Hermitian geometry of complex quotient manifolds with trivial canonical bundle

Luis Ugarte

I.U.M.A. – Univ. Zaragoza

SYMMETRY and SHAPE

celebrating the sixtieth birthday of Eduardo García Río

— **Santiago de Compostela, 23–27 September, 2024** —

Based on recent joint works with Antonio Otal (CUD-UZ) and Raquel Villacampa (UZ).

Outline

We will mainly focus in six dimensions on the classification of Lie algebras $\mathfrak{g}$ admitting a complex structure with closed (3,0)-form Ψ.

Outline

We will mainly focus in six dimensions on the classification of Lie algebras $\mathfrak{g}$ admitting a complex structure with closed (3,0)-form Ψ.

The main reason is that they allow to construct compact complex quotients $M = \Gamma \backslash G$ whose canonical bundle is trivial.

Outline

We will mainly focus in six dimensions on the classification of Lie algebras $\mathfrak{g}$ admitting a complex structure with closed (3,0)-form Ψ.

The main reason is that they allow to construct compact complex quotients $M = \Gamma\backslash G$ whose canonical bundle is trivial.

Some of them admit balanced metrics, which provide solutions of the Hull-Strominger system with respect to metric connections $\nabla^{\varepsilon,\rho}$ in the plane that contains Levi-Civita and the Gauduchon line.

Outline

We will mainly focus in six dimensions on the classification of Lie algebras $\mathfrak{g}$ admitting a complex structure with closed (3,0)-form Ψ.

The main reason is that they allow to construct compact complex quotients $M = \Gamma\backslash G$ whose canonical bundle is trivial.

Some of them admit balanced metrics, which provide solutions of the Hull-Strominger system with respect to metric connections $\nabla^{\varepsilon,\rho}$ in the plane that contains Levi-Civita and the Gauduchon line.

We will show when $\nabla^{\varepsilon,\rho}$ satisfies the Hermitian-Yang-Mills condition.

We will mainly focus in six dimensions on the classification of Lie algebras $\mathfrak{g}$ admitting a complex structure with closed (3,0)-form Ψ.

The main reason is that they allow to construct compact complex quotients $M = \Gamma\backslash G$ whose canonical bundle is trivial.

Some of them admit balanced metrics, which provide solutions of the Hull-Strominger system with respect to metric connections $\nabla^{\varepsilon,\rho}$ in the plane that contains Levi-Civita and the Gauduchon line.

We will show when $\nabla^{\varepsilon,\rho}$ satisfies the Hermitian-Yang-Mills condition.

- Nilmanifolds and solvmanifolds

We will mainly focus in six dimensions on the classification of Lie algebras $\mathfrak{g}$ admitting a complex structure with closed (3,0)-form Ψ.

The main reason is that they allow to construct compact complex quotients $M = \Gamma\backslash G$ whose canonical bundle is trivial.

Some of them admit balanced metrics, which provide solutions of the Hull-Strominger system with respect to metric connections $\nabla^{\varepsilon,\rho}$ in the plane that contains Levi-Civita and the Gauduchon line.

We will show when $\nabla^{\varepsilon,\rho}$ satisfies the Hermitian-Yang-Mills condition.

- Nilmanifolds and solvmanifolds
- The non-solvable case

Outline

We will mainly focus in six dimensions on the classification of Lie algebras $\mathfrak{g}$ admitting a complex structure with closed (3,0)-form Ψ.

The main reason is that they allow to construct compact complex quotients $M = \Gamma\backslash G$ whose canonical bundle is trivial.

Some of them admit balanced metrics, which provide solutions of the Hull-Strominger system with respect to metric connections $\nabla^{\varepsilon,\rho}$ in the plane that contains Levi-Civita and the Gauduchon line.

We will show when $\nabla^{\varepsilon,\rho}$ satisfies the Hermitian-Yang-Mills condition.

- Nilmanifolds and solvmanifolds
- The non-solvable case
- Hull-Strominger system and invariant solutions

Outline

We will mainly focus in six dimensions on the classification of Lie algebras $\mathfrak{g}$ admitting a complex structure with closed (3,0)-form Ψ.

The main reason is that they allow to construct compact complex quotients $M = \Gamma\backslash G$ whose canonical bundle is trivial.

Some of them admit balanced metrics, which provide solutions of the Hull-Strominger system with respect to metric connections $\nabla^{\varepsilon,\rho}$ in the plane that contains Levi-Civita and the Gauduchon line.

We will show when $\nabla^{\varepsilon,\rho}$ satisfies the Hermitian-Yang-Mills condition.

- Nilmanifolds and solvmanifolds
- The non-solvable case
- Hull-Strominger system and invariant solutions
- The Hermitian-Yang-Mills condition for $\nabla^{\varepsilon,\rho}$

Complex structures with trivial canonical bundle

Given a compact complex manifold X, $\dim_{\mathbb{C}} X = n$, the canonical bundle K_X is holomorphically trivial if there is a nowhere vanishing $(n, 0)$-form Ψ such that $d\Psi = 0$.

Complex structures with trivial canonical bundle

Given a compact complex manifold X, $\dim_{\mathbb{C}} X = n$, the canonical bundle K_X is holomorphically trivial if there is a nowhere vanishing $(n,0)$-form Ψ such that $d\Psi = 0$.

If $\dim_{\mathbb{C}} X = 2$ then, K_X is trivial if and only if X is

- Torus (Kähler and nilmanifold), $\mathbb{T} = \Gamma\backslash\mathbb{C}^2$,
- Kodaira surface (non-Kähler and nilmanifold), KT,
- K3 surface (Kähler).

Given a compact complex manifold X, $\dim_{\mathbb{C}} X = n$, the canonical bundle K_X is holomorphically trivial if there is a nowhere vanishing $(n, 0)$-form Ψ such that $d\Psi = 0$.

If $\dim_{\mathbb{C}} X = 2$ then, K_X is trivial if and only if X is

- Torus (Kähler and nilmanifold), $\mathbb{T} = \Gamma \backslash \mathbb{C}^2$,
- Kodaira surface (non-Kähler and nilmanifold), KT,
- K3 surface (Kähler).

- No classification for $\dim_{\mathbb{C}}(X) \geq 3$.

Complex structures with trivial canonical bundle

Given a compact complex manifold X, $\dim_{\mathbb{C}} X = n$, the canonical bundle K_X is holomorphically trivial if there is a nowhere vanishing $(n, 0)$-form Ψ such that $d\Psi = 0$.

If $\dim_{\mathbb{C}} X = 2$ then, K_X is trivial if and only if X is

- Torus (Kähler and nilmanifold), $\mathbb{T} = \Gamma \backslash \mathbb{C}^2$,
- Kodaira surface (non-Kähler and nilmanifold), KT,
- K3 surface (Kähler).

• No classification for $\dim_{\mathbb{C}}(X) \geq 3$.

• An interesting class: compact quotients $\Gamma \backslash G$ endowed with J coming from complex structures on the Lie algebra $\mathfrak{g}$ of G having a non-zero closed (n,0)-form $\Psi \in \Lambda^{n,0}(\mathfrak{g}^*)$

Given a compact complex manifold X, $\dim_{\mathbb{C}} X = n$, the canonical bundle K_X is holomorphically trivial if there is a nowhere vanishing $(n, 0)$-form Ψ such that $d\Psi = 0$.

If $\dim_{\mathbb{C}} X = 2$ then, K_X is trivial if and only if X is

- Torus (Kähler and nilmanifold), $\mathbb{T} = \Gamma\backslash\mathbb{C}^2$,
- Kodaira surface (non-Kähler and nilmanifold), KT,
- K3 surface (Kähler).

- No classification for $\dim_{\mathbb{C}}(X) \geq 3$.

- An interesting class: compact quotients $\Gamma\backslash G$ endowed with J coming from complex structures on the Lie algebra $\mathfrak{g}$ of G having a non-zero closed (n,0)-form $\Psi \in \Lambda^{n,0}(\mathfrak{g}^*)$

Problem (n≥3): classify (unimodular) Lie algebras admitting such a J

The Lie algebra $\mathfrak{g}$ of G has an endomorphism $J\colon \mathfrak{g} \longrightarrow \mathfrak{g}$ such that $J^2 = -Id$, and $[JU, JV] = J[JU, V] + J[U, JV] + [U, V]$, $\quad U, V \in \mathfrak{g}$.

The Lie algebra $\mathfrak{g}$ of G has an endomorphism $J\colon \mathfrak{g} \longrightarrow \mathfrak{g}$ such that $J^2 = -Id$, and $[JU, JV] = J[JU, V] + J[U, JV] + [U, V]$, $\quad U, V \in \mathfrak{g}$.

Equivalently, $d(\mathfrak{g}^{1,0}) \subset \bigwedge^{2,0}(\mathfrak{g}^*) \oplus \bigwedge^{1,1}(\mathfrak{g}^*)$

The Lie algebra $\mathfrak{g}$ of G has an endomorphism $J\colon \mathfrak{g} \longrightarrow \mathfrak{g}$ such that $J^2 = -Id$, and $[JU, JV] = J[JU, V] + J[U, JV] + [U, V]$, $\quad U, V \in \mathfrak{g}$.

Equivalently, $d(\mathfrak{g}^{1,0}) \subset \bigwedge^{2,0}(\mathfrak{g}^*) \oplus \bigwedge^{1,1}(\mathfrak{g}^*)$

- Suppose J is almost complex (i.e. $J^2 = -Id$) and that there is a non-zero closed (n,0)-form Ψ.

