On the Geometry of Three-dimensional Homogeneous Lorentzian Manifolds

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INTRODUCTION

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Homogeneous spaces are a central topic of Geometry.
Because of their uniform structure, investigations of homogeneous spaces attracted the attention of many researchers.

As the geometry of a homogeneous manifold (M, g) is "the same" around each point, analytic objects on (M, g) can be investigated algebraically.

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Therefore, the first relevant case to consider is the classification of three-dimensional homogeneous pseudo-Riemannian manifolds.

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One possibility is to understand which kind of geometric properties are shared by the homogeneous pseudo-Riemannian manifolds of a given dimension.

Another is to treat the problem "algebraically", classifying all the pairs $(\mathfrak{g}, \mathfrak{h})$, formed by a Lie algebra $\mathfrak{g}$ and an isotropy subalgebra $\mathfrak{h} \subset \mathfrak{so}(p, q)$, such that $\dim(\mathfrak{g}/\mathfrak{h}) = p + q = \dim M$.

Theorem [Sekigawa, 1977]

A 3D (connected, simply connected, complete) homogeneous Riemannian manifold is either symmetric or isometric to a 3D Riemannian Lie group (a Lie group equipped with a left-invariant Riemannian metric).

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Dimension four is already a completely different story...due to the existence of non-reductive homogeneous pseudo-Riemannian manifolds [Fels-Renner, 2006].

3D symmetric spaces

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- 3D Cahen-Wallach spaces (symmetric spaces admitting a parallel lightlike vector field).

Unimodular Lie groups

Cross product

Let $\mathfrak{g}$ denote a 3D Lie algebra, equipped with an either Riemannian or Lorentzian scalar product $\langle , \rangle : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$. The cross product

$$\times : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

is defined by condition

$$\langle e_i \times e_j, e_k \rangle = \det(e_i, e_j, e_k),$$

where $\{e_1, e_2, e_3\}$ denotes an orthonormal basis of the Lie algebra $\mathfrak{g}$.

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Structure operator

The bracket product operation in $\mathfrak{g}$ is related to the cross product operation by

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where $L : \mathfrak{g} \to \mathfrak{g}$ is a uniquely defined linear map.
G is unimodular if and only if L is self-adjoint.

Unimodular Riemannian Lie groups

If the scalar product $\langle , \rangle$ is Riemannian, L being self-adjoint, $\mathfrak{g}$ admits an orthonormal basis $\{e_l, e_2, e_3\}$ of eigenvectors for L and so,

$$[e_2, e_3] = \lambda_1 e_1, \quad [e_3, e_1] = \lambda_2 e_2, \quad [e_1, e_2] = \lambda_3 e_3.$$

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Non-isometric unimodular Riemannian Lie groups

G	λ_1	λ_2	λ_3
$SU(2)$	$+$	$+$	$+$
$\widetilde{SL}(2,\mathbb{R})$	$+$	$+$	$-$
$\tilde{E}(2)$	$+$	$+$	0
$E(1,1)$	$+$	$-$	0
H_3	$+$	0	0
$\mathbb{R}^3$	0	0	0

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Ia L is diagonalizable:
$[e_1, e_2] = -\lambda_3 e_3, \quad [e_1, e_3] = -\lambda_2 e_2, \quad [e_2, e_3] = \lambda_1 e_1.$

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Ib L admits complex eigenvalues:
$[e_1, e_2] = -\beta e_2 - \alpha e_3, \quad [e_1, e_3] = -\alpha e_2 + \beta e_3, \quad [e_2, e_3] = \lambda e_1,$
$(\beta \neq 0)$.

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$(\beta \neq 0).$

II L has a double root in its minimal polynomial:
$[u_1, u_2] = \lambda_2 u_3, \quad [u_1, u_3] = -\lambda_1 u_1 - \varepsilon u_2, \quad [u_2, u_3] = \lambda_1 u_2,$
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$(\varepsilon^2 = 1)$.

