Geometric analysis of the Lorentzian distance function

Luis J. Alías[1]

Departamento de Matemáticas
Universidad de Murcia, Spain

Symmetry and shape 2024

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- Our results where strongly based on a previous work by Erkekoglu, García-Río and Kupeli, where they established the basis for the comparison analysis of the (Lorentzian) Hessian and Laplacian operators of the Lorentzian distance funtion:

    On level sets of Lorentzian distance function, General Relativity and Gravitation **35** (2003), 1597–1615.

Causality relations in a spacetime

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- Obviously, $p \ll q$ implies $p < q$. As usual, $p \leq q$ means that either $p < q$ or $p = q$.
- For a subset $S \subset M$, one defines the chronological future of S as

$$I^+(S) = \{q \in M : p \ll q \text{ for some } p \in S\},$$

and the causal future of S as

$$J^+(S) = \{q \in M : p \leq q \text{ for some } p \in S\}.$$

Thus $S \cup I^+(S) \subset J^+(S)$.

Causality relations

- In a dual way, $I^-(S) = \{q \in M : q \ll p \text{ for some } p \in S\}$ and $J^-(S) = \{q \in M : q \leq p \text{ for some } p \in S\}$ are the chronological past and causal past of S.

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- For instance, for a point $p \in \overline{M}$ in Minkowski space, $I^+(p)$ is just the future timecone of p,

$$I^+(p) = \{q \in \overline{M} \ : \langle q-p, q-p\rangle < 0 \text{ and } \langle q-p, e_{n+1}\rangle < 0\},$$

and

$$J^+(p) = \overline{I^+(p)} = \{p\} \cup \{q : \langle q-p, q-p\rangle \leq 0 \text{ and } \langle q-p, e_{n+1}\rangle < 0\}$$

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- $I^{+}(p)$ is always open. $J^{+}(p)$ is neither open nor closed in general.

Lorentzian distance function

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- Recall that a spacetime M is said to be globally hyperbolic if
  - (i) it is causal, that is, there exists no causal loop in M, and
  - (ii) the intersections $J^+(p) \cap J^-(q)$ are compact for every $p, q \in M$.

Lorentzian distance function from a point

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- Let $T_{-1}M|_p$ be the fiber of the unit future observer bundle of M at p, that is,

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- Define the function $s_p : T_{-1}M|_p \to [0, +\infty]$ by

$$s_p(v) = \sup\{t \geq 0 : d_p(\gamma_v(t)) = t\},$$

where $\gamma_v : [0, a) \to M$ is the future inextendible geodesic starting at p with initial velocity v.

- Then, one can define the subset $\tilde{\mathcal{I}}^+(p) \subset T_pM$ given by

$$\tilde{\mathcal{I}}^+(p) = \{tv : \text{ for all } v \in T_{-1}M|_p \text{ and } 0 < t < s_p(v)\}$$

and consider the subset $\mathcal{I}^+(p) \subset M$ given by

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Lemma 1 (Erkekoglu, García-Río and Kupeli, GRG 2003)

Let M be a spacetime and $p \in M$.

1. If M is strongly causal at p, then $s_p(v) > 0$ for all $v \in T_{-1}M|_p$ and $\mathcal{I}^+(p) \neq \emptyset$.

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2. If $\mathcal{I}^+(p) \neq \emptyset$, then the Lorentzian distance function d_p is smooth on $\mathcal{I}^+(p)$ and its gradient $\overline{\nabla} d_p$ is a past-directed timelike (geodesic) unit vector field on $\mathcal{I}^+(p)$.

Hessian comparison results for the Lorentzian distance

- For every $c \in \mathbb{R}$, let us define

$$h_c(t) = \begin{cases} \frac{1}{\sqrt{c}} \sinh(\sqrt{c}\, t) & \text{if } c > 0 \text{ and } t > 0 \\ t & \text{if } c = 0 \text{ and } t > 0 \\ \frac{1}{\sqrt{-c}} \sin(\sqrt{-c}\, t) & \text{if } c < 0 \text{ and } 0 < t < \pi/\sqrt{-c}. \end{cases}$$

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- Observe that the index of a Jacobi field along a timelike geodesic in a Lorentzian space form of constant curvature c is given by