Then, $d\Psi = 0 \quad \Rightarrow \quad J$ is complex (integrable):

The Lie algebra $\mathfrak{g}$ of G has an endomorphism $J\colon \mathfrak{g} \longrightarrow \mathfrak{g}$ such that $J^2 = -Id$, and $[JU, JV] = J[JU, V] + J[U, JV] + [U, V]$, $\quad U, V \in \mathfrak{g}$.

Equivalently, $d(\mathfrak{g}^{1,0}) \subset \bigwedge^{2,0}(\mathfrak{g}^*) \oplus \bigwedge^{1,1}(\mathfrak{g}^*)$

- Suppose J is almost complex (i.e. $J^2 = -Id$) and that there is a non-zero closed (n,0)-form Ψ.

Then, $d\Psi = 0 \quad \Rightarrow \quad J$ is complex (integrable):

For any $\omega \in \mathfrak{g}^{1,0}$ one has $\omega \wedge \Psi = 0$,

then $0 = d(\omega \wedge \Psi) = d\omega \wedge \Psi = (d\omega)^{0,2} \wedge \Psi$,

which implies that $d(\mathfrak{g}^{1,0})$ has zero component in $\bigwedge^{0,2}(\mathfrak{g}^*)$.

The goal is to *classify* the unimodular *Lie algebras* $\mathfrak{g}$ with $\dim_{\mathbb{R}} \mathfrak{g} = 6$ admitting a complex structure with *closed* $(3,0)$*-form* $\Psi \neq 0$

The goal is to *classify* the unimodular *Lie algebras* $\mathfrak{g}$ with $\dim_{\mathbb{R}} \mathfrak{g} = 6$ admitting a complex structure with *closed* $(3,0)$*-form* $\Psi \neq 0$

$\mathfrak{g}$ NILPOTENT	$\mathfrak{g}$ SOLVABLE	$\mathfrak{g}$ NON-SOLVABLE

The goal is to *classify* the unimodular *Lie algebras* $\mathfrak{g}$ with $\dim_{\mathbb{R}} \mathfrak{g} = 6$ admitting a complex structure with *closed* $(3,0)$*-form* $\Psi \neq 0$

$\mathfrak{g}$ NILPOTENT	$\mathfrak{g}$ SOLVABLE	$\mathfrak{g}$ NON-SOLVABLE
$\Downarrow$	$\Downarrow$	$\Downarrow$
Nilmanifolds	Solvmanifolds	Quotient manifolds

with complex structures and holomorphically trivial canonical bundle

The goal is to *classify* the unimodular *Lie algebras* $\mathfrak{g}$ with $\dim_{\mathbb{R}} \mathfrak{g} = 6$ admitting a complex structure with *closed* $(3,0)$*-form* $\Psi \neq 0$

$\mathfrak{g}$ NILPOTENT	$\mathfrak{g}$ SOLVABLE	$\mathfrak{g}$ NON-SOLVABLE
[Salamon]	[Fino-Otal-U]	[Otal-U]
$\Downarrow$	$\Downarrow$	$\Downarrow$
Nilmanifolds	Solvmanifolds	Quotient manifolds

with complex structures and holomorphically trivial canonical bundle

Nilmanifolds ($\mathfrak{g}$ nilpotent)

The Lie algebra $\mathfrak{g}$ of G is *s-step nilpotent*, i.e. the descending series $\{\mathfrak{g}_i\}$ satisfies

$$\mathfrak{g}_0 = \mathfrak{g} \supset \mathfrak{g}_1 = [\mathfrak{g}, \mathfrak{g}] \supset \cdots \supset \mathfrak{g}_{i+1} = [\mathfrak{g}_i, \mathfrak{g}] \supset \cdots \supset \mathfrak{g}_s = [\mathfrak{g}_{s-1}, \mathfrak{g}] = \{0\}.$$

The Lie algebra $\mathfrak{g}$ of G is *s-step nilpotent*, i.e. the descending series $\{\mathfrak{g}_i\}$ satisfies

$$\mathfrak{g}_0 = \mathfrak{g} \supset \mathfrak{g}_1 = [\mathfrak{g}, \mathfrak{g}] \supset \cdots \supset \mathfrak{g}_{i+1} = [\mathfrak{g}_i, \mathfrak{g}] \supset \cdots \supset \mathfrak{g}_s = [\mathfrak{g}_{s-1}, \mathfrak{g}] = \{0\}.$$

Some important well-known properties:

- $\mathfrak{g}$ is unimodular.
- G has a lattice Γ if and only if the structure constants of $\mathfrak{g}$ are rational numbers [Mal'cev].

The Lie algebra $\mathfrak{g}$ of G is *s-step nilpotent*, i.e. the descending series $\{\mathfrak{g}_i\}$ satisfies

$$\mathfrak{g}_0 = \mathfrak{g} \supset \mathfrak{g}_1 = [\mathfrak{g}, \mathfrak{g}] \supset \cdots \supset \mathfrak{g}_{i+1} = [\mathfrak{g}_i, \mathfrak{g}] \supset \cdots \supset \mathfrak{g}_s = [\mathfrak{g}_{s-1}, \mathfrak{g}] = \{0\}.$$

Some important well-known properties:

- $\mathfrak{g}$ is unimodular.
- G has a lattice Γ if and only if the structure constants of $\mathfrak{g}$ are rational numbers [Mal'cev].
- By [Nomizu], [Deligne-Griffiths-Morgan-Sullivan], [Hasegawa],

for complex nilmanifolds: Kähler $\Longleftrightarrow$ $\partial\bar{\partial}$-property. Hence,

$\Gamma\backslash G$ cannot admit Kähler metrics, except for a torus

The Lie algebra $\mathfrak{g}$ of G is *s-step nilpotent*, i.e. the descending series $\{\mathfrak{g}_i\}$ satisfies

$$\mathfrak{g}_0 = \mathfrak{g} \supset \mathfrak{g}_1 = [\mathfrak{g}, \mathfrak{g}] \supset \cdots \supset \mathfrak{g}_{i+1} = [\mathfrak{g}_i, \mathfrak{g}] \supset \cdots \supset \mathfrak{g}_s = [\mathfrak{g}_{s-1}, \mathfrak{g}] = \{0\}.$$

Some important well-known properties:

- $\mathfrak{g}$ is unimodular.
- G has a lattice Γ if and only if the structure constants of $\mathfrak{g}$ are rational numbers [Mal'cev].
- By [Nomizu], [Deligne-Griffiths-Morgan-Sullivan], [Hasegawa], for complex nilmanifolds: Kähler $\Longleftrightarrow$ $\partial\bar{\partial}$-property. Hence,

$\Gamma\backslash G$ cannot admit Kähler metrics, except for a torus

- Any complex structure J on $\mathfrak{g}$ has non-zero closed (n,0)-form: $d\omega^i \in \mathcal{I}(\omega^1, \ldots, \omega^{i-1}) \;\Rightarrow\; d(\omega^1 \wedge \cdots \wedge \omega^n) = 0$ [Salamon]

The Lie algebra $\mathfrak{g}$ of G is *s-step solvable*, i.e. the series $\{\mathfrak{g}_{(i)}\}$ satisfies

$$\mathfrak{g}_{(0)} = \mathfrak{g} \supset \mathfrak{g}_{(1)} = [\mathfrak{g}, \mathfrak{g}] \supset \cdots \supset \mathfrak{g}_{(i+1)} = [\mathfrak{g}_{(i)}, \mathfrak{g}_{(i)}] \supset \cdots$$
$$\cdots \supset \mathfrak{g}_{(s)} = [\mathfrak{g}_{(s-1)}, \mathfrak{g}_{(s-1)}] = \{0\}.$$

The Lie algebra $\mathfrak{g}$ of G is *s-step solvable*, i.e. the series $\{\mathfrak{g}_{(i)}\}$ satisfies

$$\mathfrak{g}_{(0)} = \mathfrak{g} \supset \mathfrak{g}_{(1)} = [\mathfrak{g},\mathfrak{g}] \supset \cdots \supset \mathfrak{g}_{(i+1)} = [\mathfrak{g}_{(i)},\mathfrak{g}_{(i)}] \supset \cdots$$
$$\cdots \supset \mathfrak{g}_{(s)} = [\mathfrak{g}_{(s-1)},\mathfrak{g}_{(s-1)}] = \{0\}.$$

Important differences with respect to nilmanifolds:

- There is no a simple condition for the existence of lattice.

Necessary condition [Milnor]: if G has a lattice, then it is unimodular, i.e. $\operatorname{tr}\operatorname{ad} X = 0$, $\forall X \in \mathfrak{g}$.

Solvmanifolds ($\mathfrak{g}$ solvable)

The Lie algebra $\mathfrak{g}$ of G is *s-step solvable*, i.e. the series $\{\mathfrak{g}_{(i)}\}$ satisfies

$$\mathfrak{g}_{(0)} = \mathfrak{g} \supset \mathfrak{g}_{(1)} = [\mathfrak{g}, \mathfrak{g}] \supset \cdots \supset \mathfrak{g}_{(i+1)} = [\mathfrak{g}_{(i)}, \mathfrak{g}_{(i)}] \supset \cdots$$
$$\cdots \supset \mathfrak{g}_{(s)} = [\mathfrak{g}_{(s-1)}, \mathfrak{g}_{(s-1)}] = \{0\}.$$

Important differences with respect to nilmanifolds:

- There is no a simple condition for the existence of lattice.

