Vector fields and magnetic maps

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Outline

Geodesics

... are given by a second order nonlinear differential equation:
Euler-Lagrange equation of motions

More precisely, a *geodesic* γ in a Riemannian manifold (M, g) is characterized as critical point of the **kinetic energy** (also called the **action integral**)

$$E(\gamma) = \int \frac{1}{2}|\gamma'(s)|^2\, ds$$

Geodesics

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Euler-Lagrange equation of motions

$$\ddot{x}^k(s) + \Gamma^k_{ij}(x(s))\dot{x}^i(s)\dot{x}^j(s) = 0$$

$$(\textbf{EL}) \qquad \frac{d}{ds}\left(\frac{\partial L}{\partial \dot{x}^h}\right) - \frac{\partial L}{\partial x^h} = 0$$

Lagrangian : $L(x, \dot{x}) = g_{ij}(x(s))\dot{x}^i(s)\dot{x}^j(s)$

Magnetic curves

Let ω be the **potential 1-form**.

For a curve $\gamma : [a, b] \longrightarrow M$ consider the functional

$$LH(\gamma) = \int_a^b \left(\frac{1}{2} \langle \gamma'(t), \gamma'(t) \rangle + \omega(\gamma'(t)) \right) dt.$$

It is often called the **Landau Hall functional** for the curve γ with the potential 1-form ω.

Magnetic curves

Consider a variation of γ:

$\Gamma : [a, b] \times (-\upsilon, \upsilon) \to M, \quad \Gamma(t, 0) = \gamma(t), \Gamma(a, \cdot) = \gamma(a), \Gamma(b, \cdot) = \gamma(b)$

Simplify the notations: $\gamma_\epsilon : [a, b] \longrightarrow M$, $\gamma_\epsilon(t) = \Gamma(t, \epsilon)$

The variation vector on γ: $V = \frac{\partial \gamma_\epsilon}{\partial \epsilon} : [a, b] \longrightarrow M$, that is $V(a) = V(b) = 0$.

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In order to find the critical points of the functional LH we compute:

$$\frac{d}{d\epsilon} LH(\gamma_\epsilon)\Big|_{\epsilon=0} = -\int_a^b g\big(\nabla_{\gamma'}\gamma' - \phi(\gamma'), V\big)\, dt.$$

Magnetic curves

The critical points of the LH functional are solutions of the equation $\frac{d}{d\epsilon}LH(\gamma_\epsilon)\Big|_{\epsilon=0} = 0$, that is

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$$\frac{d}{d\epsilon} LH(\gamma_\epsilon)\Big|_{\epsilon=0} = -\int_a^b g\left(\nabla_{\gamma'}\gamma' - \phi(\gamma'), V\right) dt = 0,$$

which is equivalent to

$$\nabla_{\gamma'}\gamma' - \phi(\gamma') = 0$$

known as the **Lorentz equation**.

Background

(M, g) Riemannian manifold; $(\dim M = n \geq 2)$

Lorentz force ϕ: $g(\phi(X), Y) = d\omega(X, Y)$, X, Y tangent to M

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(M, g) Riemannian manifold; $(\dim M = n \geq 2)$

magnetic field: F - closed 2-form on M

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A smooth curve γ in (M, g, F) is called

magnetic curve/trajectory of (M, g, F)

if its velocity vector field γ' satisfies the **Lorentz equation**:

$$\nabla_{\gamma'}\gamma' = \phi(\gamma')$$

Examples of magnetic fields

- in dimension 2: any $f\ d\sigma$
- the Kähler 2-form (almost Kaehlerian manifolds)
- the fundamental 2-form in almost contact metric manifolds (Sasakian, cosymplectic, quasi-Sasakian manifolds)

Harmonic maps

The notion of geodesic is generalized to maps between Riemannian manifolds.

