Totally Geodesic and Parallel hypersurfaces

of Cahen-Wallach spacetimes

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Symmetry and shape

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Let ξ be an ε-unit normal vector field on M, hypersurface immersed into $\bar{M}$. The *formula of Gauss* gives

$$\bar{\nabla}_X Y = \nabla^M_X Y + h(X,Y)\xi, \quad X, Y \in TM,$$

where h is called the **second fundamental form** of the immersion.

Definition 1

M is a **totally geodesic hypersurface** if $h = 0$.

Moreover, considering the covariant derivative $\nabla^M h$, defined by

$$(\nabla^M h)(X,Y,Z) = X(h(Y,Z)) - h(\nabla^M_X Y, Z) - h(Y, \nabla^M_X Z),$$

we give the following

Definition 2

A hypersurface is said to be **parallel** if $\nabla^M h = 0$.

The second fundamental form h of M is said to be **Codazzi** if $\nabla^M h$ is totally symmetric. This is equivalent to requiring that $R(X,Y)\xi = 0$ for all vector fields X, Y tangent to M^n.

Clearly, parallel hypersurfaces have a Codazzi second fundamental form.

The knowledge of totally geodesic and parallel hypersurfaces improves our understanding of the geometry of a given ambient space.

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General setup

- Geometrical description of the **ambient space**;
- *M* **Codazzi** $\Rightarrow$ different cases for algebraic conditions on the components of the **unit normal vector field**;
- Geometrical investigation of each **subcase**;
- Totally Geodesicity and Parallelism.

Cahen-Wallach spaces

Indecomposable symmetric Lorentzian manifolds of non-constant sectional curvature are knows as **Cahen-Wallach spaces**. Explicitly, an arbitrary four-dimensional Cahen-Wallach symmetric space is described as $\mathbb{R}^4$ equipped with the Lorentzian metric

$$g = \left(k_3 x_3^2 + k_4 x_4^2\right) dx_1^2 + 2 dx_1 dx_2 + dx_3^2 + dx_4^2,$$

where $k_3, k_4 \neq 0$ are some real constants.

In the special case where $k_3 = k_4 = k$, these spaces are also known as **ε-spaces**. They are locally conformally flat and admit a large group of isometries.

Brinkmann manifolds

A **Brinkmann manifold** is a Lorentzian manifold (M, g) admitting a *parallel null vector field* U, that is, such that $g(U, U) = \nabla U = 0$.

Cahen-Wallach symmetric spaces are examples of Brinkmann manifolds. In fact, ∂_2 is a parallel null vector field on any Cahen-Wallach space.

A Brinkmann manifold is a special kind of **Walker manifold**, namely a pseudo-Riemannian manifold which admits a *non-trivial distribution* D which is

- **parallel**: if $X \in D$ then $\nabla X \in D$;
- and **null**: the metric restricted to D vanishes identically.

Three-dimensional Walker manifold

In dimension three, a *Walker manifold* admits local coordinates (t,x,y), with respect to which its Lorentzian metric tensor is expressed as follows:

$$\tilde{g} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \varepsilon & 0 \\ 1 & 0 & f(t,x,y) \end{pmatrix}$$

for some function $f(t,x,y)$, where $\varepsilon = \pm 1$ and the parallel degenerate line field becomes $D = \langle \frac{\partial}{\partial_t} \rangle$.
If ∂_t is a parallel null vector field then $f = f(x,y)$ and conversely.

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M is locally symmetric $\Longleftrightarrow$ $f(x,y) = \alpha x^2 + x\beta(y) + \gamma(y)$, where α is a real constant and β, γ are arbitrary functions.

M. Chaichi, E. Garcia-Rio and M. E. Vazquez-Abal. *Three-dimensional Lorentz manifolds admitting a parallel null vector field*, J. Phys. A: Math. Gen., **38** (2005), 841–850.

Codazzi hypersurfaces

Consider the normal vector field $\xi = a\partial_1 + b\partial_2 + c\partial_3 + d\partial_4$, for some functions $a, b, c, d : U \to \mathbb{R}$. Then, the following vector fields are tangent to the hypersurface:

$$X_1 = a\partial_1 - \rho\partial_2, \qquad X_2 = c\partial_1 - \rho\partial_3, \qquad X_3 = d\partial_1 - \rho\partial_4,$$

$$X_4 = c\partial_2 - a\partial_3, \qquad X_5 = d\partial_2 - a\partial_4, \qquad X_6 = d\partial_3 - c\partial_4,$$

where we put $\rho = a\left(k_3 x_3^2 + k_4 x_4^2\right) + b$. If **$h$ is Codazzi**, then $R(X_i, X_j)\xi = 0$ for every $i, j \in \{1, \ldots, 6\}$.