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- On the other hand, $\dfrac{h_c'(t)}{h_c(t)}$ is the future mean curvature of the level set

$$\Sigma_c(t) = \{q \in \mathcal{I}^+(p) : d_p(q) = t\} \subset M_c^n.$$

Lemma 2

Let M be a spacetime such that $K_M(\Pi) \geq c$, $c \in \mathbb{R}$, for all timelike planes in M. Assume that there exists a point $p \in M$ such that $\mathcal{I}^+(p) \neq \emptyset$, and let $q \in \mathcal{I}^+(p)$ (with $d_p(q) < \pi/\sqrt{-c}$ when $c < 0$). Then for every spacelike vector $x \in T_qM$ orthogonal to $\overline{\nabla} d_p(q)$

$$\overline{\nabla}^2 d_p(x,x) \leq -\frac{h_c'}{h_c}(d_p(q))\langle x,x\rangle,$$

where $\overline{\nabla}^2$ stands for the Hessian operator on M.

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where $\overline{\nabla}^2$ stands for the Hessian operator on M.

- The proof of Lemma 2 follows from the fact that

$$\overline{\nabla}^2 d_p(x,x) = I_\gamma(J,J)$$

where γ is the radial future directed unit timelike geodesic from p to q and J is the Jacobi field along γ with $J(0) = 0$ and $J(s) = x$, and is strongly based on the maximality of the index of Jacobi fields.

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- On the other hand, under the assumption that the sectional curvatures of the timelike planes of M are bounded from above by a constant c, we get the following result.

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- The proof is similar to that of Lemma 2.

Spacelike submanifolds contained in $\mathcal{I}^+(p)$

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- Our first objective is to compute the Hessian of u. To do that, observe that

$$\overline{\nabla} r = \nabla u + (\overline{\nabla} r)^\perp$$

along Σ, where $\nabla u = (\overline{\nabla} r)^\top$ stands for the gradient of u on Σ and $(\overline{\nabla} r)^\perp$ denotes the normal component of $\overline{\nabla} r$.

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  along Σ, where $\nabla u = (\overline{\nabla} r)^{\top}$ stands for the gradient of u on Σ and $(\overline{\nabla} r)^{\perp}$ denotes the normal component of $\overline{\nabla} r$.
- By Gauss and Weingarten formulae we get

$$\overline{\nabla}_X \overline{\nabla} r = \nabla_X \nabla u - A_{(\overline{\nabla} r)^{\perp}} X + \mathrm{II}(X, \nabla u) + \nabla^{\perp}_X (\overline{\nabla} r)^{\perp},$$

  for every tangent vector $X \in T\Sigma$, where II denotes the second fundamental form of the submanifold and, for every normal vector η, A_η denotes the Weingarten endomorphism with respect to η.

- It follows from here that

$$\nabla^2 u(X,Y) = \overline{\nabla}^2 r(X,Y) + \langle \mathrm{II}(X,Y), \overline{\nabla} r\rangle$$

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- Tracing this expression, one gets that the Laplacian of u is given by

$$\Delta u = \sum_{i=1}^{m} \overline{\nabla}^2 r(E_i, E_i) + m \langle \mathrm{H}, \overline{\nabla} r \rangle,$$

where $\{E_1, \dots, E_m\}$ is a local orthonormal frame on Σ, and

$$\mathrm{H} := \frac{1}{m} \mathrm{tr}(\mathrm{II}) = \frac{1}{m} \sum_{i=1}^{m} \mathrm{II}(E_i, E_i)$$

defines the mean curvature vector field of the submanifold.

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- Let A stand for the shape operator of Σ with respect to N. Then

$$\mathrm{H} = -\frac{1}{m}\mathrm{tr}(A)N = HN,$$

where $H = -(1/m)\mathrm{tr}(A)$ is the future mean curvature function of Σ.