III L has a triple root in its minimal polynomial:
$[u_1, u_2] = u_1 + \lambda u_3, \quad [u_1, u_3] = -\lambda u_1, \quad [u_2, u_3] = \lambda u_2 + u_3.$

In the cases above, $\{e_i\}$ is an orthonormal basis with e_3 timelike and $\{u_i\}$ is a pseudo-orthonormal basis with $\langle u_1, u_2 \rangle = \langle u_3, u_3 \rangle = 1$.

Unimodular Lorentzian Lie groups

With Lie algebra *Ia*

Lie group	λ_1	λ_2	λ_3
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$SU(2)$	$+$	$+$	$-$
$\widetilde{E}(2)$	$+$	$+$	0
$\widetilde{E}(2)$	$+$	0	$-$
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$E(1,1)$	$+$	0	$+$
H_3	$+$	0	0
H_3	0	0	$-$
$\mathbb{R}^3$	0	0	0

Unimodular Lorentzian Lie groups

With Lie algebra either *Ib* or *III*

Lie group	λ
$\widetilde{SL}(2,\mathbb{R})$	$\neq 0$
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Unimodular Lorentzian Lie groups

With Lie algebra either *Ib* or *III*

Lie group	λ
$\widetilde{SL}(2,\mathbb{R})$	$\neq 0$
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With Lie algebra *II*

Lie group	λ_1	λ_2
$\widetilde{SL}(2,\mathbb{R})$	$\neq 0$	$\neq 0$
$E(1,1)$	0	$\neq 0$
$E(2)$	$\neq 0$	0
H_3	0	0

Non-unimodular Riemannian Lie groups

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Non-unimodular Riemannian Lie groups

If L is not self-adjoint, then the *unimodular kernel* $\mathfrak{u}$ of $\mathfrak{g}$ is two-dimensional.

When the Lie algebra $\mathfrak{g}$ of a 3D non-unimodular Lie group is Riemannian, so is $\mathfrak{u}$, and $\mathfrak{g}$ admits an orthonormal basis $\{e_l, e_2, e_3\}$ such that

$$[e_1, e_2] = \alpha e_2 + \beta e_3, \quad [e_1, e_3] = \gamma e_2 + \delta e_3, \quad [e_2, e_3] = 0,$$

with $\alpha + \delta \neq 0$ and $\alpha\gamma + \beta\delta = 0$

(the latter condition follows from the possibility of choosing an orthonormal basis $\{e_i\}$ with $\mathrm{ad}_{e_1} e_2 \perp \mathrm{ad}_{e_1} e_3$).

Non-unimodular Lorentzian Lie groups

If L is not self-adjoint, being $\mathfrak{g}$ Lorentzian, the 2D *unimodular kernel* $\mathfrak{u}$ of $\mathfrak{g}$ is either **Lorentzian, Riemannian or degenerate**.

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Non-unimodular Lorentzian 3D algebras

IV.1 $\mathfrak{u}$ Lorentzian: $\alpha + \delta \neq 0$,

$$[e_1, e_2] = 0, \quad [e_1, e_3] = \alpha e_1 + \beta e_2, \quad [e_2, e_3] = \gamma e_1 + \delta e_2$$

($\{e_i\}$ orthonormal basis with e_1 timelike).

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IV.3 $\mathfrak{u}$ degenerate: $\alpha+\delta \neq 0$,

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($\{u_i\}$ pseudo-orthonormal with $\langle u_1,u_1\rangle=\langle u_2,u_3\rangle=1$).

Riemannian vs Lorentzian classification

3D homogeneous Riemannian manifolds

- 3 space forms;
- 2 direct products;
- 1 standard form for the Riemannian unimodular Lie algebras,
- and 1 for the non-unimodular ones.

[Milnor, 1976]

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3D homogeneous Lorentzian manifolds

- 3 space forms;
- 4 direct products;
- Cahen-Wallach spaces;
- 4 standard forms for the Lorentzian unimodular Lie algebras,
- and 3 for the non-unimodular ones.