A map $f:(N,h)\to(M,g)$ between Riemannian manifolds is said to be **harmonic** if it is a critical point of the energy functional:

$$E(f)=\int_N \frac{1}{2}|df|^2 dv_h$$

under compactly supported variations. The Euler-Lagrange equation of this variational problem is given by

$$\tau(f)=\mathrm{div}\, df=0.$$

Here $\tau(f)$ is called the tension field of f.

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$$\tau(f)=h^{ij}(x)\left(\frac{\partial f^\alpha}{\partial x^i\partial x^j}-{}^N\Gamma^k_{ij}(x)\frac{\partial f^\alpha}{\partial x^k}+{}^M\Gamma^\alpha_{\beta\epsilon}(f(x))\frac{\partial f^\beta}{\partial x^i}\frac{\partial f^\epsilon}{\partial x^j}\right)\left.\frac{\partial}{\partial y^\alpha}\right|_{f(x)}=0$$

Here $\tau(f)$ is called the tension field of f.

The Landau Hall functional for maps

Let $f : N \longrightarrow M$ be a smooth maps between two Riemannian manifolds (N, h) of dimension n and (M, g) of dimension m.

Let ξ be a divergence free vector field on N and ω be a 1-form on M.

The energy of f is $E(f) = \dfrac{1}{2}\displaystyle\int_N |df|^2 dv_h.$

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The energy of f is $E(f) = \frac{1}{2}\int_N |df|^2 dv_h.$

Let us define the following functional for f associated to ξ and ω

$$LH(f) \ = \ E(f) + \int_N \omega(df(\xi))dv_h.$$

First variation for the Landau Hall functional

A smooth variation $\{\mathcal{F}_\epsilon\}$ of f means a smooth map $\mathcal{F} : N \times I \longrightarrow M$, such that $\mathcal{F}(p, 0) = f(p)$. For the sake of simplicity we use to write $f_\epsilon(p) = \mathcal{F}(p, \epsilon)$.

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Definition. The map f is called **magnetic** with respect to ξ and ω if it is a critical point of the Landau Hall integral defined above, i.e. the first variation

$$\frac{d}{d\epsilon} LH(f_\epsilon)\Big|_{\epsilon=0}$$

is zero.

Magnetic maps

Theorem (Inoguchi, M. - 2014)

Let $f : (N, h) \longrightarrow (M, g)$ be a magnetic map with respect to ξ and ω. Then f satisfies the Lorentz equation

$$\tau(f) = \phi(f_*\xi).$$

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Remark (remove assumptions)

A magnetic map is defined without assumptions N compact and F exact. And sometimes remove also $\mathrm{div}\xi = 0$.

Examples of magnetic maps

1. A *constant map* $f : N \longrightarrow M$ is magnetic with respect to any $\xi \in \chi(N)$ and any closed 2-form F on M.

Examples of magnetic maps

2. Let $N = [a, b]$, and t be the parameter on N. Take $h = dt^2$ and $\xi = \frac{d}{dt}$. If F is a magnetic field on M and γ a magnetic curve on M corresponding to F, then γ is a magnetic map associated to ξ and F. This allows us to say that **magnetic maps extend magnetic curves.**

Examples of magnetic maps

3. In the absence of a magnetic field the magnetic equation becomes $\tau(f) = 0$; hence f is a harmonic map. Therefore one may say that **magnetic maps extend harmonic maps.**

Isometric immersions

Let $f : (N, h) \longrightarrow (M, g)$ be an isometric immersion between two Riemannian manifolds N and M.

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Isometric immersions

Let $f : (N, h) \longrightarrow (M, g)$ be an isometric immersion between two Riemannian manifolds N and M. Then, the tension field $\boxed{\tau(f) = n\mathbf{H}}$, where $\mathbf{H}$ is the mean curvature vector field of N in M. We have the following

Proposition (new form of the magnetic equation)

If ξ is a global vector field on N and ϕ is a Lorentz force on M, then f is magnetic if and only if

$$\mathbf{H} = \frac{1}{n}\,\phi(f_*\xi).$$

Magnetic maps in almost contact geometry

Example.