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- Since M^{m+1} is time-oriented and Σ^m is spacelike, there is a unique future-directed timelike unit normal field N globally defined on Σ.
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- Therefore, the Laplacian of u becomes in this case

$$\begin{aligned}\Delta u &= \sum_{i=1}^{m} \overline{\nabla}^2 r(E_i, E_i) + mH\langle N, \overline{\nabla}r\rangle \\ &= \sum_{i=1}^{m} \overline{\nabla}^2 r(E_i, E_i) + mH\sqrt{1 + |\nabla u|^2}.\end{aligned}$$

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$$f_c(t) = \frac{h_c'(t)}{h_c(t)} = \begin{cases} \sqrt{c}\coth(\sqrt{c}\,t) & \text{if } c > 0 \text{ and } t > 0 \\ 1/t & \text{if } c = 0 \text{ and } t > 0 \\ \sqrt{-c}\cot(\sqrt{-c}\,t) & \text{if } c < 0 \text{ and } 0 < t < \pi/\sqrt{-c}. \end{cases}$$

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- Before stating our main results, we need to introduce some terminology.

The Omori-Yau maximum principle

- Following the terminology introduced by Pigola, Rigoli and Setti (2005), the Omori-Yau maximum principle is said to hold on an n-dimensional Riemannian manifold Σ if, for any smooth function $u \in \mathcal{C}^2(\Sigma)$ with $u^* = \sup_\Sigma u < +\infty$ there exists a sequence of points $\{p_k\}_{k\in\mathbb{N}}$ in Σ with the properties

$$\text{(i)}\;\; u(p_k) > u^* - \frac{1}{k}, \;\; \text{(ii)}\;\; |\nabla u(p_k)| < \frac{1}{k}, \;\text{and}\; \text{(iii)}\;\; \Delta u(p_k) < \frac{1}{k}.$$

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- Equivalently, for any $u \in \mathcal{C}^2(\Sigma)$ with $u_* = \inf_\Sigma u > -\infty$ there exists a sequence of points $\{p_k\}_{k\in\mathbb{N}}$ in Σ satisfying

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- In this sense, the classical maximum principle given by Omori (1967) and Yau (1975) stays that the Omori-Yau maximum principle holds on every complete Riemannian manifold with Ricci curvature bounded from below.

First applications: Hypersurfaces bounded by a level set of the Lorentzian distance. Case $K_M(\Pi) \geq c$

Theorem 1 (Alías, Hurtado, Palmer, TAMS 2010)

Let M^{m+1} be an $(m+1)$-dimensional spacetime such that $K_M(\Pi) \geq c$, $c \in \mathbb{R}$, for all timelike planes in M. Let $p \in M$ be such that $\mathcal{I}^+(p) \neq \emptyset$, and let $\psi : \Sigma^m \to M^{m+1}$ be a spacelike hypersurface such that $\psi(\Sigma) \subset \mathcal{I}^+(p)$. If the Omori-Yau maximum principle holds on Σ (and $\inf_\Sigma u < \pi/\sqrt{-c}$ when $c < 0$), then its future mean curvature H satisfies

$$\sup_\Sigma H \geq f_c(\inf_\Sigma u),$$

where u denotes the Lorentzian distance d_p along the hypersurface. In particular, if $\inf_\Sigma u = 0$ then $\sup_\Sigma H = +\infty$.

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Corollary 1 (Alías, Hurtado, Palmer, TAMS 2010)

Under the assumptions of Theorem 1, if H is bounded from above on Σ, then there exists some $\delta > 0$ such that $\psi(\Sigma) \subset O^+(p, \delta)$, where $O^+(p, \delta)$ denotes the future outer ball of radius δ,

$$O^+(p, \delta) = \{q \in I^+(p) : d_p(q) > \delta\}.$$

Proof of Theorem 1

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- The last assertion follows from the fact that $\lim_{s \to 0} f_c(s) = +\infty$.

Hypersurfaces in Lorentzian space forms

Theorem 2 (Alías, Hurtado, Palmer, TAMS 2010)

Let M_c^{m+1} be a Lorentzian space form of constant sectional curvature c and let $p \in M_c^{m+1}$. Let us consider $\psi : \Sigma^m \to M_c^{m+1}$ a spacelike hypersurface such that $\psi(\Sigma) \subset \mathcal{I}^+(p) \cap B^+(p, \delta)$ for some $\delta > 0$ (with $\delta \leq \pi/\sqrt{-c}$ if $c < 0$). If the Omori-Yau maximum principle holds on Σ, then

$$\inf_\Sigma H \leq f_c(\sup_\Sigma u) \leq f_c(\inf_\Sigma u) \leq \sup_\Sigma H,$$

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Corollary 2 (Alías, Hurtado, Palmer, TAMS 2010)

Let M_c^{m+1} be a Lorentzian space form of constant sectional curvature c and let $p \in M_c^{m+1}$. If Σ is a complete spacelike hypersurface in M_c^{m+1} with constant mean curvature H which is contained in $\mathcal{I}^+(p)$ and bounded from above by a level set of the Lorentzian distance function d_p (with $d_p < \pi/\sqrt{-c}$ if $c < 0$), then Σ is necessarily a level set of d_p.