[Rahmani, 1992] [Cordero-Parker, 1997], [Garcia-Rio et al., 2023].

Homogeneous geodesics

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Definition

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Definition

A geodesic γ through $o \in M = G/H$ is called homogeneous if it is the orbit of a one-parameter subgroup.

In general, the group G is not unique.
If γ is homogeneous with respect to some isometry group G, then it is also homogeneous with respect to the maximal connected group of isometries of (M, g), but not conversely.

Homogeneous geodesics

A homogeneous pseudo-Riemannian manifold (M, g) is said to be reductive if $M = G/H$ and the Lie algebra $\mathfrak{g}$ of G can be decomposed into a direct sum $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, where $\mathfrak{m}$ is an $\mathrm{Ad}(H)$-invariant subspace.

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Homogeneous geodesics in a reductive homogeneous pseudo-Riemannian manifold are characterized by the Geodesic Lemma [Dušek and Kowalski, 2006].

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{symmetric spaces} $\subset$ {naturally reductive spaces} $\subset$ {g.o.spaces} (inclusions are strict).

h.g. of 3D Lorentzian Lie groups

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In particular: for type *Ia* with distinct eigenvalues, if neither $\lambda_1 < \lambda_3 < \lambda_2$ nor $\lambda_2 < \lambda_3 < \lambda_1$ hold, *there are not lightlike h.g.* (FIRST EXAMPLE).

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The cases where all geodesics are homogeneous were unimodular type *Ia* with two of the λ_i's coinciding (just like in the Riemannian case).

3D naturally reductive spaces

Theorem [Tricerri-Vanhecke, 1983], [GC-Marinosci, 2008]

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Corollary

3D connected, simply connected pseudo-Riemannian non-symmetric naturally reductive spaces: $\widetilde{SL}(2, \mathbb{R})$, $SU(2)$ and H_3, equipped with some suitable left-invariant metrics.

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Manifolds in both classes $\mathcal{A}$ and $\mathcal{B}$ have constant scalar curvature. Moreover, $\mathcal{E} \subset \mathcal{P} = \mathcal{A} \cap \mathcal{B}$, where $\mathcal{E}$ and $\mathcal{P}$ are Einstein and Ricci-parallel manifolds respectively.

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In particular, conformally flat homogeneous Lorentzian three-manifolds need not be locally symmetric.

Definition

Pseudo-Riemannian Ricci Soliton:
a pseudo-Riemannian manifold (M, g) with a vector field $X \in \mathcal{X}(M)$, such that

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A Ricci soliton is said to be either expanding, steady o shrinking, depending on whether $\lambda < 0$, $\lambda = 0$ or $\lambda > 0$.

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It is then natural to ask what happens in the 3D Lorentzian case.

Unimodular case

Theorem [Brozos-Vazquez, GC, Garcia-Rio and Gavino-Fernandez, 2012]

Let G denote a 3D unimodular Lorentzian Lie group. Then, G admits a left-invariant Ricci soliton in the following cases:

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1) $\mathfrak{g}$ is of type II:

$$[e_1, e_2] = \frac{1}{2}e_2 - (\beta - \frac{1}{2})e_3, \; [e_1, e_3] = -(\beta + \frac{1}{2})e_2 - \frac{1}{2}e_3, \; [e_2, e_3] = \alpha e_1$$

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- $\alpha = \beta \neq 0$ ($G = \widetilde{SL}(2,\mathbb{R})$): then, $X = -\frac{1}{2}\beta e_1 + \delta(e_2 + e_3)$ is a spacelike expanding Ricci soliton.

