left (nor right) invariant

3. Clairaut metric: concept

Definition

Clairaut metric (associated to a Clairaut coframe: g, (e_i)):

$$h := \sum \omega^i \otimes \omega^i$$

- h Riemannian metric on G
- Transformation rule:
  $h_p(p.u, p.v) = \sum_i \, g_1((\mathrm{Ad}_{p^{-1}})(e_i), u) \, g_1((\mathrm{Ad}_{p^{-1}})(e_i), v).$

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- Change of bases (e_i), $(\hat{e}_i)$ with Clairaut h, $\hat{h}$:
  - bi-Lipschitz bounded ($\Rightarrow$ uniformly equivalent distances)
    Moreover: Transition matrix M orthonormal $\Longrightarrow h = \hat{h}$

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$\rightsquigarrow$ **Clairaut uniformity of g**

Theorem

If the Clairaut uniformity [or metric h] is complete then g is complete.

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Proof. Let $\gamma : [0, b) \to M, b < \infty$ a geodesic:

- $h(\gamma', \gamma') \equiv C$, thus, γ has finite h-length

Theorem

If the Clairaut uniformity [or metric h] is complete then g is complete.

Proof. Let $\gamma : [0, b) \to M$, $b < \infty$ a geodesic:

- $h(\gamma', \gamma') \equiv C$, thus, γ has finite h-length
- h is complete (and Riemannian):
  γ' lies in a compact subset of TM
  $\rightsquigarrow \gamma$ extensible as a geodesic $\square$

Corollary (Special case of Marsden's)

Any left invariant metric g on a compact Lie group G is complete

Proof. Its Clairaut h is complete because M is compact □

Theorem (Alternative proof of Marsden)

A compact homogeneous semi-Riemannian manifold is complete.

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Compactness of M: finite covering $U_{p_k}, k = 1, \ldots, s$
$\longrightarrow h = \sum h_{p_k}$ is positive def. (and complete)
$\rightsquigarrow$ the uniformity of h is complete $\rightsquigarrow$ g is complete $\square$

In the remainder:

1. Given left invariant g, construct Clairaut h
   $\rightsquigarrow$ choice of basis (e_i) of $\mathfrak{g} = T_1 G$
2. Our aim will be to prove completeness of h (and thus of g)

Choices (with the aim of proving h complete)

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Expression for h:

$$h_p(p.u, p.v) = \tilde{\mathbf{g}}_1(\mathrm{Ad}^*_{p^{-1}}(\psi(u)), \mathrm{Ad}^*_{p^{-1}}(\psi(v)))$$

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Summing up, h constructed from:

- **Euclidean** $\tilde{g}_1$, its **isometry** ψ and **adjoint operator** *
- the **adjoint representation** of G: $\mathrm{Ad}_q(v) = q \cdot v \cdot q^{-1}$.

4. Linear growth and geodesic completeness

Abstract setting:

- M (non-compact, connected) mfld, g_R Riemann., complete
  g_R-norm $\|\cdot\|_R$, $d_R(x) := \text{dist}(x, x_0)$ for some $x_0 \in M$

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Optimal growth of φ to ensure completeness for h?

- Estimate for $(M, g_R) = (\mathbb{R}, dx^2)$:
  divergent curve $\gamma(x) = x$, $x_0 = 0$

$$\mathrm{length}_h(\gamma) \geq \int_0^\infty \frac{dx}{\varphi(|x|)} = \infty$$

Proposition

If $\varphi : [0,\infty[\to]0,\infty[$ *be satisfies*

$$\int_0^\infty \frac{1}{\varphi(r)} dr = \infty$$

and

$$\| v_x \|_h \geq \frac{\| v_x \|_R}{\varphi(d_R(x))}, \qquad \forall x \in M$$

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In particular, when φ *grows at most linearly,*

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Proof of the general case : reduce to dim 1,

use Lipschitz regularity of d_R at the cut locus.