- For a proof simply observe that the Ricci curvature of a spacelike hypersurface in an arbitrary spacetime M is given by

$$\begin{aligned}\mathrm{Ric}(X,X) &= \mathrm{Ric}_M(X,X) - \left(K_M(X\wedge N) + \frac{m^2H^2}{4}\right)|X|^2 + |AX + \frac{m}{2}X|\\ &\geq \mathrm{Ric}_M(X,X) - \left(K_M(X\wedge N) + \frac{m^2H^2}{4}\right)|X|^2.\end{aligned}$$

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Corollary 3 (Alías, Hurtado, Palmer, TAMS 2010)

The only complete spacelike hypersurfaces with constant mean curvature in the Lorentz-Minkowski space $\mathbb{L}^{m+1}$ which are contained in $\mathcal{I}^+(p)$ (for some fixed $p\in\mathbb{L}^{m+1}$) and bounded from above by a hyperbolic space centered at p are precisely the hyperbolic spaces centered at p.

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(i) If the future mean curvature of Σ satisfies $H \leq \dfrac{2\sqrt{m-1}}{m} f_c(u)$ with $H < f_c(u)$ at some point of Σ if $m = 2$, then Σ is hyperbolic.

(ii) If $c = 0$ and $H \leq 0$, then Σ is hyperbolic.

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In particular, every maximal hypersurface contained in $\mathcal{I}^+(p)$ (and satisfying $u < (\pi/2\sqrt{-c})$ if $c < 0$) is hyperbolic.

Proof of Theorem 3

- In order to proof (i), observe that under our assumptions on H we have
$$mH \leq 2\sqrt{m-1}\ f_c(u) \leq \frac{f_c(u)(m+|\nabla u|^2)}{\sqrt{1+|\nabla u|^2}}.$$

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- In order to proof (i), observe that under our assumptions on H we have
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- Therefore, by the Hessian comparison result in Lemma 2 we conclude
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- To prove (ii) and (iii), simply observe that $f_0(u) = 1/u > 0$ and $f_c(u) = \sqrt{c}\coth(\sqrt{c}u) > \sqrt{c}$ on Σ.

Case 2: Spacelike submanifolds contained in $\mathcal{I}^+(p)$

- Recall that the Laplacian of u is given by

$$\Delta u = \sum_{i=1}^{m} \overline{\nabla}^2 r(E_i, E_i) + m\langle \mathrm{H}, \overline{\nabla} r\rangle.$$

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- Then, the Laplacian of v is given by

$$\begin{aligned} \Delta v &= \phi_c'(u)\Delta u + \phi_c''(u)|\nabla u|^2 \\ &= h_c(u)\sum_{i=1}^{m} \overline{\nabla}^2 r(E_i, E_i) + m h_c(u)\langle \mathrm{H}, \overline{\nabla} r\rangle + h_c'(u)|\nabla u|^2. \end{aligned}$$

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$$\phi_c(t) = \begin{cases} \frac{1}{c}\cosh(\sqrt{c}\, t) & \text{if } c > 0 \text{ and } t > 0 \\ \frac{t^2}{2} & \text{if } c = 0 \text{ and } t > 0 \\ \frac{1}{c}\cos(\sqrt{-c}\, t) & \text{if } c < 0 \text{ and } 0 < t < \pi/\sqrt{-c}. \end{cases}$$

- Then, the Laplacian of v is given by

$$\begin{aligned} \Delta v &= \phi_c'(u)\Delta u + \phi_c''(u)|\nabla u|^2 \\ &= h_c(u)\sum_{i=1}^{m} \overline{\nabla}^2 r(E_i, E_i) + m h_c(u)\langle \mathrm{H}, \overline{\nabla} r\rangle + h_c'(u)|\nabla u|^2. \end{aligned}$$

- Assume now that $K_M(\Pi) \geq c$ (resp. $K_M(\Pi) \leq c$) for all timelike planes in M.