Let $(M, \varphi, \xi, \eta, g)$ be an almost contact metric manifold.

The identity map $\mathbf{1}_M : M \longrightarrow M$ is a magnetic map with respect to ξ and $F = d\eta$ if and only if

$$\iota_\xi d\eta = 0.$$

Tangent bundle

$(x; u)$, where $x \in M$ and $u \in T_x M$

$\pi : TM \longrightarrow M$ induces a foliation $\mathcal{V} = \mathrm{Ker}\,(d\pi)$

$(\bar{x}^1, \bar{x}^2, \cdots, \bar{x}^n; u^1, u^2, \cdots, u^n)$, $\bar{x}^i := x^i \circ \pi$, $u^i := dx^i(u)$

$\boldsymbol{U} = u^i \frac{\partial}{\partial u^i}$ is globally defined on TM: Liouville vector field

(M, g, ∇) be a Riemannian manifold

$$T(TM) = \mathcal{H} \oplus \mathcal{V}$$

Tangent bundle

$X \in T_x M$:

the **vertical lift** of X to u is a unique vector $X^{\mathrm{v}} \in \mathcal{V}_u$ such that $X^{\mathrm{v}}(df) = Xf$ for all $f \in C^\infty(M)$

the **horizontal lift** of X to a point $(x; u) \in TM$ is a unique vector $X^{\mathrm{h}} \in \mathcal{H}_u$ such that $\pi_{*u} X^{\mathrm{h}}_u = X$

The *Sasaki metric* g^{S} of TM and an almost complex structure

$$g^{\mathrm{S}}(X^{\mathrm{h}}, Y^{\mathrm{h}}) = g^{\mathrm{S}}(X^{\mathrm{v}}, Y^{\mathrm{v}}) = g(X, Y) \circ \pi, \; g^{\mathrm{S}}(X^{\mathrm{h}}, Y^{\mathrm{v}}) = 0$$

$$JX^{\mathrm{h}} = X^{\mathrm{v}}, \;\; JX^{\mathrm{v}} = -X^{\mathrm{h}}, \;\; X \in \Gamma(TM).$$

When a vector field is a magnetic map?

Classical result: $(T(M), g_S, J_S)$ is an almost Käherian manifold.

Hence, the Kähler 2-form

$$\Omega_S = g_S(J_S\, \cdot, \, \cdot)$$

may be considered as a magnetic field on $T(M)$.

When a vector field is a magnetic map?

Let $\xi \in \mathfrak{X}(M)$ be thought as a map from (M, g) to $(T(M), g_S, J_S)$.

Compute the differential of this map: $\xi_{*,p} : T_pM \longrightarrow T_{(p,\xi(p))}T(M)$.

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Well known result:

The map $\xi : (M, g) \longrightarrow (T(M), g_S)$ is an isometric immersion if and only if $\nabla\xi = 0$.

When a vector field is a magnetic map?

Compact case:
the energy of ξ on M is

$$E(\xi) = \frac{n}{2}\,\mathrm{vol}(M) + \frac{1}{2}\int_M ||\nabla\xi||_g^2 dv_g,$$

The number

$$\mathcal{B}(\xi) = \int_M ||\nabla\xi||_g^2 dv_g$$

is called the *total bending* of the vector field ξ.

Result. (Ishihara and Nouhaud)
$\xi : (M, g) \longrightarrow (T(M), g_S)$ is harmonic if and only if ξ is parallel. In such a case it is an absolute minimum of the energy functional $E(\xi)$.

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Remark. (Gil-Medrano) Even we restrict the variation (in the Dirichlet energy functional) to vector fields on M, the same conclusion holds.

When a vector field is a magnetic map?