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2. Choosing φ increasing: natural to estimate growth
   ...even if it loses generality
3. The uniformities of h and g_R are not equal under such bounds

Note. Some related results on completeness and growth:

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- Completeness of spacelike submanifolds of $\mathbb{L}^n$:
  at most linear (subaffine)
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- Completeness of trajectories accelerated by a potential V
  At most quadratic growth of V
  Abraham, Marsden book'87, Candela, Romero, S. ARMA'13
  Ehlers-Kundt conjecture (Flores, S. JDE'20)

5. Growth of the adjoint representation and proof of Thm

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4. **Next** : sufficient hypoth. on Ad to apply the criterion

Concept of (at most) linear growth for G

For left-invariant Riem. g_R on G with norm $\|\cdot\|$, let:
$r : G \longrightarrow \mathbb{R}, \qquad r(p) := \mathrm{dist}_R(1, p)$

$$\begin{aligned} \|\mathrm{Ad}_p\| &= \mathrm{Max}_{\|u\|=1}\{\|\mathrm{Ad}_p(u)\|\} \\ = \lambda_+(p) &:= \mathrm{Max}\{\sqrt{\Lambda_i} : \Lambda_i \text{ is a eigenvalue of } \mathrm{Ad}_p^* \circ \mathrm{Ad}_p\} \end{aligned}$$

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Definition

G has (at most) linear growth if there exist constants $a, b > 0$ such that for $p \in G$, $u \in \mathfrak{g}$, alternatively:

- $\|\mathrm{Ad}_p(u)\| \leq (a + b\, r(p))\|u\|$
- $\dfrac{\|u\|}{a + b\, r(p)} \leq \|\mathrm{Ad}_p(u)\|$
- $\dfrac{\|u\|}{a + b\, r(p)} \leq \|\mathrm{Ad}_p(u)\| \leq (a + b\, r(p))\|u\|.$

Remarks:

- Equivalences: use $r(p) = r(p^{-1})$, $\forall p \in G$.
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Lemma

Let G be of linear growth. Then, the Clairaut metric h associated to any pair of Wick rotated semi-Riemannian metrics $(g, \tilde{g})$ satisfies the criterion of completeness for $g_R = \tilde{g}$.

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$$\begin{aligned} h_p(p.u, p.u) = & \quad \tilde{g}_1(\mathrm{Ad}^*_{p^{-1}}(\psi(u)), \mathrm{Ad}^*_{p^{-1}}(\psi(u))) \\ = & \quad \tilde{g}_1(\psi(u), \mathrm{Ad}_{p^{-1}} \circ \mathrm{Ad}^*_{p^{-1}}(\psi(u))) \\ \geq & \quad \lambda_-(p^{-1})^2 \, \tilde{g}_1(\psi(u), \psi(u)) \end{aligned}$$

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$$h_p(p.u, p.u) \geq \frac{\tilde{g}_1(u, u)}{\|\mathrm{Ad}_p\|^2} = \frac{\tilde{g}_p(p.u, p.u)}{\|\mathrm{Ad}_p\|^2}.$$

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Taking roots, criterion fulfilled with $\varphi(p) = \|\mathrm{Ad}_p\|$ (linear) □

Main theorem

Theorem

All the left-invariant semi-Riemannian metrics of a Lie group with linear growth are geodesically complete.

Proof. Linear growth of $G \implies h$ complete $\implies g$ complete $\square$

6. Discussion: the case of $\mathrm{Aff}(\mathbb{R})$

First questions: linear, polynomyal, exponential growth:

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First questions: linear, polynomyal, exponential growth:

- Q1: Interest for other issues on Lie groups?
- Q2: Makes sense to consider a finer growth as $r \log^k(1+r)$ for Lie groups?

Discussion

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$\Longrightarrow$ Growth of Clairaut h (respect to g_R) independent g-signature

h **for** g **Riemannian (complete) equal growth than**
h **for** g **indefinite (possibly incomplete)!**

Example: affine group of $\mathbb{R}$

Aff($\mathbb{R}$): exponential growth and incomplete g!

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- Admits left invariant $g^{(+1)}$, $g^{(-1)}$ with Clairaut $h^{(+1)}$, $h^{(-1)}$

g^{+1} **Riemanian** (Complete)	h^{+1} (Riem.) **Complete**
$g^{(-1)}$ **Lorentzian Incomplete**	h^{-1} (Riem.) **Incomplete**

The growth of $h^{(+1)}$, $h^{(-1)}$ respect to g_R are equal!