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2) $\mathfrak{g}$ is of type *III*:

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Theorem [Brozos-Vazquez, GC, Garcia-Rio and Gavino-Fernandez, 2012, 2023]

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- $[u_1, u_2] = 0, \quad [u_1, u_3] = \alpha u_1 + \beta u_2, \quad [u_2, u_3] = (2 - \alpha) u_2 \quad (\alpha \neq 0)$,

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where $\{u_i\}$ is a pseudo-orthonormal basis with $\langle u_1, u_1\rangle = 1$. Then, $X = \frac{\alpha(1-\alpha)}{\alpha-2}u_2$ if $\alpha \neq \frac{2}{3}$ and $X = \frac{3}{2}\beta u_1 + (\frac{1}{6} - \frac{9}{8}\beta^2)u_2 + u_3$ if $\alpha = \frac{2}{3}$, is a Ricci soliton (steady, expanding or shrinking, depending on the structure constants and the components of X).

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A *homogeneous (pseudo-Riemannian) structure (h.s.)* on (M, g) is a tensor field T of type $(1, 2)$ on M, such that the connection $\tilde{\nabla} = \nabla - T$ satisfies

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More details and recent developments on h.s. in pseudo-Riemannian settings:

G. Calvaruso and M. Castrillon-Lopez, *Pseudo-Riemannian Homogeneous Structures*, Developments in Math., Springer, 2019.

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We take the space of tensors $\mathcal{S}(V)\subset\otimes^3V^*$ with the same symmetries as T:

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$\mathcal{S}(V)$ is isomorphic to $V^* \otimes \wedge^2 V^*$ and carries a non-degenerate symmetric bilinear form, defined by

$$\langle T, T' \rangle = \sum_{i,j,k=1}^{m} \varepsilon^i \varepsilon^j \varepsilon^k T(e_i, e_j, e_k) T'(e_i, e_j, e_k),$$

where $\{e_1, \ldots, e_m\}$ is any orthonormal basis of $(V, \langle\, , \rangle)$ and $\varepsilon^i = \langle e_i, e_i \rangle$.

Whenever $\dim M \geq 3$, the space $\mathcal{S}(V)$ decomposes into irreducible and mutually orthogonal $O(p,q)$-submodules as

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with $\sigma_{X,Y,Z}$ denoting the cyclic sum with respect to X, Y, Z.

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  - In Lorentzian settings, if T is nondegenerate ($\iff \xi$ is not lightlike), then M has constant sectional curvature $K = -g(\xi,\xi) \neq 0$.
    If T is degenerate (that is, $g(\xi,\xi) = 0$), then M is a *singular scale-invariant plane wave.*

We now focus on the classification of h.s. on 3D Lie groups equipped with a left-invariant pseudo-Riemannian metric.

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$(M = G, g)$ admits a *canonical homogeneous structure* T^{∇}, defined, for all $X, Y, Z \in \mathfrak{g}$, by

$$2g(T^{\nabla}(X, Y), Z) = g([X, Y], Z) - g([Y, Z], Z) + g([Z, X], Y).$$

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The corresponding Ambrose-Singer connection $\tilde{\nabla} = \nabla - T^{\nabla}$ is determined by $\tilde{\nabla}_X Y = 0$ for all $X, Y \in \mathfrak{g}$, leading to the canonical description $G = G/\{e\}$ of G as homogeneous space.

3D Riemannian h.s.: unimodular case

Theorem [Calviño-Louzao, Ferreiro-Subrido, Garcia-Rio and Vazquez-Lorenzo, 2023]

$[e_2, e_3] = \lambda_1 e_1$, $[e_3, e_1] = \lambda_2 e_2$, $[e_1, e_2] = \lambda_3 e_3$, $\{e_i\}$ orthonormal.

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When (G, g) is not symmetric, there are two mutually excluding cases:

(i) The λ_i's are all distinct. Then, the only h.s. is the canonical one. $T^\nabla \in \mathcal{S}_2$ if $\lambda_1 + \lambda_2 + \lambda_3 = 0$, otherwise $T^\nabla \in \mathcal{S}_2 \oplus \mathcal{S}_3$.