Arbitrary case:

$$\tau(\xi) = -\left\{ \left(\mathrm{trace}_g R(\nabla_{\bullet}\xi, \xi)\bullet\right)^H + \left(\overline{\Delta}_g \xi\right)^V \right\} \circ \xi$$

$\overline{\Delta}_g$ denotes the *rough Laplacian* on vector fields:

$$\overline{\Delta}_g X = -\sum_{k=1}^{n} \left[\nabla_{e_k} \nabla_{e_k} X - \nabla_{\nabla_{e_k} e_k} X \right],$$

When a vector field is a magnetic map?

Theorem (Inoguchi, M. - 2015, 2018)

Let (M, g) be a Riemannian manifold and $(T(M), g_S, J_S)$ its tangent bundle endowed with the usual almost Kählerian structure.

Let ξ be a vector field on M.

Then ξ is a magnetic map with strength q associated to ξ itself and the Kähler magnetic field Ω_S if and only if the following conditions hold:

$$(*) \qquad \mathrm{trace}_g R(\nabla_\bullet \xi, \xi)\bullet) = q\, \nabla_\xi \xi$$

$$(**) \qquad \overline{\Delta}_g \xi = -q\, \xi.$$

When a vector field is a magnetic map?

Proof.

The magnetic equation with strength q:

$$\tau(\xi) = q\, J_S(\xi_* \xi), \;\; q \in \mathbb{R}.$$

We compute

$$J_S(\xi_* \xi) = \xi^V - (\nabla_\xi \xi)^H.$$

Identify the vertical and the horizontal parts, respectively. □

When a vector field is a magnetic map?

Interesting results may be obtained in cases when the curvature tensor has a certain expression.

When a vector field is a magnetic map?

1. M is of constant sectional curvature c:

$$R(X,Y)Z = c\big(g(Y,Z)X - g(X,Z)Y\big), \text{ for all } X,Y,Z \in \mathfrak{X}(M)$$

We obtain:

$$\mathrm{trace}_g R(\nabla_\bullet \xi, \xi)\bullet = c\,[\nabla_\xi \xi - (\mathrm{div}\ \xi)\xi]\,.$$

When a vector field is a magnetic map?

2. $M = M(c)$ is a Sasakian space form

$$R(X,Y)Z = \frac{c+3}{4}\big(g(Y,Z)X - g(X,Z)Y\big)$$
$$+\frac{c-1}{4}\Big(\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi$$
$$+g(Z,\varphi Y)\varphi X - g(Z,\varphi X)\varphi Y + 2g(X,\varphi Y)\varphi Z\Big)$$

$(*)$ is automatically satisfied and $(**)$ implies: $\overline{\Delta}_g\xi = 2n\xi$

Proposition (Inoguchi, M.)

The vector field ξ is magnetic with the strength $q = -2n$.

Tangent sphere bundles

The *tangent sphere bundle* of radius $r > 0$ is the hypersurface of TM

$$T^{(r)}M := \{(x;u) \in TM \mid g_x(u,u) = r^2 \}$$

$T^{(1)}M \overset{not}{\equiv} UM$: the *unit tangent sphere bundle* of M.

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$\boldsymbol{n} := \boldsymbol{U}/r$ unit normal vector field to $T^{(r)}M$

$\bar{g}$ = the Riemannian metric on $T^{(r)}M$ induced by g^{S}

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[Boeckx and Vanhecke] : the *tangential lift* X^{t} of X

$$X^{\mathrm{t}}_u = X^{\mathrm{v}}_u - \frac{1}{r^2} g_x(X,u)\boldsymbol{U}_u$$

Tangent sphere bundle

[Boeckx and Vanhecke] The tangent space $T_u(T^{(r)}M)$ of $T^{(r)}M$ at a point $u = (x; u)$ is given by

$$T_u(T^{(r)}M) = \{X^{\mathrm{h}} + Y^{\mathrm{t}} \mid X, Y \in T_xM,\ g_x(Y, u) = 0\}.$$

Tangent sphere bundle

The a. K. str. (J, g^{S}) induces an a. ct. m. str. $(\varphi, \xi, \eta, \bar{g})$ on $T^{(r)}M$:

$JE = \varphi E + \eta(E)\boldsymbol{n}, \qquad \xi = -J\boldsymbol{n}.$

$\bar{\Omega}(E, F) := \bar{g}(E, \varphi F) = 2r d\eta(E, F), \quad E, F \in \Gamma(T(T^{(r)}M)).$

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$r = 1/2$: $(T^{(1/2)}M, \varphi, \xi, \eta, \bar{g})$ is a contact metric manifold.