Necessary condition [Milnor]: if G has a lattice, then it is unimodular, i.e. $\operatorname{tr} \operatorname{ad} X = 0$, $\forall X \in \mathfrak{g}$.

- There exist non-Kähler $\partial\bar{\partial}$-solvmanifolds.

A solvmanifold has a Kähler metric if and only if it is a finite quotient of a complex torus [Hasegawa].

The Lie algebra $\mathfrak{g}$ of G is *s-step solvable*, i.e. the series $\{\mathfrak{g}_{(i)}\}$ satisfies

$$\mathfrak{g}_{(0)} = \mathfrak{g} \supset \mathfrak{g}_{(1)} = [\mathfrak{g},\mathfrak{g}] \supset \cdots \supset \mathfrak{g}_{(i+1)} = [\mathfrak{g}_{(i)},\mathfrak{g}_{(i)}] \supset \cdots$$
$$\cdots \supset \mathfrak{g}_{(s)} = [\mathfrak{g}_{(s-1)},\mathfrak{g}_{(s-1)}] = \{0\}.$$

Important differences with respect to nilmanifolds:

- There is no a simple condition for the existence of lattice.

Necessary condition [Milnor]: if G has a lattice, then it is unimodular, i.e. $\mathrm{tr}\,\mathrm{ad}\,X = 0$, $\forall X \in \mathfrak{g}$.

- There exist non-Kähler $\partial\bar{\partial}$-solvmanifolds.

A solvmanifold has a Kähler metric if and only if it is a finite quotient of a complex torus [Hasegawa].

- There exist complex structures J with no closed (n,0)-form.

The Lie algebra $\mathfrak{g}$ of G is *s-step solvable*, i.e. the series $\{\mathfrak{g}_{(i)}\}$ satisfies

$$\mathfrak{g}_{(0)} = \mathfrak{g} \supset \mathfrak{g}_{(1)} = [\mathfrak{g}, \mathfrak{g}] \supset \cdots \supset \mathfrak{g}_{(i+1)} = [\mathfrak{g}_{(i)}, \mathfrak{g}_{(i)}] \supset \cdots$$
$$\cdots \supset \mathfrak{g}_{(s)} = [\mathfrak{g}_{(s-1)}, \mathfrak{g}_{(s-1)}] = \{0\}.$$

Important differences with respect to nilmanifolds:

• There is no a simple condition for the existence of lattice.
Necessary condition [Milnor]: if G has a lattice, then it is unimodular, i.e. $\operatorname{tr} \operatorname{ad} X = 0$, $\forall X \in \mathfrak{g}$.

• There exist non-Kähler $\partial\bar{\partial}$-solvmanifolds.
A solvmanifold has a Kähler metric if and only if it is a finite quotient of a complex torus [Hasegawa].

• There exist complex structures J with no closed (n,0)-form.

• There may exist non-invariant trivializing sections, even when J is invariant with no invariant closed (n,0)-form [Andrada-Tolcachier 2023].

THEOREM [Salamon]. In dimension 6, a nilpotent Lie algebra admitting a complex structure J is isomorphic to one of the following

$\mathfrak{n}_1 = (0,0,0,0,0,0)$,
$\mathfrak{n}_2 = (0,0,0,0,12,34)$,
$\mathfrak{n}_3 = (0,0,0,0,0,12+34)$,
$\mathfrak{n}_4 = (0,0,0,0,12,14+23)$,
$\mathfrak{n}_5 = (0,0,0,0,13+42,14+23)$,
$\mathfrak{n}_6 = (0,0,0,0,12,13)$,
$\mathfrak{n}_7 = (0,0,0,12,13,23)$,
$\mathfrak{n}_8 = (0,0,0,0,0,12)$,
$\mathfrak{n}_9 = (0,0,0,0,12,14+25)$,
$\mathfrak{n}_{10} = (0,0,0,12,13,14)$,
$\mathfrak{n}_{11} = (0,0,0,12,13,14+23)$,
$\mathfrak{n}_{12} = (0,0,0,12,13,24)$,
$\mathfrak{n}_{13} = (0,0,0,12,13+14,24)$,
$\mathfrak{n}_{14} = (0,0,0,12,14,13+42)$,
$\mathfrak{n}_{15} = (0,0,0,12,13+42,14+23)$,
$\mathfrak{n}_{16} = (0,0,0,12,14,24)$,
$\mathfrak{n}_{19}^- = (0,0,0,12,23,14-35)$,
$\mathfrak{n}_{26}^+ = (0,0,12,13,23,14+25)$.

THEOREM [Salamon]. In dimension 6, a nilpotent Lie algebra admitting a complex structure J is isomorphic to one of the following

$\mathfrak{n}_1 = (0,0,0,0,0,0)$,	$\mathfrak{n}_{10} = (0,0,0,12,13,14)$,
$\mathfrak{n}_2 = (0,0,0,0,12,34)$,	$\mathfrak{n}_{11} = (0,0,0,12,13,14+23)$,
$\mathfrak{n}_3 = (0,0,0,0,0,12+34)$,	$\mathfrak{n}_{12} = (0,0,0,12,13,24)$,
$\mathfrak{n}_4 = (0,0,0,0,12,14+23)$,	$\mathfrak{n}_{13} = (0,0,0,12,13+14,24)$,
$\mathfrak{n}_5 = (0,0,0,0,13+42,14+23)$,	$\mathfrak{n}_{14} = (0,0,0,12,14,13+42)$,
$\mathfrak{n}_6 = (0,0,0,0,12,13)$,	$\mathfrak{n}_{15} = (0,0,0,12,13+42,14+23)$,
$\mathfrak{n}_7 = (0,0,0,12,13,23)$,	$\mathfrak{n}_{16} = (0,0,0,12,14,24)$,
$\mathfrak{n}_8 = (0,0,0,0,0,12)$,	$\mathfrak{n}_{19}^- = (0,0,0,12,23,14-35)$,
$\mathfrak{n}_9 = (0,0,0,0,12,14+25)$,	$\mathfrak{n}_{26}^+ = (0,0,12,13,23,14+25)$.

$\mathfrak{n}_2$ is the product of two 3-dimensional Heisenberg algebras

THEOREM [Salamon]. In dimension 6, a nilpotent Lie algebra admitting a complex structure J is isomorphic to one of the following

$\mathfrak{n}_1 = (0,0,0,0,0,0)$,
$\mathfrak{n}_2 = (0,0,0,0,12,34)$,
$\mathfrak{n}_3 = (0,0,0,0,0,12+34)$,
$\mathfrak{n}_4 = (0,0,0,0,12,14+23)$,
$\mathfrak{n}_5 = (0,0,0,0,13+42,14+23)$,
$\mathfrak{n}_6 = (0,0,0,0,12,13)$,
$\mathfrak{n}_7 = (0,0,0,12,13,23)$,
$\mathfrak{n}_8 = (0,0,0,0,0,12)$,
$\mathfrak{n}_9 = (0,0,0,0,12,14+25)$,
$\mathfrak{n}_{10} = (0,0,0,12,13,14)$,
$\mathfrak{n}_{11} = (0,0,0,12,13,14+23)$,
$\mathfrak{n}_{12} = (0,0,0,12,13,24)$,
$\mathfrak{n}_{13} = (0,0,0,12,13+14,24)$,
$\mathfrak{n}_{14} = (0,0,0,12,14,13+42)$,
$\mathfrak{n}_{15} = (0,0,0,12,13+42,14+23)$,
$\mathfrak{n}_{16} = (0,0,0,12,14,24)$,
$\mathfrak{n}_{19}^{-} = (0,0,0,12,23,14-35)$,
$\mathfrak{n}_{26}^{+} = (0,0,12,13,23,14+25)$.

$\mathfrak{n}_3$ is the product of the 5-dimensional Heisenberg algebra by $\mathbb{R}$

THEOREM [Salamon]. In dimension 6, a nilpotent Lie algebra admitting a complex structure J is isomorphic to one of the following

$\mathfrak{n}_1 = (0,0,0,0,0,0)$,
$\mathfrak{n}_2 = (0,0,0,0,12,34)$,
$\mathfrak{n}_3 = (0,0,0,0,0,12+34)$,
$\mathfrak{n}_4 = (0,0,0,0,12,14+23)$,
$\mathfrak{n}_5 = (0,0,0,0,13+42,14+23)$,
$\mathfrak{n}_6 = (0,0,0,0,12,13)$,
$\mathfrak{n}_7 = (0,0,0,12,13,23)$,
$\mathfrak{n}_8 = (0,0,0,0,0,12)$,
$\mathfrak{n}_9 = (0,0,0,0,12,14+25)$,
$\mathfrak{n}_{10} = (0,0,0,12,13,14)$,
$\mathfrak{n}_{11} = (0,0,0,12,13,14+23)$,
$\mathfrak{n}_{12} = (0,0,0,12,13,24)$,
$\mathfrak{n}_{13} = (0,0,0,12,13+14,24)$,
$\mathfrak{n}_{14} = (0,0,0,12,14,13+42)$,
$\mathfrak{n}_{15} = (0,0,0,12,13+42,14+23)$,
$\mathfrak{n}_{16} = (0,0,0,12,14,24)$,
$\mathfrak{n}_{19}^- = (0,0,0,12,23,14-35)$,
$\mathfrak{n}_{26}^+ = (0,0,12,13,23,14+25)$.