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- **(ii)** $\lambda_1 - \lambda_2 = 0 \neq \lambda_3$.
  Then the exists a one-parameter family of h.s.

$$T_k = \lambda_3 e^1 \otimes e^2 \wedge e^3 - \lambda_3 e^2 \otimes e^1 \wedge e^3 + 2\kappa e^3 \otimes e^1 \wedge e^2, \quad \kappa \in \mathbb{R}.$$

$T_k \in \mathcal{S}_2$ for $\kappa = -\lambda_3$ and $T_k \in \mathcal{S}_3$ for $\kappa = \frac{1}{2}\lambda_3$, otherwise $T_k \in \mathcal{S}_2 \oplus \mathcal{S}_3$.

3D Riemannian h.s.: non-unimodular case

Theorem [Calviño-Louzao, Ferreiro-Subrido, Garcia-Rio and Vazquez-Lorenzo, 2023]

$[e_1, e_2] = \alpha e_2 + \beta e_3,\ [e_1, e_3] = \gamma e_2 + \delta e_3,\ [e_2, e_3] = 0,\ \alpha + \delta \neq 0.$

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(i) $\alpha\delta \neq 0$, $\beta = -\frac{\alpha\gamma}{\delta}$: only the canonical homogeneous structure. $T^{\nabla} \in \mathcal{S}_1 \oplus \mathcal{S}_2$ if $\gamma = 0$, otherwise T^{∇} is generic.

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Consequences

A non-symmetric 3D Riemannian Lie group: admits a non-canonical h.s.

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A non-symmetric 3D Riemannian Lie group: admits a non-canonical h.s. $\Leftrightarrow$ its isometry group has dimension four

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Consequences

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$\Leftrightarrow$ it admits $T \in \mathcal{S}_3$ (naturally reductive) $\Rightarrow$ it admits $T \in \mathcal{S}_2$.

3D Lorentzian h.s.: unimodular case Ia

Theorem [GC-Zaeim, 2024]

$[e_1, e_2] = -\lambda_3 e_3$, $[e_1, e_3] = -\lambda_2 e_2$, $[e_2, e_3] = \lambda_1 e_1$, e_3 timelike.

3D Lorentzian h.s.: unimodular case Ia

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(i) The λ_i's are all distinct. Then, the only homogeneous Lorentzian structure is the canonical one T^∇.
$T^\nabla \in \mathcal{S}_2$ if $\lambda_1 + \lambda_2 + \lambda_3 = 0$, otherwise $T^\nabla \in \mathcal{S}_2 \oplus \mathcal{S}_3$.

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(ii) $\lambda_1 - \lambda_2 = 0 \neq \lambda_3$. Then, there exists a one-parameter family of h.s.

$$T_k = \lambda_3 e^1 \otimes e^2 \wedge e^3 - \lambda_3 e^2 \otimes e^1 \wedge e^3 + 2\kappa e^3 \otimes e^1 \wedge e^2, \ \kappa \in \mathbb{R}.$$

$T_k \in \mathcal{S}_2$ for $\kappa = -\lambda_3$ and $T_k \in \mathcal{S}_3$ for $\kappa = \frac{1}{2}\lambda_3$, otherwise $T_k \in \mathcal{S}_2 \oplus \mathcal{S}_3$.

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$T_k \in \mathcal{S}_2$ for $\kappa = -\lambda_3$ and $T_k \in \mathcal{S}_3$ for $\kappa = \frac{1}{2}\lambda_3$, otherwise $T_k \in \mathcal{S}_2 \oplus \mathcal{S}_3$.

(iii) $\lambda_1 - \lambda_3 = 0 \neq \lambda_2$. Then, the exists a one-parameter family of h.s.

$$T_k = \lambda_2 e^1 \otimes e^2 \wedge e^3 + 2\kappa e^2 \otimes e^1 \wedge e^3 + \lambda_2 e^3 \otimes e^1 \wedge e^2, \quad \kappa \in \mathbb{R}.$$

$T_k \in \mathcal{S}_2$ when $\kappa = -\lambda_2$ and $T_k \in \mathcal{S}_3$ if $\kappa = \frac{\lambda_2}{2}$, otherwise $T_k \in \mathcal{S}_2 \oplus \mathcal{S}_3$.