Magnetic unit vector fields

Joint work with

Jun-ichi Inoguchi (University of Hokkaido, Japan),

RACSAM, Serie A. Matematicas, 117 (2023) 2, art. 71.

Harmonic unit vector fields

We have seen that the study of vector fields as harmonic maps from M to $T(M)$ (in the compact case) implies that the vector field is **parallel.** Therefore, another appropriate mapping space for Dirichlet or bending energy could be $C^\infty(M, UM)$ or the space of all smooth unit vector fields $\mathfrak{X}_1(M)$.

A unit vector field X is a critical point of $\mathcal{B}$ through (compactly supported) variations in $\mathfrak{X}_1(M)$ if and only if

$$(*) \qquad \overline{\Delta}_g X = |\nabla X|^2 X.$$

Unit vector fields satisfying $(*)$ are called *harmonic unit vector fields*. (Dragomir and Perrone)

Harmonic unit vector fields

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X is a harmonic map into UM, that is, critical point of E through (compactly supported) variations in $C^\infty(M, UM)$ if and only if X satisfies $(*)$ $\quad \overline{\Delta}_g X = |\nabla X|^2 X$ together with

$$\mathrm{tr}_g R(\nabla_\cdot X, X)\cdot = 0.$$

LH-critical vector fields

The *canonical* 1*-form* ω of TM:

$$\omega_{(p;u)}(X^{\mathrm{h}}) = g(u, X_p), \quad \omega_{(p;u)}(X^{\mathrm{v}}) = 0$$

The magnetic field on TM:

$$F = -d\omega = g_S(J\cdot, \cdot)$$

LH-critical vector fields

Variation through vector fields: $X^{(s)}$ is regarded as a map

$$X^{(s)} : (-\varepsilon, \varepsilon) \times M \to TM; \ (s, p) \longmapsto X^{(s)}(p) \in TM$$

$X^{(0)}(p) = X_p$, for any $p \in M$
$\pi(X^{(s)}(p)) = p$, for any s and for any $p \in M$, i.e. $X^{(s)}(p) \in T_pM$

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$X^{(0)}(p) = X_p$, for any $p \in M$
$\pi(X^{(s)}(p)) = p$, for any s and for any $p \in M$, i.e. $X^{(s)}(p) \in T_pM$
The *variational vector field* V of $\{X^{(s)}\}$:

$$V_p = \frac{d}{ds}\bigg|_{s=0} X^{(s)}(p) = \lim_{s \to 0} \frac{1}{s}\left(X^{(s)}(p) - X_p\right) \in T_pM$$

LH-critical vector fields

The canonical 1-form satisfies

$$\omega_{X^{(s)}(p)}((X^{(s)})_{*p}X^{(0)}(p)) = \omega_{X^{(s)}(p)}\left((X^{(0)})^{\mathrm{h}}_{X^{(s)}(p)}\right) = g_p(X^{(s)}(p), X^{(0)}(p))$$

Hence

$$\left.\frac{d}{ds}\right|_{s=0} \omega_{X^{(s)}(p)}((X^{(s)})_{*p}X^{(0)}(p)) = g_p(V, X)$$

The first variation of the magnetic term $\int_{\mathrm{D}} \omega(X_*^{(s)} X^{(0)})\, dv_g$:

$$\left.\frac{d}{ds}\right|_{s=0} \int_{\mathrm{D}} \omega(X_*^{(s)} X^{(0)})\, dv_g = \int_{\mathrm{D}} g(V, X)\, dv_g$$