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where we put $\rho = a\left(k_3 x_3^2 + k_4 x_4^2\right) + b$. If **$h$ is Codazzi**, then $R(X_i, X_j)\xi = 0$ for every $i, j \in \{1, \ldots, 6\}$. In particular, we have

$$0 = R(X_1, X_4)\xi = -a^2 k_3 X_4,$$

which gives necessarily $a = 0$.
Therefore, $g(\xi, \xi) = c^2 + d^2$ and so, M is necessarily timelike and c and d cannot both vanish.

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Therefore, $g(\xi,\xi) = c^2 + d^2$ and so, M is necessarily timelike and c and d cannot both vanish. Moreover, from

$$0 = R(X_2, X_6)\xi = cd(k_3 - k_4)X_4$$

we deduce that we have to **consider separately two cases**, depending on whether $k_3 = k_4$.

Theorem 1 (Codazzi hypersurfaces)

Let $F : M \to \bar{M}$ be a hypersurface with a ***Codazzi second fundamental form*** *and ξ the unit normal vector field, with $g(\xi,\xi) = \varepsilon \in \{-1,1\}$.*
Consider the coordinate vector fields $\{\partial_i\}$ on $\bar{M}$ introduced above. Then, $g(\xi,\partial_2) = 0$ and M is a timelike hypersurface. Moreover, some of the following holds:

(I) *every point of M has an open neighbourhood in M, on which either $\xi = b\partial_2 + \partial_3$ or $\xi = b\partial_2 + \partial_4$;*

(II) *M is an ε-space, that is, $k_3 = k_4 = k$.*

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Theorem 2

Any hypersurface $F : M \to \bar{M}$ with a Codazzi second fundamental form in a Cahen-Wallach spacetime is a ***Brinkmann manifold****.*
Moreover, if $\bar{M}$ is not an ε-space, then M is minimal.

Totally geodesic hypersurfaces - Case (I)

Case (I): $\xi = b\partial_2 + \partial_3$ (or $\xi = b\partial_2 + \partial_4$).

If $\xi = b\partial_2 + \partial_3$, for some smooth function b on M, the vector fields

$$Y_1 = \partial_2, \qquad Y_2 = \partial_1 - b\partial_3, \qquad Y_3 = \partial_4 \tag{1}$$

span the tangent space to M at every point.

The symmetry conditions for h implies $Y_1(b) = Y_3(b) = 0$).

Then h is completely determined by $h(Y_2, Y_2) = Y_2(b) + k_3 x_3$.

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Let us observe that

$$\mathrm{tr}_{g_M} h = \frac{1}{3} g_M^{-1}(Y_2, Y_2) h(Y_2, Y_2) = 0,$$

so that all such hypersurfaces with Codazzi second fundamental form are minimal. The case where $\xi = b\partial_2 + \partial_4$ is analogous.

M is **totally geodesic** if and only if

$$Y_2(b) = -k_3 x_3. \tag{2}$$

It is easy to check that $[Y_i, Y_j] = 0$ for all indices i, j. So, the vector fields

$$Y_1 = \partial_t, \quad Y_2 = \partial_y, \quad Y_3 = \partial_x \tag{3}$$

may be taken as coordinate vector fields on M. Since $Y_1(b) = Y_3(b) = 0$ we then get that $b = b(y)$ and condition (2) can be rewritten in the form

$$b'(y) = -k_3 x_3. \tag{4}$$

Denote now by $F : M \to \bar{M}$ the immersion of the hypersurface in the local coordinates (t, x, y) introduced above. We obtain

$$F(t, x, y) = (y, t, \phi(y), x), \quad \phi(y) = \begin{cases} A\cosh(\sqrt{k_3}\, y) + B\sinh(\sqrt{k_3}\, y) & \text{if } k_3 > 0, \\ A\cos(\sqrt{-k_3}\, y) + B\sin(\sqrt{-k_3}\, y) & \text{if } k_3 < 0, \end{cases} \tag{5}$$

$A, B \in \mathbb{R}$. Hence, it is an open part of the **cylindrical hypersurface** spanned by the curve of equation $x_3 = \phi(x_1)$ in the (x_1, x_3)–plane.

The case where $\xi = b\partial_2 + \partial_4$ can be treated exactly in the same way. It leads to the following explicit expression of the immersion:

$$F(t,x,y) = (y,t,x,\psi(y)),$$

where

$$\psi(y) = \begin{cases} A\cosh(\sqrt{k_4}\,y) + B\sinh(\sqrt{k_4}\,y) & \text{if } k_4 > 0, \\ A\cos(\sqrt{-k_4}\,y) + B\sin(\sqrt{-k_4}\,y) & \text{if } k_4 < 0, \end{cases} \tag{6}$$

for some real constants A, B and $b(y) = -\psi'(y)$.