$\mathfrak{n}_5$ is the underlying Lie algebra to the Iwasawa manifold

THEOREM [Fino-Otal-U]. A 6-dim. unimodular (non-nilpotent) solvable Lie algebra $\mathfrak{g}$ admitting J with closed (3,0)-form $\Psi \neq 0$ is isomorphic to

THEOREM [Fino-Otal-U]. A 6-dim. unimodular (non-nilpotent) solvable Lie algebra $\mathfrak{g}$ admitting J with closed (3,0)-form $\Psi \neq 0$ is isomorphic to

$$\mathfrak{s}_1 = (15, -25, -35, 45, 0, 0),$$
$$\mathfrak{s}_2^{\alpha \geq 0} = (\alpha \cdot 15 + 25, -15 + \alpha \cdot 25, -\alpha \cdot 35 + 45, -35 - \alpha \cdot 45, 0, 0),$$
$$\mathfrak{s}_3 = (0, -13, 12, 0, -46, -45),$$

Decomposable Lie algebras

THEOREM [Fino-Otal-U]. A 6-dim. unimodular (non-nilpotent) solvable Lie algebra $\mathfrak{g}$ admitting J with closed (3,0)-form $\Psi \neq 0$ is isomorphic to

$$\mathfrak{s}_1 = (15, -25, -35, 45, 0, 0),$$

$$\mathfrak{s}_2^{\alpha \geq 0} = (\alpha \cdot 15 + 25, -15 + \alpha \cdot 25, -\alpha \cdot 35 + 45, -35 - \alpha \cdot 45, 0, 0),$$

$$\mathfrak{s}_3 = (0, -13, 12, 0, -46, -45),$$

$\mathfrak{s}_2^{\alpha}$ with $\alpha \geq 0$ is an infinite family of non-isomorphic Lie algebras

THEOREM [Fino-Otal-U]. A 6-dim. unimodular (non-nilpotent) solvable Lie algebra $\mathfrak{g}$ admitting J with closed (3,0)-form $\Psi \neq 0$ is isomorphic to

$$\mathfrak{s}_1 = (15, -25, -35, 45, 0, 0),$$
$$\mathfrak{s}_2^{\alpha\geq 0} = (\alpha\cdot 15+25, -15+\alpha\cdot 25, -\alpha\cdot 35+45, -35-\alpha\cdot 45, 0, 0),$$
$$\mathfrak{s}_3 = (0, -13, 12, 0, -46, -45),$$
$$\mathfrak{s}_4 = (23, -36, 26, -56, 46, 0),$$
$$\mathfrak{s}_5 = (24 + 35, 26, 36, -46, -56, 0),$$
$$\mathfrak{s}_6 = (24 + 35, -36, 26, -56, 46, 0),$$
$$\mathfrak{s}_7 = (24 + 35, 46, 56, -26, -36, 0),$$
$$\mathfrak{s}_8 = (16 - 25, 15 + 26, -36 + 45, -35 - 46, 0, 0),$$
$$\mathfrak{s}_9 = (45, 15 + 36, 14 - 26 + 56, -56, 46, 0).$$

Indecomposable Lie algebras

6-dimensional Lie algebras underlying complex solvmanifolds

THEOREM [Fino-Otal-U]. A 6-dim. unimodular (non-nilpotent) solvable Lie algebra $\mathfrak{g}$ admitting J with closed (3,0)-form $\Psi \neq 0$ is isomorphic to

$$\mathfrak{s}_1 = (15, -25, -35, 45, 0, 0),$$
$$\mathfrak{s}_2^{\alpha\geq 0} = (\alpha\cdot 15+25, -15+\alpha\cdot 25, -\alpha\cdot 35+45, -35-\alpha\cdot 45, 0, 0),$$
$$\mathfrak{s}_3 = (0, -13, 12, 0, -46, -45),$$
$$\mathfrak{s}_4 = (23, -36, 26, -56, 46, 0),$$
$$\mathfrak{s}_5 = (24+35, 26, 36, -46, -56, 0),$$
$$\mathfrak{s}_6 = (24+35, -36, 26, -56, 46, 0),$$
$$\mathfrak{s}_7 = (24+35, 46, 56, -26, -36, 0),$$
$$\mathfrak{s}_8 = (16-25, 15+26, -36+45, -35-46, 0, 0),$$
$$\mathfrak{s}_9 = (45, 15+36, 14-26+56, -56, 46, 0).$$

$\mathfrak{s}_8$ is the underlying Lie algebra to the Nakamura manifold

THEOREM [Fino-Otal-U]. A 6-dim. unimodular (non-nilpotent) solvable Lie algebra $\mathfrak{g}$ admitting J with closed (3,0)-form $\Psi \neq 0$ is isomorphic to

$$\mathfrak{s}_1 = (15, -25, -35, 45, 0, 0),$$
$$\mathfrak{s}_2^{\alpha\geq 0} = (\alpha\cdot 15+25, -15+\alpha\cdot 25, -\alpha\cdot 35+45, -35-\alpha\cdot 45, 0, 0),$$
$$\mathfrak{s}_3 = (0, -13, 12, 0, -46, -45),$$
$$\mathfrak{s}_4 = (23, -36, 26, -56, 46, 0),$$
$$\mathfrak{s}_5 = (24 + 35, 26, 36, -46, -56, 0),$$
$$\mathfrak{s}_6 = (24 + 35, -36, 26, -56, 46, 0),$$
$$\mathfrak{s}_7 = (24 + 35, 46, 56, -26, -36, 0),$$
$$\mathfrak{s}_8 = (16 - 25, 15 + 26, -36 + 45, -35 - 46, 0, 0),$$
$$\mathfrak{s}_9 = (45, 15 + 36, 14 - 26 + 56, -56, 46, 0).$$

PROPOSITION [Fino-Otal-U]. The corresponding connected and simply-connected solvable Lie groups admit a lattice (for a countable number of α's, including $\alpha = 0$).

The non-solvable case in dimension 6

THEOREM [Otal-U]. Let $\mathfrak{g}$ be an unimodular non-solvable Lie algebra of dimension 6. Then, $\mathfrak{g}$ admits a complex structure with a non-zero closed $(3,0)$-form if and only if it is isomorphic to $\mathfrak{so}(3,1)$.

THEOREM [Otal-U]. Let $\mathfrak{g}$ be an unimodular non-solvable Lie algebra of dimension 6. Then, $\mathfrak{g}$ admits a complex structure with a non-zero closed $(3,0)$-form if and only if it is isomorphic to $\mathfrak{so}(3,1)$.

The real Lie algebra $\mathfrak{so}(3,1)$ underlies the 3-dimensional complex Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ given by the complex structure equations

$$d\omega^1 = \omega^2 \wedge \omega^3, \quad d\omega^2 = -\omega^1 \wedge \omega^3, \quad d\omega^3 = \omega^1 \wedge \omega^2.$$

Clearly, $d(\omega^1 \wedge \omega^2 \wedge \omega^3) = 0$.

THEOREM [Otal-U]. Let $\mathfrak{g}$ be an unimodular non-solvable Lie algebra of dimension 6. Then, $\mathfrak{g}$ admits a complex structure with a non-zero closed $(3,0)$-form if and only if it is isomorphic to $\mathfrak{so}(3,1)$.

The real Lie algebra $\mathfrak{so}(3,1)$ underlies the 3-dimensional complex Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ given by the complex structure equations

$$d\omega^1 = \omega^2 \wedge \omega^3, \quad d\omega^2 = -\omega^1 \wedge \omega^3, \quad d\omega^3 = \omega^1 \wedge \omega^2.$$

Clearly, $d(\omega^1 \wedge \omega^2 \wedge \omega^3) = 0$.

Consider the real basis $\{e^j\}_{j=1}^6$ given by

$$\omega^1 = e^3 - i\,e^6, \quad \omega^2 = e^1 - i\,e^4, \quad \omega^3 = e^2 - i\,e^5.$$

Then we get

$$\mathfrak{so}(3,1) = (23-56,\ -13+46,\ 12-45,\ 26-35,\ -16+34,\ 15-24)$$

Key obs.: the $(3,0)$-form Ψ is determined by its real part $\rho = \mathfrak{Re}\,\Psi$, which is a stable form.

The idea of the proofs: stable forms in dimension 6

Key obs.: the $(3,0)$-form Ψ is determined by its real part $\rho = \mathfrak{Re}\,\Psi$, which is a stable form.

Let (V, ν) be an oriented 6-dimensional vector space.

A form $\rho \in \Lambda^3 V^*$ is stable if its orbit under the action of $\mathrm{GL}(V)$ is open.

The idea of the proofs: stable forms in dimension 6

Key obs.: the $(3,0)$-form Ψ is determined by its real part $\rho = \mathfrak{Re}\,\Psi$, which is a stable form.

Let (V, ν) be an oriented 6-dimensional vector space.

A form $\rho \in \Lambda^3 V^*$ is stable if its orbit under the action of GL(V) is open.

Consider the isomorphism $k : \Lambda^5 V^* \to V$, given by $\eta \mapsto y$,
where y is such that $\iota_y \nu = \eta$,
and the endomorphism $K_\rho : V \to V$, given by $x \mapsto k(\iota_x \rho \wedge \rho)$.

[Reichel; Hitchin]: ρ stable $\iff \lambda(\rho) = \frac{1}{6}\,\mathrm{trace}\,(K_\rho^2) \neq 0$.

The sign of $\lambda(\rho)$ only depends on ρ (not on ν).