- Then by the Hessian comparison results for r, one gets that

$$\overline{\nabla}^2 r(X,X) \leq (\geq) -\frac{h_c'}{h_c}(u)(1+\langle X,\nabla u\rangle^2)$$

for every unit tangent vector field $X \in T\Sigma$.

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- Therefore,

$$h_c(u)\sum_{i=1}^{m}\overline{\nabla}^2 r(E_i,E_i) \leq (\geq) - h_c'(u)(m+|\nabla u|^2),$$

which, jointly with the expression above, gives the following inequality for the Laplacian of v

$$\Delta v \leq (\geq) - m h_c'(u) + m h_c(u)\langle \mathrm{H}, \overline{\nabla} r\rangle.$$

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- Summarizing:
  - $K_M(\Pi) \geq c$ implies that

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where $v = \phi_c(u)$ and u is the Lorentzian distance function of M restricted on the spacelike submanifold Σ.
- For statement of our main results, we introduce some terminology.

The weak maximum principle

- The *weak* maximum principle is said to hold on Σ if, for any $u \in \mathcal{C}^2(\Sigma)$ with $u^* < +\infty$ there is a sequence $\{p_k\}_{k\in\mathbb{N}}$ in Σ with

$$\text{(i)} \quad u(p_k) > u^* - \frac{1}{k}, \quad \text{and} \quad \text{(iii)} \quad \Delta u(p_k) < \frac{1}{k}.$$

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- Σ is said to be stochastically complete if its Brownian motion is stochastically complete, i.e, the probability of a particle to be found in the state space is constantly equal to 1.
- This is equivalent (among other conditions) to the fact that for every $\lambda > 0$, the only non-negative bounded smooth solution u of $\Delta u \geq \lambda u$ on Σ is the constant $u = 0$.
- In particular, every parabolic manifold is stochastically complete. Hence, the weak max principle holds on every parabolic manifold.

Trapped submanifolds in a spacetime

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- Finally, Σ is said to be weakly future trapped if H is causal (that is, timelike or lightlike) and future-pointing everywhere.

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- Similarly, Σ is said to be marginally past trapped if H is lightlike and past-pointing on Σ.
- Finally, Σ is said to be weakly future trapped if H is causal (that is, timelike or lightlike) and future-pointing everywhere.
- Analogously, Σ is said to be weakly past trapped if H is causal and past-pointing on Σ.

Weakly trapped submanifolds in the chronological future of a point. Case $K_M(\Pi) \geq c$

Theorem 4 (Alías, Bessa, de Lira, CQG 2016)

Let M be a spacetime with a reference point $p \in M$ such that $\mathcal{I}^+(p) \neq \emptyset$, and assume $K_M(\Pi) \geq c$, $c \in \mathbb{R}$, for all timelike planes in M.

1. If $c \geq 0$ there exists no stochastically complete, weakly past trapped submanifold contained in $\mathcal{I}^+(p)$.
2. If $c < 0$ and Σ is a stochastically complete, weakly past trapped submanifold contained in $\mathcal{I}^+(p) \cap B^+(p, \pi/\sqrt{-c})$, then

$$u_* = \inf_{\Sigma} u \geq \frac{\pi}{2\sqrt{-c}},$$

where u denotes the Lorentzian distance d_p along the hypersurface. In other words, Σ is contained in $B^+(p, \pi/\sqrt{-c}) \cap O^+(p, \pi/2\sqrt{-c})$.