3D Lorentzian h.s.: unimodular case Ib

Theorem [GC-Zaeim, 2024]

$[e_1, e_2] = -\beta e_2 - \alpha e_3$, $[e_1, e_3] = -\alpha e_2 + \beta e_3$, $[e_2, e_3] = \lambda e$, $\beta \neq 0$, e_3 timelike.

3D Lorentzian h.s.: unimodular case Ib

Theorem [GC-Zaeim, 2024]

$[e_1, e_2] = -\beta e_2 - \alpha e_3$, $[e_1, e_3] = -\alpha e_2 + \beta e_3$, $[e_2, e_3] = \lambda e_,$ $\beta \neq 0$, e_3 timelike.
(G, g), if not symmetric, only admits the canonical homogeneous structure.

$T^{\nabla} \in \mathcal{S}_2$ when $\lambda + 2\alpha = 0$, otherwise $T^{\nabla} \in \mathcal{S}_2 \oplus \mathcal{S}_3$.

3D Lorentzian h.s.: unimodular case II

Theorem [GC-Zaeim, 2024]

$[u_1, u_2] = \lambda_2 u_3$, $[u_1, u_3] = -\lambda_1 u_1 - \varepsilon u_2$, $[u_2, u_3] = \lambda_1 u_2$
$(\varepsilon^2 = 1,\ \langle u_1, u_2\rangle = \langle u_3, u_3\rangle = 1)$.

3D Lorentzian h.s.: unimodular case II

Theorem [GC-Zaeim, 2024]

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$(\varepsilon^2 = 1, \langle u_1, u_2 \rangle = \langle u_3, u_3 \rangle = 1)$. (G, g) admits the following h.s.:

(i) The canonical h.s. T^∇, of type $\mathcal{S}_2$ when $2\lambda_1 + \lambda_2 = 0$, otherwise $T^\nabla \in \mathcal{S}_2 \oplus \mathcal{S}_3$.

3D Lorentzian h.s.: unimodular case II

Theorem [GC-Zaeim, 2024]

$[u_1, u_2] = \lambda_2 u_3$, $[u_1, u_3] = -\lambda_1 u_1 - \varepsilon u_2$, $[u_2, u_3] = \lambda_1 u_2$
($\varepsilon^2 = 1$, $\langle u_1, u_2\rangle = \langle u_3, u_3\rangle = 1$). (G, g) admits the following h.s.:

(i) The canonical h.s. T^∇, of type $\mathcal{S}_2$ when $2\lambda_1 + \lambda_2 = 0$, otherwise $T^\nabla \in \mathcal{S}_2 \oplus \mathcal{S}_3$.

(ii) When $\lambda_2 = \lambda_1 \neq 0$:

$$\begin{aligned} T_k &= \lambda_1 u^3 \otimes u^1 \wedge u^2 + 2\kappa u^1 \otimes u^1 \wedge u^3 - \lambda_1 u^2 \otimes u^1 \wedge u^3 \\ &\quad + \lambda_1 u^1 \otimes u^2 \wedge u^3, \ \kappa \in \mathbb{R}. \end{aligned}$$

$T_k \in \mathcal{S}_3$ when $\kappa = 0$, otherwise $T_k \in \mathcal{S}_2 \oplus \mathcal{S}_3$.

3D Lorentzian h.s.: unimodular case III

Theorem [GC-Zaeim, 2024]

$[u_1, u_2] = u_1 + \lambda u_3, \; [u_1, u_3] = -\lambda u_1, \; [u_2, u_3] = \lambda u_2 + u_3.$

3D Lorentzian h.s.: unimodular case III

Theorem [GC-Zaeim, 2024]

$[u_1, u_2] = u_1 + \lambda u_3$, $[u_1, u_3] = -\lambda u_1$, $[u_2, u_3] = \lambda u_2 + u_3$. (G, g) admits the following h.s.:

(i) The canonical h.s., $T^{\nabla} \in \mathcal{S}_2$ when $\lambda = 0$, otherwise $T^{\nabla} \in \mathcal{S}_2 \oplus \mathcal{S}_3$.