Key obs.: the $(3,0)$-form Ψ is determined by its real part $\rho = \mathfrak{Re}\,\Psi$, which is a stable form.

Let (V,ν) be an oriented 6-dimensional vector space.

A form $\rho \in \Lambda^3 V^*$ is stable if its orbit under the action of $\mathrm{GL}(V)$ is open.

Consider the isomorphism $k\colon \Lambda^5 V^* \to V$, given by $\eta \mapsto y$,
where y is such that $\iota_y \nu = \eta$,
and the endomorphism $K_\rho\colon V \to V$, given by $x \mapsto k(\iota_x\rho \wedge \rho)$.

[Reichel; Hitchin]: ρ stable $\iff \lambda(\rho) = \frac{1}{6}\,\mathrm{trace}\,(K_\rho^2) \neq 0$.

The sign of $\lambda(\rho)$ only depends on ρ (not on ν).

If $\lambda(\rho) < 0$, then $J_\rho := \frac{1}{\sqrt{|\lambda(\rho)|}} K_\rho$ is an almost complex structure on V, which is given by:

$$\sqrt{|\lambda(\rho)|}\,(J_\rho^*\alpha)(x)\,\nu = \alpha \wedge \iota_x\rho \wedge \rho, \quad \alpha \in V^*, \quad x \in V.$$

Key obs.: the $(3,0)$-form Ψ is determined by its real part $\rho = \mathfrak{Re}\,\Psi$, which is a stable form.

Let (V, ν) be an oriented 6-dimensional vector space.

A form $\rho \in \Lambda^3 V^*$ is stable if its orbit under the action of GL(V) is open.

Consider the isomorphism $k: \Lambda^5 V^* \to V$, given by $\eta \mapsto y$,
where y is such that $\iota_y \nu = \eta$,
and the endomorphism $K_\rho: V \to V$, given by $x \mapsto k(\iota_x \rho \wedge \rho)$.

[Reichel; Hitchin]: ρ stable $\iff \lambda(\rho) = \frac{1}{6}\,\mathrm{trace}\,(K_\rho^2) \neq 0$.

The sign of $\lambda(\rho)$ only depends on ρ (not on ν).

If $\lambda(\rho) < 0$, then $J_\rho := \frac{1}{\sqrt{|\lambda(\rho)|}} K_\rho$ is an almost complex structure on V, which is given by:

$$\sqrt{|\lambda(\rho)|}\,(J_\rho^* \alpha)(x)\,\nu = \alpha \wedge \iota_x \rho \wedge \rho, \quad \alpha \in V^*, \ \ x \in V.$$

The complex form $\Psi = \rho + i J_\rho^*(\rho)$ has bidegree $(3,0)$ w.r.t. J_ρ.

STEP 1. Classify real Lie algebras $\mathfrak{g}$ admitting complex structures.
Given a Lie algebra $\mathfrak{g}$, let $Z^3(\mathfrak{g}) = \{\rho \in \Lambda^3\mathfrak{g}^* \mid d\rho = 0\}$
and look for $\rho \in Z^3(\mathfrak{g})$ such that $\lambda(\rho) < 0$ and $d(J_\rho^* \rho) = 0$.

STEP 1. Classify real Lie algebras $\mathfrak{g}$ admitting complex structures.
Given a Lie algebra $\mathfrak{g}$, let $Z^3(\mathfrak{g}) = \{\rho \in \Lambda^3\mathfrak{g}^* \mid d\rho = 0\}$
and look for $\rho \in Z^3(\mathfrak{g})$ such that $\lambda(\rho) < 0$ and $d(J_\rho^*\rho) = 0$.

STEP 2. Moduli space $\mathcal{M}(\mathfrak{g})$ of complex structures for every $\mathfrak{g}$.

$$\mathcal{M}(\mathfrak{g}) = \mathcal{C}(\mathfrak{g})/\sim$$

where $J \sim J'$ iff $\exists\, A \in \mathrm{Aut}(\mathfrak{g})$ such that $A \circ J = J' \circ A$.

STEP 1. Classify real Lie algebras $\mathfrak{g}$ admitting complex structures.
Given a Lie algebra $\mathfrak{g}$, let $Z^3(\mathfrak{g}) = \{\rho \in \Lambda^3\mathfrak{g}^* \mid d\rho = 0\}$
and look for $\rho \in Z^3(\mathfrak{g})$ such that $\lambda(\rho) < 0$ and $d(J_\rho^* \rho) = 0$.

STEP 2. Moduli space $\mathcal{M}(\mathfrak{g})$ of complex structures for every $\mathfrak{g}$.

$$\mathcal{M}(\mathfrak{g}) = \mathcal{C}(\mathfrak{g})/\sim$$

where $J \sim J'$ iff $\exists\, A \in \mathrm{Aut}(\mathfrak{g})$ such that $A \circ J = J' \circ A$.

Conclusion: many compact complex 3-folds $X = (M, J)$ defined by a quotient of G by a lattice Γ, where one can study different questions, as

STEP 1. Classify real Lie algebras $\mathfrak{g}$ admitting complex structures.
Given a Lie algebra $\mathfrak{g}$, let $Z^3(\mathfrak{g}) = \{\rho \in \Lambda^3\mathfrak{g}^* \mid d\rho = 0\}$
and look for $\rho \in Z^3(\mathfrak{g})$ such that $\lambda(\rho) < 0$ and $d(J_\rho^*\rho) = 0$.

STEP 2. Moduli space $\mathcal{M}(\mathfrak{g})$ of complex structures for every $\mathfrak{g}$.

$$\mathcal{M}(\mathfrak{g}) = \mathcal{C}(\mathfrak{g})/\sim$$

where $J \sim J'$ iff $\exists\, A \in \mathrm{Aut}(\mathfrak{g})$ such that $A \circ J = J' \circ A$.

Conclusion: many compact complex 3-folds $X = (M, J)$ defined by a quotient of G by a lattice Γ, where one can study different questions, as

- complex invariants (Dolbeault, Bott-Chern, Aeppli cohomologies), Frölicher spectral sequence $H_{\bar\partial}^{*,*}(X) \to H_{dR}^*(X)$,...

STEP 1. Classify real Lie algebras $\mathfrak{g}$ admitting complex structures.
Given a Lie algebra $\mathfrak{g}$, let $Z^3(\mathfrak{g}) = \{\rho \in \Lambda^3\mathfrak{g}^* \mid d\rho = 0\}$
and look for $\rho \in Z^3(\mathfrak{g})$ such that $\lambda(\rho) < 0$ and $d(J_\rho^* \rho) = 0$.

STEP 2. Moduli space $\mathcal{M}(\mathfrak{g})$ of complex structures for every $\mathfrak{g}$.

$$\mathcal{M}(\mathfrak{g}) = \mathcal{C}(\mathfrak{g})/\sim$$

where $J \sim J'$ iff $\exists\, A \in \mathrm{Aut}(\mathfrak{g})$ such that $A \circ J = J' \circ A$.

Conclusion: many compact complex 3-folds $X = (M, J)$ defined by a quotient of G by a lattice Γ, where one can study different questions, as

- complex invariants (Dolbeault, Bott-Chern, Aeppli cohomologies), Frölicher spectral sequence $H^{*,*}_{\bar\partial}(X) \to H^*_{dR}(X)$,...
- small deformations of complex structure, behaviour in central fiber,...

STEP 1. Classify real Lie algebras $\mathfrak{g}$ admitting complex structures.
Given a Lie algebra $\mathfrak{g}$, let $Z^3(\mathfrak{g}) = \{\rho \in \Lambda^3\mathfrak{g}^* \mid d\rho = 0\}$
and look for $\rho \in Z^3(\mathfrak{g})$ such that $\lambda(\rho) < 0$ and $d(J_\rho^* \rho) = 0$.

STEP 2. Moduli space $\mathcal{M}(\mathfrak{g})$ of complex structures for every $\mathfrak{g}$.

$$\mathcal{M}(\mathfrak{g}) = \mathcal{C}(\mathfrak{g})/\sim$$

where $J \sim J'$ iff $\exists\, A \in \mathrm{Aut}(\mathfrak{g})$ such that $A \circ J = J' \circ A$.

Conclusion: many compact complex 3-folds $X = (M, J)$ defined by a quotient of G by a lattice Γ, where one can study different questions, as

- complex invariants (Dolbeault, Bott-Chern, Aeppli cohomologies), Frölicher spectral sequence $H_{\bar\partial}^{*,*}(X) \to H_{dR}^*(X)$,...
- small deformations of complex structure, behaviour in central fiber,...
- existence of different types of special Hermitian metrics (Kähler, lcK, pluriclosed, generalized Gauduchon, balanced,...)

Let F be a Hermitian metric on X, i.e. a positive $(1,1)$-form on X (note that $F(\cdot,\cdot) = g(\cdot, J\cdot)$).

Let F be a Hermitian metric on X, i.e. a positive $(1,1)$-form on X (note that $F(\cdot,\cdot) = g(\cdot, J\cdot)$).

Recall the Lee form of a Hermitian metric F is the 1-form $\theta = J\,d^* F$.

Let F be a Hermitian metric on X, i.e. a positive $(1,1)$-form on X (note that $F(\cdot,\cdot) = g(\cdot, J\cdot)$).

Recall the Lee form of a Hermitian metric F is the 1-form $\theta = J\, d^* F$.

A Hermitian metric F on X is balanced if the Lee 1-form $\theta = 0$, equivalently, $dF^{n-1} = 0$ $\;(n = \dim_{\mathbb{C}} X)$.