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- Recall that the subsets $B^+(p,\delta)$ and $O^+(p,\delta)$ denote the future inner ball and the future outer ball of radius $\delta > 0$, that is,

$$\begin{aligned} B^+(p,\delta) &= \{q \in I^+(p) : d_p(q) < \delta\} \\ O^+(p,\delta) &= \{q \in I^+(p) : d_p(q) > \delta\}. \end{aligned}$$

Proof of Theorem 4

- As $K_M(\Pi) \geq c$, we know that

$$\Delta v \leq -mh_c'(u) + mh_c(u)\langle \mathrm{H}, \overline{\nabla} r \rangle.$$

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$$-\frac{1}{k} < \Delta v(p_k) \leq -mh'(u(p_k)) + mh(u(p_k))\langle \mathrm{H}, \overline{\nabla} r\rangle(p_k),$$
for $\{p_k\} \subset \Sigma$ with $\lim_{k\to\infty} v(p_k) = v_*$ and $\lim_{k\to\infty} u(p_k) = u_*$.

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- Observe that, since Σ is weakly past trapped, then

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and, making $k \to \infty$ here we get $h_c'(u_*) \leq 0$.
- The result then follows by observing that, when $c \geq 0$ then $h_c'(t) > 0$, and if $c < 0$ then $h_c'(t) \leq 0$ when $\pi/2\sqrt{-c} \leq t < \pi/\sqrt{-c}$.

Marginally trapped submanifolds in the chronological future of a point. Case $K_M(\Pi) \geq c$

Theorem 5 (Alías, Bessa, de Lira, CQG 2016)

Let M be a spacetime with a reference point $p \in M$ such that $\mathcal{I}^+(p) \neq \emptyset$, and assume $K_M(\Pi) \geq c$, $c \in \mathbb{R}$, for all timelike planes in M. Let Σ be a stochastically complete, marginally trapped submanifold contained in $\mathcal{I}^+(p)$ (with $u_* < \pi/2\sqrt{-c}$ in the case $c < 0$). Then

$$\sup_\Sigma |\mathrm{H}_0| \geq \frac{h_c'}{h_c}(u_*),$$

where H_0 stands for the spacelike component of the lightlike vector field H which is orthogonal to $\overline{\nabla} r$, and $u_* = \inf_\Sigma u$. In particular, if $u_* = 0$ then $\sup_M |\mathrm{H}_0| = +\infty$.

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Corollary 4 (Alías, Bessa, de Lira, CQG 2016)

Under the assumptions of Theorem 5, if $|\mathrm{H}_0|$ is bounded from above on Σ, then there exists some $\delta > 0$ such that $\Sigma \subset O^+(p, \delta)$, where $O^+(p, \delta)$ denotes the future outer ball of radius δ.

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$$\Delta v \leq -m h_c'(u) + m h_c(u) |\mathrm{H}_0|.$$

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- Otherwise, let us write

$$\Delta v \leq -m h_c'(u) + m h_c(u) |\mathrm{H}_0| \leq -m h_c'(u) + m h_c(u) \sup_\Sigma |\mathrm{H}_0|.$$

Proof of Theorem 5

- Applying again the weak maximum principle on Σ to the function $v = \phi_c(u)$, with $v_* = \inf_\Sigma v = \phi_c(u_*)$, we have

$$-\frac{1}{k} < \Delta v(p_k) \leq -mh_c'(u(p_k)) + mh_c(u(p_k)) \sup_\Sigma |\mathrm{H}_0|,$$

for $\{p_k\} \subset \Sigma$ with $\lim_{k\to\infty} v(p_k) = v_*$ and $\lim_{k\to\infty} u(p_k) = u_*$.

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$$\sup_\Sigma |\mathrm{H}_0| \geq \frac{h_c'(u_*)}{h_c(u_*)}.$$

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- Letting $k \to +\infty$ we conclude that

$$\sup_\Sigma |\mathrm{H}_0| \geq \frac{h'_c(u_*)}{h_c(u_*)}.$$

- The last assertion follows from the fact that $h_c(0) = 0$ and $h'_c(0) = 1$.

Marginally trapped submanifolds in the chronological future of a point. Case $K_M(\Pi) \leq c$

Theorem 6 (Alías, Bessa, de Lira, CQG 2016)

Let M be a spacetime with a reference point $p \in M$ such that $\mathcal{I}^+(p) \neq \emptyset$, and assume $K_M(\Pi) \leq c$, $c \in \mathbb{R}$, for all timelike planes in M. Let Σ be a stochastically complete, marginally future trapped submanifold contained in $\mathcal{I}^+(p) \cap B^+(p, \delta)$ for some $\delta > 0$ (with $\delta \leq \pi/\sqrt{-c}$ when $c < 0$). Then

$$\inf_{\Sigma} |\mathrm{H}_0| \leq \frac{h_c'}{h_c}(u^*),$$

where H_0 stands for the spacelike component of the lightlike vector field H which is orthogonal to $\overline{\nabla} r$, and $u^* = \sup_\Sigma u$.