3D Lorentzian h.s.: unimodular case III

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(i) The canonical h.s., $T^\nabla \in \mathcal{S}_2$ when $\lambda = 0$, otherwise $T^\nabla \in \mathcal{S}_2 \oplus \mathcal{S}_3$.

(ii) When $\lambda = 0$:

$$T_k = -2u^2 \otimes u^1 \wedge u^2 + 2\kappa u^2 \otimes u^2 \wedge u^3 + 2u^3 \otimes u^2 \wedge u^3, \quad \kappa \in \mathbb{R}.$$

$T_k \in \mathcal{S}_1$ when $\kappa = 0$, otherwise $T_k \in \mathcal{S}_1 \oplus \mathcal{S}_2$.

3D Lorentzian h.s.: non-unimodular case IV.1 (Lorentzian kernel)

Theorem [GC-Zaeim, 2024]

$[e_1, e_2] = 0$, $[e_1, e_3] = \alpha e_1 + \beta e_2$, $[e_2, e_3] = \gamma e_1 + \delta e_2$, $\alpha + \delta \neq 0$, e_3 timelike.

3D Lorentzian h.s.: non-unimodular case IV.1 (Lorentzian kernel)

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(G, g), if not symmetric, admits the following h.s.:

(i) The canonical h.s. $T^{\nabla} \in \mathcal{S}_1 \oplus \mathcal{S}_2$ if $\gamma + \beta = 0$, otherwise it is generic.

3D Lorentzian h.s.: non-unimodular case IV.1 (Lorentzian kernel)

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(G, g), if not symmetric, admits the following h.s.:

- **(i)** The canonical h.s. $T^{\nabla} \in \mathcal{S}_1 \oplus \mathcal{S}_2$ if $\gamma + \beta = 0$, otherwise it is generic.
- **(ii)** If $\beta = \frac{\alpha\delta}{\gamma}, \gamma \neq 0$: a one-parameter family of h.s. T_k. In general $T_k \in \mathcal{S}_2 \oplus \mathcal{S}_3$, but $T_k \in \mathcal{S}_2$ and $T_k \in \mathcal{S}_3$ for special values of the structure constants.

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(iii) If $\delta = \gamma = 0 \neq \beta$:

$$T_k = -\beta e^1 \otimes e^2 \wedge e^3 + 2\kappa e^2 \otimes e^1 \wedge e^3 - \beta e^3 \otimes e^1 \wedge e^2, \ \ \kappa \in \mathbb{R}.$$

$T_k \in \mathcal{S}_2$ when $\kappa = -\beta$ and $T_k \in \mathcal{S}_3$ when $\kappa = \frac{\beta}{2}$, otherwise $T_k \in \mathcal{S}_2 \oplus \mathcal{S}_3$.

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(G, g), if not symmetric, admits the following h.s.:

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$$T_k = -\beta e^1 \otimes e^2 \wedge e^3 + 2\kappa e^2 \otimes e^1 \wedge e^3 - \beta e^3 \otimes e^1 \wedge e^2, \;\; \kappa \in \mathbb{R}.$$

$T_k \in \mathcal{S}_2$ when $\kappa = -\beta$ and $T_k \in \mathcal{S}_3$ when $\kappa = \frac{\beta}{2}$, otherwise $T_k \in \mathcal{S}_2 \oplus \mathcal{S}_3$.

(iv) If $\alpha = \gamma = 0 \neq \beta\delta$: a one-parameter family of h.s. T_k.
In general $T_k \in \mathcal{S}_2 \oplus \mathcal{S}_3$, but $T_k \in \mathcal{S}_2$ and $T_k \in \mathcal{S}_3$ for special values of the structure constants.