Existence of balanced metrics on $X = (\Gamma\backslash G, J)$

Let F be a Hermitian metric on X, i.e. a positive $(1,1)$-form on X (note that $F(\cdot,\cdot) = g(\cdot, J\cdot)$).

Recall the Lee form of a Hermitian metric F is the 1-form $\theta = J\, d^* F$.

A Hermitian metric F on X is balanced if the Lee 1-form $\theta = 0$, equivalently, $dF^{n-1} = 0$ $(n = \dim_{\mathbb{C}} X)$.

Balanced metrics correspond to the Gray-Hervella class $\mathcal{W}_3$ (semi-Kähler). They are studied by many authors (Michelsohn,...) and play an important role in geometry and physics (string theory).

Existence of balanced metrics on $X = (\Gamma\backslash G, J)$

Let F be a Hermitian metric on X, i.e. a positive $(1,1)$-form on X (note that $F(\cdot,\cdot) = g(\cdot, J\cdot)$).

Recall the Lee form of a Hermitian metric F is the 1-form $\theta = J\,d^*F$.

A Hermitian metric F on X is balanced if the Lee 1-form $\theta = 0$, equivalently, $dF^{n-1} = 0$ $(n = \dim_{\mathbb{C}} X)$.

Balanced metrics correspond to the Gray-Hervella class $\mathcal{W}_3$ (semi-Kähler). They are studied by many authors (Michelsohn,...) and play an important role in geometry and physics (string theory).

PROPOSITION. Let $X = (\Gamma\backslash G, J)$ endowed with any invariant complex structure J. If F is a balanced metric on X, then there is a balanced metric on the Lie algebra $\mathfrak{g}$ (obtained by symmetrization).

Let F be a Hermitian metric on X, i.e. a positive $(1,1)$-form on X (note that $F(\cdot,\cdot) = g(\cdot, J\cdot)$).

Recall the Lee form of a Hermitian metric F is the 1-form $\theta = J\,d^*F$.

A Hermitian metric F on X is balanced if the Lee 1-form $\theta = 0$, equivalently, $dF^{n-1} = 0$ $\;(n = \dim_{\mathbb{C}} X)$.

Balanced metrics correspond to the Gray-Hervella class $\mathcal{W}_3$ (semi-Kähler). They are studied by many authors (Michelsohn,...) and play an important role in geometry and physics (string theory).

PROPOSITION. Let $X = (\Gamma\backslash G, J)$ endowed with any invariant complex structure J. If F is a balanced metric on X, then there is a balanced metric on the Lie algebra $\mathfrak{g}$ (obtained by symmetrization).

REDUCTION TO THE LIE ALGEBRA: For each pair $(\mathfrak{g}, J)$, we are reduced to study the existence of J-Hermitian inner products F on $\mathfrak{g}$ satisfying the balanced condition.

Existence of balanced metrics: nilpotent case

[U]	Balanced
$\mathfrak{n}_1 = (0,0,0,0,0,0)$	✓
$\mathfrak{n}_2 = (0,0,0,0,12,34)$	$\checkmark_{(J)}$
$\mathfrak{n}_3 = (0,0,0,0,0,12+34)$	$\checkmark_{(J)}$
$\mathfrak{n}_4 = (0,0,0,0,12,14+23)$	$\checkmark_{(J)}$
$\mathfrak{n}_5 = (0,0,0,0,13+42,14+23)$	$\checkmark_{(J)}$
$\mathfrak{n}_6 = (0,0,0,0,12,13)$	✓
$\mathfrak{n}_7 = (0,0,0,12,13,23)$	–
$\mathfrak{n}_8 = (0,0,0,0,0,12)$	–
$\mathfrak{n}_9 = (0,0,0,0,12,14+25)$	–
$\mathfrak{n}_{10} = (0,0,0,12,13,14)$	–
$\mathfrak{n}_{11} = (0,0,0,12,13,14+23)$	–
$\mathfrak{n}_{12} = (0,0,0,12,13,24)$	–
$\mathfrak{n}_{13} = (0,0,0,12,13+14,24)$	–
$\mathfrak{n}_{14} = (0,0,0,12,14,13+42)$	–
$\mathfrak{n}_{15} = (0,0,0,12,13+42,14+23)$	–
$\mathfrak{n}_{16} = (0,0,0,12,14,24)$	–
$\mathfrak{n}_{19}^{-} = (0,0,0,12,23,14-35)$	✓
$\mathfrak{n}_{26}^{+} = (0,0,12,13,23,14+25)$	–

Existence of balanced metrics: non-nilpotent case

[FOU]	Balanced
$\mathfrak{s}_1 = (15, -25, -35, 45, 0, 0)$	✓
$\mathfrak{s}_2^0 = (25, -15, 45, -35, 0, 0)$	✓
$\mathfrak{s}_2^\alpha = (\alpha\cdot 15+25, -15+\alpha\cdot 25, -\alpha\cdot 35+45, -35-\alpha\cdot 45, 0, 0), \quad \alpha > 0$	✓
$\mathfrak{s}_3 = (0, -13, 12, 0, -46, -45)$	✓
$\mathfrak{s}_4 = (23, -36, 26, -56, 46, 0)$	–
$\mathfrak{s}_5 = (24+35, 26, 36, -46, -56, 0)$	✓
$\mathfrak{s}_6 = (24+35, -36, 26, -56, 46, 0)$	–
$\mathfrak{s}_7 = (24+35, 46, 56, -26, -36, 0)$	✓
$\mathfrak{s}_8 = (16-25, 15+26, -36+45, -35-46, 0, 0)$	$\checkmark_{(J)}$
$\mathfrak{s}_9 = (45, 15+36, 14-26+56, -56, 46, 0)$	–

$\mathfrak{sl}(2,\mathbb{C})$	Balanced
$\mathfrak{so}(3,1) = (23-56,\ -13+46,\ 12-45,\ 26-35,\ -16+34,\ 15-24)$	✓

Hull and Strominger proposed a 10-dim. space-time $L^{1,9-d} \times X^d$, X compact, in order to compactify the heterotic strings with torsion.

It was proposed independently by

- C.M. Hull (Compactifications of the heterotic superstring, *Physics Letters* **B 178** (4):357-364, 1986.)
- A. Strominger (Superstrings with torsion, *Nuclear Phys.* **B 274** (1986), 253.)

Hull and Strominger proposed a 10-dim. space-time $L^{1,9-d} \times X^d$, X compact, in order to compactify the heterotic strings with torsion.

It was proposed independently by

- C.M. Hull (Compactifications of the heterotic superstring, *Physics Letters* **B 178** (4):357-364, 1986.)
- A. Strominger (Superstrings with torsion, *Nuclear Phys.* **B 274** (1986), 253.)

X is a compact complex manifold, $\dim_{\mathbb{C}} X = 3$, with holomorphically trivial canonical bundle. The system consists of three equations:

(I) The conformally balanced equation.

Let Ψ be a nowhere vanishing holomorphic (3,0)-form on X.

Let F be a Hermitian metric on X and denote by $||\Psi||_F$ the norm of Ψ with respect to the metric F.

The first equation is $d(||\Psi||_F \cdot F^2) = 0$,

in the formulation of [Li-Yau, 2005].

- The equation $d(||\Psi||_F \cdot F^2) = 0$ implies that $\tilde{F} = e^{-f/2}F$, with $f = -\log ||\Psi||_F$, is a balanced metric.

The function f is known as the dilaton function.

• The equation $d(||\Psi||_F \cdot F^2) = 0$ implies that $\tilde{F} = e^{-f/2}F$, with $f = -\log ||\Psi||_F$, is a balanced metric.

The function f is known as the dilaton function.

Let ∇^+ be the Bismut connection, i.e. the unique Hermitian connection (i.e. $\nabla J = 0$ and $\nabla F = 0$) with totally skew-symmetric torsion:

$$\nabla^+ = \nabla^{LC} + \frac{1}{2}T, \qquad \text{where } T = JdF \text{ torsion 3-form.}$$

- The equation $d(||\Psi||_F \cdot F^2) = 0$ implies that $\tilde{F} = e^{-f/2}F$, with $f = -\log ||\Psi||_F$, is a balanced metric.

The function f is known as the dilaton function.

Let ∇^+ be the Bismut connection, i.e. the unique Hermitian connection (i.e. $\nabla J = 0$ and $\nabla F = 0$) with totally skew-symmetric torsion:

$$\nabla^+ = \nabla^{LC} + \frac{1}{2}T, \qquad \text{where } T = JdF \text{ torsion 3-form.}$$

- The equation $d(||\Psi||_F \cdot F^2) = 0$ implies that the (3,0)-form $e^f\Psi$, $f = -\log ||\Psi||_F$, is parallel with respect to ∇^+, i.e.

$$\nabla^+(e^f\Psi) = 0, \qquad \text{hence } \mathrm{Hol}(\nabla^+) \subset \mathrm{SU}(3).$$

(II) The instanton equation.

A Hermitian vector bundle E over X equipped with an instanton, i.e. a connection A with curvature 2-form Ω^A satisfying the Hermitian-Yang-Mills equation

$$\Omega^A \wedge F^2 = 0, \quad (\Omega^A)^{0,2} = (\Omega^A)^{2,0} = 0.$$

More explicitly, with respect to a local orthonormal basis $\{e_k\}$

$$(\Omega^A)^i_j(Je_k, Je_l) = (\Omega^A)^i_j(e_k, e_l), \qquad \sum_{k=1}^{6}(\Omega^A)^i_j(e_k, Je_k) = 0.$$

It is allowed (and also of interest) to take the trivial case $\Omega^A = 0$.