Totally geodesic hypersurfaces - Case (II)

Case (II): $a = 0$ and $k_3 = k_4 = k$. In this case, $\xi = b\partial_2 + c\partial_3 + d\partial_4$, where b, c, d are smooth functions on M. As $||\xi||^2 = c^2 + d^2 = 1$, there exists a smooth function θ on M such that $c = \cos\theta$ and $d = \sin\theta$.

There exists a dense open subset Ω of M, such that each point $p \in \Omega$ either admits a neighbourhood where $b = 0$ or a neighbourhood where $b \neq 0$ at any point. Thus, we consider separately the two cases.

Case (II.a): $a = b = 0$ and $k_3 = k_4 = k$. In this case, $\xi = \cos\theta\partial_3 + \sin\theta\partial_4$ and M is **totally geodesic** if and only if θ **is constant** and

$$x_3 \cos\theta + x_4 \sin\theta = 0. \tag{7}$$

The map

$$\begin{aligned} \Lambda : \bar{M} &\rightarrow \bar{M}, \\ (x_1, x_2, x_3, x_4) &\mapsto (x_1, x_2, \cos\theta x_3 + \sin\theta x_4, -\sin\theta x_3 + \cos\theta x_4) \end{aligned} \tag{8}$$

is an isometry of the ambient ε-space $\bar{M}$, for every real constant θ. Then it suffices to consider the case where $\theta = 0$ and so, $\xi = \partial_3$ (equivalently, we can set $\theta = \frac{\pi}{2}$ and get $\xi = \partial_4$).
Hence, this case corresponds to the special solution of **Case (I)** with

$$b = g(\xi, \partial_1) = 0.$$

Correspondingly, the totally geodesic condition written in (7), namely, applying the above isometry reads $x_3 = 0$ (equivalently, $x_4 = 0$) so that the totally geodesic hypersurface is an open part of the **hyperplane** $x_3 = 0$ (respectively, $x_4 = 0$).

Case (II.b): $a = 0 \neq b$ and $k_3 = k_4 = k$. We now have $\xi = b\partial_2 + \cos\theta\partial_3 + \sin\theta\partial_4$. Next, assuming that M **is totally geodesic**, we get that θ **is constant**. Therefore, as in Case (II.a), it suffices to consider the case where $\theta = 0$ and so, $\xi = b\partial_2 + \partial_3$ (or, equivalently, $\theta = \frac{\pi}{2}$, whence, $\xi = b\partial_2 + \partial_4$). Thus, we obtain again the totally geodesic hypersurfaces described in Case (I).

Case (II.b): $a = 0 \neq b$ and $k_3 = k_4 = k$. We now have $\xi = b\partial_2 + \cos\theta\partial_3 + \sin\theta\partial_4$. Next, assuming that M **is totally geodesic**, we get that θ **is constant**. Therefore, as in Case (II.a), it suffices to consider the case where $\theta = 0$ and so, $\xi = b\partial_2 + \partial_3$ (or, equivalently, $\theta = \frac{\pi}{2}$, whence, $\xi = b\partial_2 + \partial_4$). Thus, we obtain again the totally geodesic hypersurfaces described in Case (I).

Theorem 3 (Classification of Totally Geodesic hypersurfaces)

*Let M denote a **totally geodesic hypersurface** of a Cahen-Wallach spacetime $\bar{M}$. If $\bar{M}$ is not an ε-space, then one of the following holds:*

(a) *M is an open part of the **cylindrical hypersurface** of equation $x_3 = \phi(x_1)$, where ϕ is given by* (5).

(b) *M is an open part of the **cylindrical hypersurface** of equation $x_4 = \psi(x_1)$, where ψ is given by* (6).

If $\bar{M}$ is an ε-space, then M admits an open dense subset Ω such that any point $p \in \Omega$ admits a neighbourhood as described in one of above cases (a) and (b).

Parallel hypersurfaces of *proper* Cahen-Wallach spaces

When, $\xi = b\partial_2 + \partial_3$, the second fundamental form is completely determined by

$$h(Y_2, Y_2) = -(Y_2(b) + k_3 x_3).$$

Next, $\nabla^M h = 0$ **if and only if** $Y_i(h(Y_2, Y_2)) = 0$ for all indices $i = 1, 2, 3$.