- **Growth of Ad (and h)** $\rightsquigarrow$ eigenvalues $\lambda_+^2(p)(=\|\mathrm{Ad}\|^2)$
  independent of signature
- **Completeness of h (and g)** $\rightsquigarrow$ eigendirections
  do depend on signature
  (adjoint operator * , Euclidean isometry ψ)
  and conspire to ensure or destroy completeness

Explicit computations: background

$\mathrm{Aff}(\mathbb{R})$ affine transformations of the line $f(x) = ax + b$, $a \neq 0$.

$$\mathrm{Aff}^+(\mathbb{R}) = \left\{ \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} : x > 0, y \in \mathbb{R} \right\}$$

$$\mathfrak{aff}(\mathbb{R}) \left(= T_1(\mathrm{Aff}^+(\mathbb{R}))\right) = \left\{ \begin{pmatrix} u & v \\ 0 & 0 \end{pmatrix} : u, v \in \mathbb{R} \right\}$$

Basis at $\mathfrak{aff}(\mathbb{R})$

$$e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad e_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad [e_1, e_2] = e_2$$

Left invariant vector basis: $X_1 = x\partial_x, X_2 = x\partial_y$
Left invariant g: $g(X_i, X_j) \equiv$ constant, matrix

$$\begin{pmatrix} c_1 & c_2 \\ c_2 & c_3 \end{pmatrix}. \qquad c_1 c_3 - c_2^2 \neq 0$$

Explicit computations: left invariant $g^{(\pm 1)}$

General left invariant g

$$g = \frac{1}{x^2}(c_1\, dx^2 + c_2\,(dxdy + dydx) + c_3\, dy^2), \qquad c_1 c_3 - c_2^2 \neq 0$$

Choices

$$c_1 = 1, c_2 = 0, c_3 = \epsilon = \pm 1$$

$$g^{(\epsilon)} = \frac{1}{x^2}(dx^2 + \epsilon dy^2).$$

- $g^{(+1)}$: left-invariant Riemannian metric $\Longrightarrow$ complete (hyperbolic space)
- $g^{(-1)}$: left-invariant, Lorentz
  - incomplete geodesic $\gamma(t) = \left(\frac{1}{1-t}, \frac{1}{1-t}\right)$

Explicit computations: Clairaut $h^{(\pm 1)}$

Right-invariant (Killing) v.f. induced by e_1 and e_2:

$$Y_1 = x\partial_x + y\partial_y, \quad Y_2 = \partial_y$$

Clairaut forms

$$\omega^1 = \frac{1}{x^2}(xdx + \epsilon dy) \quad \text{and} \quad \omega^2 = \frac{\epsilon}{x^2}dy$$

Clairaut metrics ($h^{(\epsilon)} = (\omega^1)^2 + (\omega^2)^2$)

$$h^{(\epsilon)} = \frac{1}{x^4}\left(x^2dx^2 + (1+y^2)dy^2 + \epsilon xy(dxdy + dydx)\right).$$

$$\text{Matrix:} \qquad \frac{1}{x^4}\begin{pmatrix} x^2 & \epsilon xy \\ \epsilon xy & 1+y^2 \end{pmatrix} \qquad (\text{recall } x > 0)$$

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(Aim: equal growth, but complete $\epsilon = 1$, incomplete $\epsilon = -1$!)

Explicit computations: growth of Clairaut $h^{(\pm 1)}$/ Ad

Eigenvalues $\mathrm{evl}_{\pm}$ independent of ϵ (determinant $= \epsilon^2/x^6$)

$$\frac{1}{2x^4}\left(x^2+\epsilon^2(1+y^2)\pm\sqrt{(x^2+\epsilon^2(1+y^2))^2-4\epsilon^2x^2}\right)$$

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- They measure the growth of Ad in coordinates respect to Euclidean dx^2+dy^2 (non left-invariant)
- The growth is exponential respect to $g^{(+1)} = (dx^2+dy^2)/x^2$ (hyperbolic)

As expected, growth independent of $\epsilon = \pm 1$

Explicit computations: completeness of Clairaut $h^{(\pm 1)}$

Incompleteness of $h^{(-1)}$ ($\Leftarrow g^{(-1)}$ was incomplete): the curve

$$\gamma(t) = (x(t), y(t)) = (\cosh t, \sinh t) \qquad \forall t \geq 0$$

(there is a heuristic way to arrive at it!)