LH-critical vector fields

Since [Dragomir and Perrone, Theorem 2.8]

$$\left.\frac{d}{ds}\right|_{s=0} E(X^{(s)}; \mathrm{D}) = \int_{\mathrm{D}} g(V, \overline{\Delta}_g X)\, dv_g$$

we have

$$\left.\frac{d}{ds}\right|_{s=0} \mathrm{LH}(X^{(s)}; \mathrm{D}) = \int_{\mathrm{D}} g(V, \overline{\Delta}_g X + qX)\, dv_g$$

LH-critical vector fields

Theorem (Inoguchi, M. - 2023)

A vector field X on an oriented Riemannian manifold (M, g) is a critical point of the Landau-Hall functional under compact supported variations in $\mathfrak{X}(M)$ if and only if

$$\overline{\Delta}_g X = -qX.$$

Magnetic unit vector fields

Recall: UM is a hypersurface of TM with unit normal vector field $\boldsymbol{U}$:

$$\boldsymbol{U}_{(p;w)} = w_w^{\mathrm{v}}$$

for any $w \in T_pM$.

The magnetic field: $F_U(\cdot,\cdot) = g_s(\phi\cdot,\cdot)$

X: a **unit vector field** on M : smooth map $X : M \to UM$

tension field [Dragomir and Perrone: page 59]; [Han and Yim]:

$$\tau_1(X) = -\left\{(\mathrm{tr}_g R(\nabla_{\cdot} X, X)\cdot)^{\mathrm{h}} + (\overline{\Delta}_g X)^{\mathrm{t}}\right\} \circ X$$

Magnetic unit vector fields

magnetic map equation: $\tau_1(X) = q\phi(X_* X)$

$$\phi(X_* X)_X = X^{\mathrm{t}}_X - (\nabla_X X)^{\mathrm{h}}_X$$

Theorem (Inoguchi, M. - 2023)

A unit vector field X on (M, g) is a magnetic map into UM if and only if

$$\mathrm{tr}_g R(\nabla_\cdot X, X)\cdot = q\nabla_X X, \quad \overline{\Delta}_g X = |\nabla X|^2 X.$$

LH functional: variation through unit vector fields

X a unit vector field; $\{X^{(s)}\}$ a variation through unit vector fields

About Dirichlet energy the following result is known:

Theorem (Han and Yim; C.M. Wood; Wiegmink)

A unit vector field X on an oriented Riemannian manifold (M, g) is a critical point of the Dirichlet energy with respect to compactly supported variations in $\mathfrak{X}_1(M)$ if and only if

$$\overline{\Delta}_g X = |\nabla X|^2 X.$$

(**harmonic unit vector field**)

LH functional: variation through unit vector fields

Landau-Hall functional under compact support variations in $\mathfrak{X}_1(M)$:

$$\mathrm{LH}(s) := \mathrm{LH}(X^{(s)}; \mathrm{D}) = E(X^{(s)}; \mathrm{D}) + q \int_{\mathrm{D}} \eta_{X^{(s)}}(X_*^{(s)} X^{(0)}) dv_g$$

Theorem (Inoguchi, M. - 2023)

A unit vector field X on an oriented Riemannian manifold (M, g) is a critical point of the Landau-Hall functional under compact support variations in $\mathfrak{X}_1(M)$ if and only if it is a critical point of the Dirichlet energy under compact support variations in $\mathfrak{X}_1(M)$.

The first variation:

$$\mathrm{LH}'(0) = \int_{\mathrm{D}} g(V, \bar{\Delta}_g X + qX) dv_g$$

Magnetic unit vector fields

Conclusion:

A unit vector field X is a magnetic map into UM with charge q if and only if it is a critical point of the Landau-Hall functional under compact support variations in $\mathfrak{X}_1(M)$ and, in addition, it satisfies

$$\mathrm{tr}_g R(\nabla_{\cdot} X, X)\cdot = q \nabla_X X.$$

Magnetic unit vector fields

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Examples: unimodular Lie groups

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