3D Lorentzian h.s.: non-unimodular case IV.2 (Riemannian kernel)

Theorem [GC-Zaeim, 2024]

$[e_1, e_2] = 0$, $[e_1, e_3] = \alpha e_1 + \beta e_2$, $[e_2, e_3] = \gamma e_1 + \delta e_2$, $\alpha + \delta \neq 0$, $\alpha\gamma + \beta\delta = 0$, e_1 timelike.

3D Lorentzian h.s.: non-unimodular case IV.2 (Riemannian kernel)

Theorem [GC-Zaeim, 2024]

$[e_1, e_2] = 0$, $[e_1, e_3] = \alpha e_1 + \beta e_2$, $[e_2, e_3] = \gamma e_1 + \delta e_2$, $\alpha + \delta \neq 0$, $\alpha\gamma + \beta\delta = 0$, e_1 timelike. (G, g), if not symmetric, admits the following h.s.:

(i) The canonical h.s. T^{∇} for $\alpha = -\frac{\beta\delta}{\gamma}, \beta\gamma \neq 0$. T^{∇} is generic.

(ii) The canonical h.s. T^{∇} for $\beta = \gamma = 0$. $T^{\nabla} \in \mathcal{S}_1 \oplus \mathcal{S}_2$.

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$$T_k = 2\kappa e^1 \otimes e^2 \wedge e^3 - \gamma e^2 \otimes e^1 \wedge e^3 + \gamma e^3 \otimes e^1 \wedge e^2, \quad \kappa \in \mathbb{R}.$$

$T_k \in \mathcal{S}_2$ if $\kappa = -\gamma$ and $T_k \in \mathcal{S}_3$ if $\kappa = \frac{\gamma}{2}$, otherwise $T_k \in \mathcal{S}_2 \oplus \mathcal{S}_3$.

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$$T_k = -\beta e^1 \otimes e^2 \wedge e^3 + 2\kappa e^2 \otimes e^1 \wedge e^3 - \beta e^3 \otimes e^1 \wedge e^2, \quad \kappa \in \mathbb{R}.$$

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3D Lorentzian h.s.: non-unimodular case IV.3 (Degenerate kernel)

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(ii) If $\beta=\frac{\alpha\delta}{\gamma}$, $\gamma\neq 0$:

$$\begin{aligned}T_k=&-\frac{\alpha}{\gamma}(\gamma-2\kappa)(u^1\otimes u^1\wedge u^3-u^3\otimes u^2\wedge u^3)+2\kappa u^1\otimes u^2\wedge u^3\\&-\gamma(u^2\otimes u^1\wedge u^3-u^3\otimes u^1\wedge u^2)+\frac{\alpha^2}{\gamma^2}(\gamma-2\kappa)u^3\otimes u^1\wedge u^3,\quad \kappa\in\mathbb{R}.\end{aligned}$$

$T_k\in\mathcal{S}_2$ when $\kappa=-\gamma$ and $T_k\in\mathcal{S}_3$ when $\kappa=\frac{\gamma}{2}$, otherwise $T\in\mathcal{S}_2\oplus\mathcal{S}_3$.

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(iii) If $\gamma = 0$:

$$T_k = -2\delta(u^1 \otimes u^1 \wedge u^3 + u^3 \otimes u^2 \wedge u^3) + 2\kappa u^3 \otimes u^1 \wedge u^3, \quad \kappa \in \mathbb{R}.$$

$T_k \in \mathcal{S}_1$ if $\kappa = 0$ and $T_k \in \mathcal{S}_2$ if $\delta = 0$, otherwise $T_k \in \mathcal{S}_1 \oplus \mathcal{S}_2$.

Consequences

Differently from the Riemannian case, there exist 3D homogeneous Lorentzian manifolds admitting a h.s. of type $\mathcal{S}_3$ but none of type $\mathcal{S}_2$ (in a special case of unimodular type *II*).

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- These special cases also admit a h.s. of degenerate type $\mathcal{S}_1$, so that they are singular scale-invariant plane waves.

¡Gracias por su atención!

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And to Eduardo: ¡Feliz cumpleaños!