- Clearly diverging
- Easy to show it has finite length for $h^{(-1)}$

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Explicit computations: completeness of Clairaut $h^{(\pm 1)}$

Completeness of $h^{(+1)}$ (is there a reason to ensure this?):

1. For $\gamma(t) = (x(t) > 0, y(t)), t \in [0, b), b \leq \infty$ diverging $\rightsquigarrow$ check infinite length
2. Bound for the minimum eigenvalue of $h^{(+1)}$

$$\begin{aligned} \text{evl}_- &= \tfrac{1}{2x^4}\left((1+x^2+y^2) - \sqrt{(1+x^2+y^2)^2 - 4x^2}\right) \\ &\geq \frac{1}{x^2(1+x^2+y^2)} \end{aligned} \tag{1}$$

3. So, for bounded "Euclidean" radius $r^2(t) := x^2(t) + y^2(t) < 2C^2$,

$$h^{(1)} \geq \frac{dx^2+dy^2}{x^2(1+x^2+y^2)} \geq \frac{dx^2+dy^2}{x^2}\frac{1}{(1+2C^2)}$$

which is a (complete) hyperbolic metric.

4 Finer bound using $r^2 = x^2 + y^2$, $(r^2)' = 2x\dot{x} + 2y\dot{y}$:

$$\begin{array}{rcl} h^{(\epsilon=1)}(\dot\gamma(t),\dot\gamma(t)) & = & \frac{1}{x^4}(x^2\dot{x}^2 + (1+y^2)\dot{y}^2 + 2x\dot{x}y\dot{y}) \\ & = & \frac{1}{x^4}\left((x\dot{x}+y\dot{y})^2 + \dot{y}^2\right) \\ & \geq & \frac{1}{x^4}(x\dot{x}+y\dot{y})^2 = \frac{1}{x^4}(\frac{1}{2}(r^2)')^2 \\ & \geq & \frac{1}{r^4}(\frac{1}{2}(r^2)')^2 = (\frac{1}{2r^2}(r^2)')^2 \\ & = & \left(\frac{1}{2}(\ln(r^2))'\right)^2. \end{array}$$

(sharp when $y \equiv 0$)

5 Taking $t_n \nearrow b$ such that $\{\gamma(t_n)\}_n$ (thus $r(_n)$) is unbounded:

$$\begin{array}{rcl} \text{length}(\gamma) & \geq & \lim_{t_n\to b} \frac{1}{2}\int_0^{t_n}(\ln(r^2))'(t)dt \\ & = & \frac{1}{2}\left(\lim_{t_n\to b}\ln(r^2(t_n)) - \ln(r^2(0))\right) \\ & = & \lim_{t_n\to b}\ln(\mathbf{r}(\mathbf{t_n})) - \ln(r(0)) = \infty, \end{array}$$

i.e., it **goes to infinity (albeit it seems slowly!)**

Explicit computations: questions

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- Q3: If left. inv. g is Riemannian
  must its Clairaut h be complete?

(*If positive answer*)

- Q4: if left-inv. g is **complete**
  must its Clairaut h be complete? (we know the converse)
  If negative, is a geometric interpretation of the Cauchy boundary of h possible?

Technical available

Natural action $\mathrm{Aut}(\mathfrak{g})$ on $\mathrm{Sym}^*(\mathfrak{g})$

- $\mathrm{Aut}(\mathfrak{g})$: Lie algebra automorphisms of $\mathfrak{g}$
- $\mathrm{Sym}^*(\mathfrak{g})$: scalar products (of any signature) on $\mathfrak{g}$

$$\varphi \in \mathrm{Aut}(\mathfrak{g}) \to g^\varphi \quad (g^\varphi)_1 = \varphi.g_1. \quad (\text{pushforward})$$

$\rightsquigarrow$ orbit of g_1 in $\mathfrak{g}$
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and, thus of g (in the open set of left invar. metr.)