Proof of Theorem 6

- Since $K_M(\Pi) \leq c$ and $\langle \mathrm{H}, \overline{\nabla} r \rangle = |\mathrm{H}_0| > 0$ on Σ, we have

$$\Delta v \geq -mh_c'(u) + mh_c(u)|\mathrm{H}_0|.$$

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- If $\inf_\Sigma |\mathrm{H}_0| = -\infty$ then there is nothing to prove.

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- Otherwise, let us write

$$\Delta v \geq -mh_c'(u) + mh_c(u)|\mathrm{H}_0| \geq -mh_c'(u) + mh_c(u) \inf_\Sigma |\mathrm{H}_0|.$$

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$$\Delta v \geq -mh_c'(u) + mh_c(u)|\mathrm{H}_0| \geq -mh_c'(u) + mh_c(u) \inf_\Sigma |\mathrm{H}_0|.$$

- Applying the weak maximum principle on Σ to the function $v = \phi_c(u)$, with $v^* = \sup_\Sigma v = \phi_c(u^*)$, we have

$$\frac{1}{k} > \Delta v(p_k) \geq -mh_c'(u(p_k)) + mh_c(u(p_k)) \inf_\Sigma |\mathrm{H}_0|,$$

for $\{p_k\} \subset \Sigma$ with $\lim_{k\to\infty} v(p_k) = v^*$ and $\lim_{k\to\infty} u(p_k) = u^*$.

Proof of Theorem 6

- Since $K_M(\Pi) \leq c$ and $\langle \mathrm{H}, \overline{\nabla} r \rangle = |\mathrm{H}_0| > 0$ on Σ, we have

$$\Delta v \geq -m h_c'(u) + m h_c(u) |\mathrm{H}_0|.$$

- If $\inf_\Sigma |\mathrm{H}_0| = -\infty$ then there is nothing to prove.
- Otherwise, let us write

$$\Delta v \geq -m h_c'(u) + m h_c(u) |\mathrm{H}_0| \geq -m h_c'(u) + m h_c(u) \inf_\Sigma |\mathrm{H}_0|.$$

- Applying the weak maximum principle on Σ to the function $v = \phi_c(u)$, with $v^* = \sup_\Sigma v = \phi_c(u^*)$, we have

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  for $\{p_k\} \subset \Sigma$ with $\lim_{k\to\infty} v(p_k) = v^*$ and $\lim_{k\to\infty} u(p_k) = u^*$.
- Making $k \to +\infty$ we conclude that

$$\inf_\Sigma |\mathrm{H}_0| \leq \frac{h_c'(u^*)}{h_c(u^*)}.$$

Marginally future trapped submanifolds in Lorentzian space forms

- In particular, when the ambient spacetime is a Lorentzian space form, by putting together Theorems 4, 5 and 6 we obtain the following consequence.

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Let M^n_c be a Lorentzian space form of constant sectional curvature c and let $p \in M^n_c$. Let Σ be a stochastically complete, marginally trapped submanifold of M^n_c which is contained in $\mathcal{I}^+(p) \cap B^+(p,\delta)$ for some $\delta > 0$ (with $\delta \leq \pi/2\sqrt{-c}$ if $c < 0$). Then

$$\inf_\Sigma |\mathrm{H}_0| \leq \frac{h'_c(u^*)}{h_c(u^*)} \leq \frac{h'_c(u_*)}{h_c(u_*)} \leq \sup_\Sigma |\mathrm{H}_0|,$$

where $u_* = \inf_\Sigma u$ and $u^* = \sup_\Sigma u$.

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where $u_* = \inf_\Sigma u$ and $u^* = \sup_\Sigma u$.

- The estimates are sharp as proved by considering Σ as a constant mean curvature hypersurface of a level set of the Lorentzian distance in M^n_c.

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and congratulations to Eduardo for his forthcoming first 60 years.