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Next, $\nabla^M h = 0$ **if and only if** $Y_i(h(Y_2, Y_2)) = 0$ for all indices $i = 1, 2, 3$.
Taking into account the symmetry conditions for h, these equations hold if and only if $Y_2(Y_2(b)) = k_3 b$, that is, with respect to the local coordinates (t, x, y) introduced in (3), $b''(y) = k_3 b$.
After a reparametrization, we then get the immersion

$$F(t, x, y) = (y, t, \phi(y) + C, x).$$

By Theorem 3, this immersion is totally geodesic if and only if $C = 0$.
The case $\xi = b\partial_2 + \partial_4$ can be treated exactly in the same way.

Theorem 4 (Parallel hypersurfaces of *proper* Cahen-Wallach spaces)

Let $F: M \to \bar{M}$ be a ***proper parallel hypersurface*** *of a Cahen-Wallach spacetime. If $\bar{M}$ is not an ε-space, then there exist local coordinates (t, x, y) on M such that up to isometries, one of the following holds:*

(a) *M is an open part of the* ***cylindrical hypersurface*** *of equation $x_3 = \phi(x_1) + C$.*

(b) *M is an open part of the* ***cylindrical hypersurface*** *of equation $x_4 = \psi(x_1) + C$.*

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Remark 1

When $g(\xi, \partial_1) = 0$, as $\phi = \psi = 0$, from Theorem 4 we get as special cases of proper parallel hypersurfaces the **hyperplanes** of equation $x_3 = C$ and $x_4 = C$, where $C \neq 0$ is a real constant.

Parallel hypersurfaces as Brinkmann manifolds

We already know from Theorem 2 that **parallel hypersurfaces** M, as described in Theorem 4, are **Brinkmann manifolds and minimal**. Moreover, being parallel hypersurfaces in a locally symmetric space, they must be locally symmetric.
In fact, with respect to local coordinates (t, x, y), the function $f(x, y)$ which appearsd in the metric g_M can be expressed by

$$f(x,y) = \begin{cases} k_4 x^2 + \left[k_3 \phi(y)^2 + 2k_3 C\phi(y) + C^2 + (\phi'(y))^2\right] & \text{in case (a),} \\ k_3 x^2 + \left[k_4 \psi(y)^2 + 2k_4 C\psi(y) + C^2 + (\psi'(y))^2\right] & \text{in case (b),} \end{cases}$$

so that they are locally symmetric Brinkmann manifolds.

Parallel hypersurfaces of ε-spaces

Parallel hypersurfaces also exist in ε-spaces, but in this case they do not provide a full classification as for general Cahen-Wallach spacetimes.
Let us assume that $b = g(\xi, \partial_1) = 0$; then the unit vector field normal to M is given by $\xi = \cos\theta\partial_3 + \sin\theta\partial_4$ and a direct calculation yields that M is **parallel** if and only if

$$k(x_3 \sin\theta - x_4 \cos\theta)\, Y_3(\theta) = 0, \quad Y_3(Y_3(\theta)) = 0. \tag{9}$$

Using the symmetry condition and the coordinates (t, x, y) already introduced, we get that $\theta = \theta(x)$ and, from (9) we have the following cases.

- *Case (1):* $\theta'(x) = 0$.
- *Case (2):* $\theta'(x) = \lambda \neq 0$.

Case (1): $\theta'(x) = 0$.

Then, θ is a real constant. By the isometry Λ of an ε-space described in (8), without loss of generality we reduce to the case where $\xi = \partial_3$ (equivalently, $\xi = \partial_4$). So, we obtain the special cases of **proper parallel hypersurfaces** listed in Theorem 4, which we already described in Remark 1.

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Case (2): $\theta'(x) = \lambda \neq 0$**.**

Integrating, we then have $\theta(x) = \lambda x + \mu$, for some real constants $\lambda \neq 0$ and μ. Moreover, from (9) we deduce

$$x_3 \sin\theta = x_4 \cos\theta.$$

Therefore, applying isometries of the ambient space, we obtain the following parametrization:

$$F(t,x,y) = \left(y, t, \frac{\cos(\lambda x + \mu)}{\lambda}, \frac{\sin(\lambda x + \mu)}{\lambda}\right),$$

which is never totally geodesic.

Theorem 5

*Let M be a **proper parallel hypersurface** of a four-dimensional $\overline{\varepsilon\text{-space}}$. Assume that the normal unit vector field ξ of M satisfies $g(\xi, \partial_1) = 0$. If the immersion F is not included in one of the cases listed in Theorem 4, then there exist local coordinates (t, x, y) on M, such that up to isometries, M is the **cylindrical hypersurface** of equation*

$$x_3^2 + x_4^2 = \frac{1}{\lambda^2}.$$

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With regard to geometric properties, we have the following.

Proposition 1

Proper parallel hypersurfaces M of an ε-space $\bar{M}$, as described in Theorem *5*, are flat and CMC.

THANK YOU
FOR LISTENING!

GRACIAS
por escuchar!

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