Proposition

1. *all the g's in the same orbit are either complete or incomplete.*
   i.e. g^φ complete $\Longleftrightarrow$ g complete.
2. *all Clairaut h's associated to left-invariant g's on the same orbit are bi-Lipschitz bounded, thus,*
   either complete or incomplete

Explicit computations: classes of metrics in $\mathrm{Aff}(\mathbb{R})$

Three classes of left invariant metrics in $\mathrm{Aff}(\mathbb{R})$ (up to scaling)

1. $g^{(+1)}$, Riemannian (complete).
2. $g^{(-1)}$, Lorentzian, incomplete.
3. $g^{(0)}$, Lorentzian, incomplete.

$$g^{(0)} := \frac{2dxdy}{x^2}$$

(choice $c_1 = 0, c_2 = 1, c_3 = 0$ before)

Trivial cases

Proposition

G is of linear growth in the following cases:

- *Abelian ($Ad_p = Id$ for all p, $\|Ad_p\| \equiv 1$)*
- *or compact ($G \ni p \mapsto \|Ad_p\|$ has a maximum)*

7. Groups of linear growth: subgroups

Proposition

If G is of linear growth then so is any subgroup $H < G$

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Proof. R_G, d_G Riem, distance; R_H, d_H restrictions to H; $p \in H$.

- $d_G(1, p) \leq d_H(1, p)$.
- $\|\mathrm{Ad}_p^H\| \leq \|\mathrm{Ad}_p^G\|$

$$\|\mathrm{Ad}_p^H\| \leq \|\mathrm{Ad}_p^G\| \leq a + b\, d_G(1, p) \leq a + b\, d_H(1, p) \quad \square$$

Direct and semi-direct products

Proposition

$G = G_1 \times G_2$ with G_1, G_2 Lie groups with linear growth
$\Longrightarrow G$ has linear growth.

Idea of the proof.
Linear bounds $a_i + b_i r$ of G_i's
$\longrightarrow$ single one $(a_1 + a_2) + (b_1 + b_2)r$ for G.
□

Direct and semi-direct products

Proposition

Let G be the semidirect product $K \ltimes_\rho V$, with

- *K: pseudo-compact, i.e. product of compact and linear groups ($\Leftrightarrow$ admits a bi-invariant Riem. metric)*
- *V: linear group,*
- *$\rho : K \longrightarrow \mathrm{GL}(V)$ representation with $\rho(K)$ precompact .*

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Steps of the proof.

1. $\exists g_R$ common left-invariant Riem. for $K \times V$ and $K \ltimes_\rho V$ (Precompactness $\rightarrow$ G admits Ad(K)-invariant Riem. met. $\rightsquigarrow$ take a direct product by one on V)
   - $\Rightarrow$ Left-invariant Riem. met. on $K \times V$ and $K \ltimes_\rho V$ bi-Lipschitz (with g_R and, then, among them)
2. $\rightsquigarrow$ Follow as in products using a bound for $\|\rho(K)\|$

2-step nilpotent groups

Proposition

If G is 2-step nilpotent, then it has linear growth.

Suggested as 2-step nilpotent $\Longrightarrow$
$\mathrm{Ad}_{\exp(\mathbf{ta})} = \exp^{\mathrm{ad}_{\mathbf{ta}}} = \mathrm{I} + \mathbf{t}\,\mathrm{ad}_{\mathbf{a}}$

2-step nilpotent groups

Proposition

If G is 2-step nilpotent, then it has linear growth.

Steps of the (non-trivial) proof

1. Z center $\rightsquigarrow \pi : G \longrightarrow G/Z$ fibration $\ker d\pi_1 = \mathfrak{z}$ (G/Z Lie)
   2-step nilpot. $\Rightarrow G/Z$ Abelian and $\mathbb{R}^d$ (as $\Pi_1(G) \subset$ center $\tilde{G}$)
2. Choose left-inv. Rieman. g_R met.:
   - $\mathfrak{p} := \mathfrak{z}^\perp \equiv T_1(G/Z)$ (horizontal v.)
   - $\pi : G \longrightarrow G/Z$ is a Riem. submersion
     $\Rightarrow$ contracting map: $d_G \geq d_{G/Z}$
3. Any (unit) geodesic γ initially horizontal:
   - remains horizontal
   - project onto a geod (globally minimizing) of $G/Z \equiv \mathbb{R}^d$

   $\rightsquigarrow \gamma$ minimizing and
   $d_G(\gamma(t), \gamma(s)) = |t - s| = d_{G/Z}(\pi(\gamma(t)), \pi(\gamma(s)))$, while
   $z \in Z \rightsquigarrow \gamma(t)z \in \pi^{-1}(\pi(\gamma(t)) \Rightarrow d_G(\gamma(t)z, \gamma(s)) \geq |t - s|$