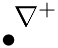
(III) The anomaly cancellation equation.

$$\boxed{dT = \frac{\alpha'}{4}\left(\mathrm{tr}\,\Omega\wedge\Omega - \mathrm{tr}\,\Omega^A\wedge\Omega^A\right)} \qquad \text{for } \alpha' \neq 0 \text{ constant}$$

α' slope parameter (string tension $\alpha' > 0$)

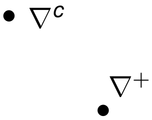
Ω is the curvature form of some metric connection ∇

- Strominger-Bismut $\nabla^+ = \nabla^{\mathrm{LC}} + \frac{1}{2}T$: torsion $T(\cdot,\cdot,\cdot) = JdF(\cdot,\cdot,\cdot)$

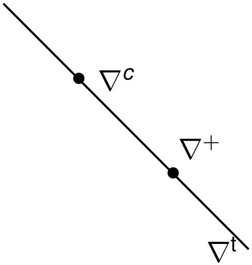


Metric connections proposed for ∇

- Strominger-Bismut $\nabla^+ = \nabla^{\text{LC}} + \frac{1}{2}T$: torsion $T(\cdot,\cdot,\cdot) = JdF(\cdot,\cdot,\cdot)$
- Chern $\nabla^c = \nabla^{\text{LC}} + \frac{1}{2}C$: torsion $C(\cdot,\cdot,\cdot) = dF(J\cdot,\cdot,\cdot)$

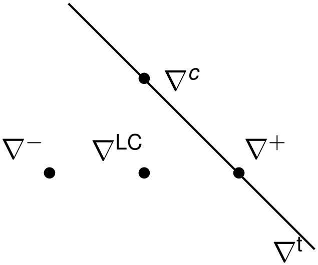


- Strominger-Bismut $\nabla^{+} = \nabla^{\mathrm{LC}} + \frac{1}{2}T$: torsion $T(\cdot,\cdot,\cdot) = JdF(\cdot,\cdot,\cdot)$
- Chern $\nabla^{c} = \nabla^{\mathrm{LC}} + \frac{1}{2}C$: torsion $C(\cdot,\cdot,\cdot) = dF(J\cdot,\cdot,\cdot)$
- Gauduchon (Hermitian) line $\nabla^{\mathrm{t}} = \nabla^{\mathrm{LC}} + \frac{1-\mathrm{t}}{4}\,T + \frac{1+\mathrm{t}}{4}\,C, \quad \mathrm{t} \in \mathbb{R}$.



Metric connections proposed for ∇

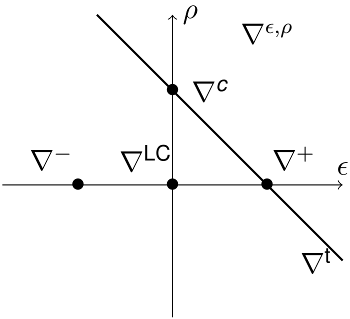
- Strominger-Bismut $\nabla^+ = \nabla^{LC} + \frac{1}{2}T$: torsion $T(\cdot,\cdot,\cdot) = JdF(\cdot,\cdot,\cdot)$
- Chern $\nabla^c = \nabla^{LC} + \frac{1}{2}C$: torsion $C(\cdot,\cdot,\cdot) = dF(J\cdot,\cdot,\cdot)$
- Gauduchon (Hermitian) line $\nabla^t = \nabla^{LC} + \frac{1-t}{4}\,T + \frac{1+t}{4}\,C, \quad t \in \mathbb{R}$.
- Hull (metric) connection $\nabla^- = \nabla^{LC} - \frac{1}{2}T$



The (ϵ, ρ)-plane of metric connections $\nabla^{\epsilon,\rho}$ [Otal-U-Villacampa]

Let (M^{2n}, J, g) be any Hermitian manifold. For any $(\varepsilon, \rho) \in \mathbb{R}^2$, define:

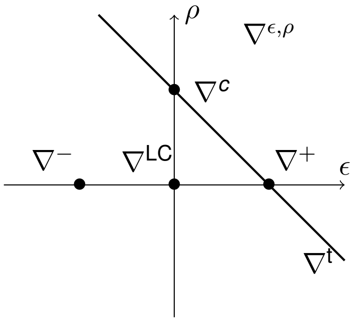
$$g(\nabla^{\varepsilon,\rho}_X Y, Z) = g(\nabla^{LC}_X Y, Z) + \varepsilon\, T(X, Y, Z) + \rho\, C(X, Y, Z).$$



Let (M^{2n}, J, g) be any Hermitian manifold. For any $(\varepsilon, \rho) \in \mathbb{R}^2$, define:

$$g(\nabla^{\varepsilon,\rho}_X Y, Z) = g(\nabla^{LC}_X Y, Z) + \varepsilon\, T(X, Y, Z) + \rho\, C(X, Y, Z).$$

Then, $\nabla^{\varepsilon,\rho} g = 0$, and $\nabla^{\varepsilon,\rho} J = -2\left(\varepsilon + \rho - \frac{1}{2}\right)\nabla^{LC} J$.



1. Any Kähler CY manifold X solves the HS system (by Yau's solution to the Calabi conjecture X admits a Ricci-flat Kähler metric) taking e.g. $\Omega^A = \Omega$.

Solutions of the Hull-Strominger system

1. Any Kähler CY manifold X solves the HS system (by Yau's solution to the Calabi conjecture X admits a Ricci-flat Kähler metric) taking e.g. $\Omega^A = \Omega$.

2. [Li-Yau, 2005] obtained the first non-Kähler solutions to the HS system on a Kähler CY 3-fold (further extended by [Andreas-García, 2012]).

1. Any Kähler CY manifold X solves the HS system (by Yau's solution to the Calabi conjecture X admits a Ricci-flat Kähler metric) taking e.g. $\Omega^A = \Omega$.

2. [Li-Yau, 2005] obtained the first non-Kähler solutions to the HS system on a Kähler CY 3-fold (further extended by [Andreas-García, 2012]).

3. On non-Kähler 3-folds:

[Fu-Yau, 2008] proved the existence of solutions on non-Kähler 3-folds given as a $\mathbb{T}^2$-bundle over a K3 surface.

[Fernández-Ivanov-U-Villacampa, 2009] first explicit invariant solutions.

[Fei-Yau, 2015] invariant solutions on complex Lie groups.

[Phong-Picard-Zhang, 2016] recover the Fu-Yau results via the anomaly flow.

[Fei-Huang-Picard, 2017] on hyperkähler fibrations over a Riemann surface.

[Otal-U-Villacampa, 2017] invariant solutions on solvmanifolds.

[Fino-Grantcharov-Vezzoni, 2021] Fu-Yau solution is generalized to torus bundles over K3 orbifolds.

Theorem [Ivanov 2010]

A solution to the HS system satisfies the heterotic equations of motion if and only if the connection ∇ is Hermitian-Yang-Mills (i.e. instanton).

Theorem [Ivanov 2010]

A solution to the HS system satisfies the heterotic equations of motion if and only if the connection ∇ is Hermitian-Yang-Mills (i.e. instanton).

The Hull-Strominger-Ivanov system

(I) $d(||\Psi||_F \cdot F^2) = 0,$

(II.a) $\Omega^A \wedge F^2 = 0, \quad (\Omega^A)^{0,2} = (\Omega^A)^{2,0} = 0,$

(II.b) $\Omega \wedge F^2 = 0, \quad \Omega^{0,2} = \Omega^{2,0} = 0,$

(III) $d(JdF) = \frac{\alpha'}{4}\left(\mathrm{tr}\,\Omega \wedge \Omega - \mathrm{tr}\,\Omega^A \wedge \Omega^A\right).$

From HS to the heterotic equations of motion

Theorem [Ivanov 2010]

A solution to the HS system satisfies the heterotic equations of motion if and only if the connection ∇ is Hermitian-Yang-Mills (i.e. instanton).

The Hull-Strominger-Ivanov system

(I) $d(||\Psi||_F \cdot F^2) = 0,$

(II.a) $\Omega^A \wedge F^2 = 0, \quad (\Omega^A)^{0,2} = (\Omega^A)^{2,0} = 0,$

(II.b) $\Omega \wedge F^2 = 0, \quad \Omega^{0,2} = \Omega^{2,0} = 0,$

(III) $d(JdF) = \frac{\alpha'}{4}\left(\mathrm{tr}\, \Omega \wedge \Omega - \mathrm{tr}\, \Omega^A \wedge \Omega^A\right).$

Theorem (Existence of invariant solutions)

The first solutions are given in [FIUV 2009] on the "nilmanifold" $\mathfrak{n}_3$, and later in [Otal-U-Villacampa 2017] on the "solvmanifold" $\mathfrak{s}_7$, and on the quotient of $(SO(3,1), J_0)$ = $SL(2,\mathbb{C})$. Moreover, $\nabla = \nabla^+$, and it is a non-flat instanton.

Simplest solutions given by the "nilmanifold" $\mathfrak{n}_3 = (0,0,0,0,0,12+34)$.