4. Horizontal geodesics through 1 are one-parameter subgroups
   Use 2-step nilpotency in Euler-Arnold eqn. for geodesics

$$\dot{x}(t) = \mathrm{ad}^*_{x(t)} x(t)$$

   (first orden eqn in $\mathfrak{g}$; $x(t) := \gamma^{-1}(t)\dot{\gamma}(t) \in \mathfrak{g}$; $*$ g_R-adjoint)

5. For $p \in G \setminus Z$, $\exists$ minim., hor. geod. γ from 1 to $\pi^{-1}(\pi(p))$
   - $p = \exp w$, $w \in \mathfrak{g}$ (for 2-step 1-connected, exp diffeo)
   - $w = u + v$, $u \in \mathfrak{p}(= \mathfrak{z}^{\perp})$, $v \in \mathfrak{z}$, let $z = \exp(-v)$
   - $\mathbf{pz} = \exp(\mathbf{u})$ (Baker-Campbell-Hausdorff with $[w, v] = 0$)
   - $\exp(u)$ lies in horiz. geod. $\gamma(t) = \exp(ta)$, $a := u/\|u| \in \mathfrak{p}$.

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   Thus, using γ unit: $r_G(p) = d_G(1, p) \geq d_G(1, pz) \geq t$
6. Check this proves affine growth of G
   For $p = z \in Z$, $\mathrm{Ad}_z = I$, $\|\mathrm{Ad}_z\| = 1$.
   For $p \in G \setminus Z$:
   - $\mathrm{Ad}_{pz} = \mathrm{Ad}_p$ (Z center)
     $\mathrm{Ad}_{\exp(\mathbf{ta})} = \exp^{\mathrm{ad}_{\mathbf{ta}}} = \mathrm{I} + \mathbf{t}\,\mathrm{ad}_{\mathbf{a}}$, ($G$ 2-step nilpotent)
   - Putting $\alpha = 1, \beta = \|\mathrm{ad}_a\|$, for $u \in G$
     $\|\mathrm{Ad}_p u\| = \|\mathrm{Ad}_{pz} u\| \leq (\alpha|t| + \beta)\|u\| = \alpha r(pz) + \beta \leq \alpha r(p) + \beta$

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- **Q6: Give a complete classification of groups of linear growth (and extend to quadratic, cubic... exponential).**

Other results on growths

- No k-step nilpotent with $k > 2$ is of linear growth
  Idea proof: expand $\mathrm{Ad}_{\exp(ta)}(u)$ in terms of powers of t
  (coefficients $\mathrm{ad}_{ta}^{k'}(u)/k'! = t^{k'} ad_a^{k'}(u)/k'!$, with $k' \leq k-1$)
  $\rightsquigarrow$ For t large: $\frac{\|\mathrm{Ad}_{p(t)}(u)\|}{t^{k-1}} \geq C$
  ($p(t) = \exp(ta)$ diverges and use $t \geq d(1, p(t))$)

Euler-Arnold eqn for geod. $\dot{x} = \mathrm{ad}^*_x x$

- Idempotent: $y_0 \neq 0$ such that $\mathrm{ad}^*_{y_0} y_0 = y_0$
- Easily: Idempotent $\Rightarrow$ incomplete geodesic

(this happened in Aff($\mathbb{R}$))

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(this happened in $\mathrm{Aff}(\mathbb{R})$)

Proposition

If G can be equipped with a semi-Riemannian metric g admitting an idempotent $\Rightarrow$ G exponential growth. (in its direction)

Idea of proof. Power series in the direction of the idempotent y_0:

$$\mathrm{Ad}_{p(t)=expta}(y_0) = e^t y_0$$

Exponential growth: $\|\mathrm{Ad}_{p(t)}(y_0)\| \geq t^m \|y_0\| \geq d_R(1, p(t))^m \|y_0\|$

Summary of open questions

In blue, questions on growth independent of completeness

- Q1: Interest of growth for other issues on a Lie group G?
- Q2: Makes sense finer growths (as $r \log^k(1+r)$) for G?
- Q3: If left inv. g is Riem., must its Clairaut h be complete?
- Q4: If left inv. g is complete, must Clairaut h be complete? If negative, is a geometric interpretation of the Cauchy boundary of h possible?
- Q5: Can the growth of (1-connected) G be deduced from $\mathfrak{g}$?
- **Q6: Give a complete classification of groups of linear growth (and extend to quadratic, cubic... exponential).**

Thank you for your attention!

Happy Anniversary Eduardo!