Existence of invariant solutions of the H-S-I system: an example

Simplest solutions given by the "nilmanifold" $\mathfrak{n}_3 = (0,0,0,0,0,12+34)$.
Two complex structures (up to equivalence), one is locally conformally Kähler, and the other one is given by:

$$d\omega^1 = d\omega^2 = 0, \quad d\omega^3 = \omega^1 \wedge \omega^{\bar{1}} - \omega^2 \wedge \omega^{\bar{2}}.$$

Simplest solutions given by the "nilmanifold" $\mathfrak{n}_3 = (0,0,0,0,0,\, 12+34)$.
Two complex structures (up to equivalence), one is locally conformally Kähler, and the other one is given by:

$$d\omega^1 = d\omega^2 = 0, \quad d\omega^3 = \omega^1 \wedge \omega^{\bar{1}} - \omega^2 \wedge \omega^{\bar{2}}.$$

The (3,0)-form $\Psi = \omega^1 \wedge \omega^2 \wedge \omega^3$ is holomorphic.

Existence of invariant solutions of the H-S-I system: an example

Simplest solutions given by the "nilmanifold" $\mathfrak{n}_3 = (0,0,0,0,0,12+34)$.
Two complex structures (up to equivalence), one is locally conformally Kähler, and the other one is given by:

$$d\omega^1 = d\omega^2 = 0, \quad d\omega^3 = \omega^1 \wedge \omega^{\bar{1}} - \omega^2 \wedge \omega^{\bar{2}}.$$

The (3,0)-form $\Psi = \omega^1 \wedge \omega^2 \wedge \omega^3$ is holomorphic.

(I) An infinite family F_t of balanced metrics with $\mathrm{Hol}(\nabla_t^+) \subset \mathrm{SU}(3)$

$$F_t = \frac{i}{2}(\omega^1 \wedge \omega^{\bar{1}} + \omega^2 \wedge \omega^{\bar{2}} + t^2\, \omega^3 \wedge \omega^{\bar{3}}), \quad t \in \mathbb{R} - \{0\}.$$

Simplest solutions given by the "nilmanifold" $\mathfrak{n}_3 = (0,0,0,0,0,12+34)$.
Two complex structures (up to equivalence), one is locally conformally Kähler, and the other one is given by:

$$d\omega^1 = d\omega^2 = 0, \quad d\omega^3 = \omega^1 \wedge \omega^{\bar{1}} - \omega^2 \wedge \omega^{\bar{2}}.$$

The (3,0)-form $\Psi = \omega^1 \wedge \omega^2 \wedge \omega^3$ is holomorphic.

(I) An infinite family F_t of balanced metrics with $\mathrm{Hol}(\nabla_t^+) \subset \mathrm{SU}(3)$

$$F_t = \frac{i}{2}(\omega^1 \wedge \omega^{\bar{1}} + \omega^2 \wedge \omega^{\bar{2}} + t^2\, \omega^3 \wedge \omega^{\bar{3}}), \quad t \in \mathbb{R} - \{0\}.$$

(II.a) There is an instanton A with $\operatorname{tr} \Omega^A \wedge \Omega^A = \omega^1 \wedge \omega^{\bar{1}} \wedge \omega^2 \wedge \omega^{\bar{2}}$.

Simplest solutions given by the "nilmanifold" $\mathfrak{n}_3 = (0,0,0,0,0,12+34)$.
Two complex structures (up to equivalence), one is locally conformally Kähler, and the other one is given by:

$$d\omega^1 = d\omega^2 = 0, \quad d\omega^3 = \omega^1 \wedge \omega^{\bar{1}} - \omega^2 \wedge \omega^{\bar{2}}.$$

The (3,0)-form $\Psi = \omega^1 \wedge \omega^2 \wedge \omega^3$ is holomorphic.

(I) An infinite family F_t of balanced metrics with $\mathrm{Hol}(\nabla_t^+) \subset \mathrm{SU}(3)$

$$F_t = \frac{i}{2}(\omega^1 \wedge \omega^{\bar{1}} + \omega^2 \wedge \omega^{\bar{2}} + t^2\, \omega^3 \wedge \omega^{\bar{3}}), \quad t \in \mathbb{R} - \{0\}.$$

(II.a) There is an instanton A with $\mathrm{tr}\, \Omega^A \wedge \Omega^A = \omega^1 \wedge \omega^{\bar{1}} \wedge \omega^2 \wedge \omega^{\bar{2}}$.

(II.b) ∇_t^+ is an instanton with $\mathrm{tr}\, \Omega_t^+ \wedge \Omega_t^+ = 16\, t^4\, \omega^1 \wedge \omega^{\bar{1}} \wedge \omega^2 \wedge \omega^{\bar{2}}$.

Simplest solutions given by the “nilmanifold” $\mathfrak{n}_3 = (0,0,0,0,0,12+34)$.
Two complex structures (up to equivalence), one is locally conformally Kähler, and the other one is given by:

$$d\omega^1 = d\omega^2 = 0, \quad d\omega^3 = \omega^1 \wedge \omega^{\bar 1} - \omega^2 \wedge \omega^{\bar 2}.$$

The (3,0)-form $\Psi = \omega^1 \wedge \omega^2 \wedge \omega^3$ is holomorphic.

(I) An infinite family F_t of balanced metrics with $\mathrm{Hol}(\nabla_t^+) \subset \mathrm{SU}(3)$

$$F_t = \frac{i}{2}(\omega^1 \wedge \omega^{\bar 1} + \omega^2 \wedge \omega^{\bar 2} + t^2\, \omega^3 \wedge \omega^{\bar 3}), \quad t \in \mathbb{R} - \{0\}.$$

(II.a) There is an instanton A with $\mathrm{tr}\, \Omega^A \wedge \Omega^A = \omega^1 \wedge \omega^{\bar 1} \wedge \omega^2 \wedge \omega^{\bar 2}$.

(II.b) ∇_t^+ is an instanton with $\mathrm{tr}\, \Omega_t^+ \wedge \Omega_t^+ = 16\, t^4\, \omega^1 \wedge \omega^{\bar 1} \wedge \omega^2 \wedge \omega^{\bar 2}$.

(III) The anomaly cancellation equation

$$dT_t = 2t^2\, \omega^1 \wedge \omega^{\bar 1} \wedge \omega^2 \wedge \omega^{\bar 2} = \frac{\alpha'}{4} \left(\mathrm{tr}\, \Omega_t^+ \wedge \Omega_t^+ - \mathrm{tr}\, \Omega^A \wedge \Omega^A\right),$$

is equivalent to $\alpha' = 8t^2/(16t^4 - 1)$. Any $t > \frac{1}{2}$ solves the H-S-I system.

An uniqueness result: the spaces $\mathfrak{n}_3$, $\mathfrak{s}_7$ and $\mathfrak{so}(3,1)$ are the unique admitting invariant solutions of the H-S-I system. More concretely,

An uniqueness result: the spaces $\mathfrak{n}_3$, $\mathfrak{s}_7$ and $\mathfrak{so}(3,1)$ are the unique admitting invariant solutions of the H-S-I system. More concretely,

Theorem [Otal-U, 2023]

$M = \Gamma\backslash G$ six-dimensional compact quotient of a simply connected Lie group G by a lattice Γ. Suppose that M possesses an invariant balanced Hermitian structure (J, F) with invariant non-zero closed $(3,0)$-form. Let $\nabla^{\varepsilon,\rho}_{(J,F)}$ be any metric connection in the (ε, ρ)-plane.

If $\nabla^{\varepsilon,\rho}_{(J,F)}$ is a non-flat instanton, then $\mathfrak{g}$ is isomorphic to $\mathfrak{n}_3$, $\mathfrak{s}_7$, or $\mathfrak{so}(3,1)$.

An uniqueness result: the spaces $\mathfrak{n}_3$, $\mathfrak{s}_7$ and $\mathfrak{so}(3,1)$ are the unique admitting invariant solutions of the H-S-I system. More concretely,

Theorem [Otal-U, 2023]

$M = \Gamma\backslash G$ six-dimensional compact quotient of a simply connected Lie group G by a lattice Γ. Suppose that M possesses an invariant balanced Hermitian structure (J, F) with invariant non-zero closed $(3,0)$-form. Let $\nabla^{\varepsilon,\rho}_{(J,F)}$ be any metric connection in the (ε, ρ)-plane.

If $\nabla^{\varepsilon,\rho}_{(J,F)}$ is a non-flat instanton, then $\mathfrak{g}$ is isomorphic to $\mathfrak{n}_3$, $\mathfrak{s}_7$, or $\mathfrak{so}(3,1)$.

PROOF: We studied the whole space of balanced Hermitian structures on the remaining Lie algebras from the previous classifications: $\mathfrak{n}_2$, $\mathfrak{n}_4$, $\mathfrak{n}_5$, $\mathfrak{n}_6$, $\mathfrak{n}_{19}^-$, $\mathfrak{s}_1$, $\mathfrak{s}_2^0$, $\mathfrak{s}_2^\alpha$ $(\alpha > 0)$, $\mathfrak{s}_3$, $\mathfrak{s}_5$, and $\mathfrak{s}_8$.
We found that if $\nabla^{\varepsilon,\rho}_{(J,F)}$ satisfies the Hermitian-Yang-Mills condition, then it is flat (and in this case, the connection is necessarily the Chern connection ∇^c).

New Developments in Differential Geometry, Budapest, July 1996

New Developments in Differential Geometry, Budapest, July 1996

HAPPY BIRTHDAY (in advance), EDUARDO!

And congratulations for your outstanding contributions to Mathematics!