Torsion-free connections with prescribed curvature

Efraín Basurto-Arzate

Fakultät für Mathematik

technische universität dortmund

Symmetry and Shape, September 2024

Outline

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Curvature of torsion-free connections

Basic background

- Given a connection on a smooth, connected manifold M, its *curvature tensor* is defined as the tensor field $R^\nabla \in \Gamma(\bigwedge^2 T^*M \otimes \mathrm{End}(TM))$ defined as

$$R^\nabla(X,Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

- If ∇ is *torsion-free* (i.e. it satisfies $\nabla_X Y - \nabla_Y X = [X,Y]$), R^∇ satisfies the *Bianchi identities:*

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1. $$\sum_{\mathrm{cyc}\,(X,Y,Z)} R^{\nabla}(X,Y)Z = 0 \in \mathfrak{X}(M),$$

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Algebraic curvature tensors

The Bianchi identities can be algebraically encoded:

Let V a finite real vector space. Let $\mathfrak{h} \subset \mathfrak{g} := \mathrm{End}(V)$ be a Lie subalgebra. The *space of algebraic curvature tensors* is defined as the subspace

$$K(\mathfrak{h}) := \left\{ R \in \bigwedge^2 V^* \otimes \mathfrak{h} \;\middle|\; \sum_{\mathrm{cyc}\,(x,y,z)} R(x,y)z = 0 \in V \; \forall x, y, z \in V \right\}$$

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$$K^1(\mathfrak{h}) := \left\{ \phi \in V^* \otimes K(\mathfrak{h}) \;\middle|\; \sum_{\mathrm{cyc}\,(x,y,z)} \phi(x)(y,z) = 0 \in \mathfrak{h} \; \forall x, y, z \in V \right\}.$$

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Algebraic curvature tensors

A consequence of the Ambrose-Singer Holonomy Theorem is the fact that the curvature tensor R^{∇} satisfies, for every $x \in M$:

$$
\begin{aligned}
R^{\nabla}_x &\in K(\mathfrak{hol}_x(\nabla)), \\
(\nabla R^{\nabla})_x &\in K^1(\mathfrak{hol}_x(\nabla)).
\end{aligned}
$$

The problem

These observations naturally lead to the following question:

Under which conditions can it be guaranteed, that a given curvature map $S : U \subset V \longrightarrow K(\mathfrak{g})$ can be induced by the curvature tensor of a torsion-free connection?

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Local problem

By choosing a normal coordinate system around a fixed p on the manifold M we can restrict ourselves, without loss of generality, to the case in which $M = U$, where U denotes an open subset of V, which is also star-shaped around 0 such that the exponential map $\exp_0 \colon U \subset V \cong T_0U \longrightarrow U$ associated to a torsion-free connection ∇ on $TU = U \times V$ is the identity.

The principal bundle setting

- Let $F(U) = U \times \overbrace{\mathrm{GL}(V)}^{=:G}$ denote the frame bundle of U.
- Let $\theta \in \Omega^1(F(U), V)$ denote the tautological form of $F(U)$.
- $\omega^\nabla \in \Omega^1(F(U), \mathfrak{g})$ denotes the connection form associated to ∇.
- $F^\nabla := \mathrm{d}\omega^\nabla + \omega^\nabla \wedge \omega^\nabla$ denotes the curvature form.
- In this setting, the fact that ∇ is torsion-free is equivalent to

$$\mathrm{d}\theta + \omega^\nabla \wedge \theta = 0.$$

- The Bianchi identities can be restated as follows:

$$F^\nabla \wedge \theta = 0,$$
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- Let $R \in C^\infty(F(U), K(\mathfrak{g}))^G$ be the unique G-equivariant map associated to R^∇ that satisfies

$$F^\nabla = R(\theta, \theta).$$

- Parallel translation along radial geodesics defines a smooth section of $F(U)$. Concretely, the map $h\colon U \longrightarrow G$ defined by $h(v) = P_{\gamma_v}$ is a smooth map, and so $\sigma := (v, h) \in \Gamma(F(U))$.
- Along this section we obtain:

$$\begin{aligned}\hat{\theta} :=& \sigma^*\theta = h^{-1}\mathrm{d}v,\\ \hat{\omega} :=& \sigma^*\omega^\nabla = \mathrm{Ad}(h^{-1}) \circ \Gamma + h^*\mu_G,\\ \hat{F} :=& \sigma^* F^\nabla = \mathrm{Ad}(h^{-1}) \circ R^\nabla.\end{aligned}$$

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From the relation $F^\nabla = R(\theta, \theta)$, we thus obtain:

$$\hat{F} = \hat{R}(\hat{\theta}, \hat{\theta}), \tag{1}$$

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An initial value problem

Let $W \subset \bigotimes^m V^* \otimes \mathfrak{g}$ be a subspace.

Let $\mathcal{L}_{\mathscr{E}} \colon \Omega^k(U, W) \longrightarrow \Omega^k(U, W)$ denote the Lie derivative along the Euler vector field $\mathscr{E} := e_i e^i$.

Proposition

There exists an integration map $I \colon \Omega^k(U, W) \longrightarrow \Omega^k(U, W)$ *such that for* $\eta \in \Omega^k(U, W)$*:*

$$I \mathcal{L}_{\mathscr{E}} \eta = \mathcal{L}_{\mathscr{E}} I \eta = \begin{cases} \eta & k \geq 1, \\ \eta - \eta_0 & k = 0. \end{cases} \tag{2}$$

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The integration map I gives us the following characterization for the pulled-back forms $\hat{\theta}$, $\hat{\omega}$:

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The main consequence of Proposition 1.2 is the following:

Proposition

The pulled-back form $\hat{\theta}$ is a solution to the singular initial value problem

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Main Theorem

The key observation is the fact that solutions to the singular initial value problem (3), for suitable curvature maps $U \longrightarrow K(\mathfrak{g})$, are decisive for the existence of torsion-free connections.

We break down the Main Result in a series of 3 principal statements:

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Proposition

Let $S\colon U \longrightarrow K(\mathfrak{g})$ *be a real analytic map. Then, the singular initial value problem*

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admits a unique real analytic solution.

Main Theorem

Proposition

Let $S\colon U \longrightarrow K(\mathfrak{g})$ *be a real analytic map. Let* θ *be the real analytic solution to the singular initial value problem* (4)*. Set* $\omega = I(S(\mathscr{E}, \theta))$*. It then holds:*

i) $\theta = \mathrm{d}v + I(\omega \cdot \mathscr{E})$,

ii) $\mathcal{L}_{\mathscr{E}}(\mathrm{d}\theta + \omega \wedge \theta) = (\mathrm{d}\omega + \omega \wedge \omega - S(\theta, \theta)) \cdot \mathscr{E}$.

Main Theorem

Theorem

Let $U \subset V$ a star-shaped around 0 open subset, let $S\colon U \longrightarrow K(\mathfrak{g})$ be a real-analytic map, θ the analytic solution to the singular initial value problem (4), and suppose $\omega := I(S(\mathscr{E}, \theta))$ satisfies the consistency relation $\mathrm{d}\omega + \omega \wedge \omega = S(\theta, \theta)$. Then there exists a unique torsion-free analytic connection ∇ on a sufficiently small open neighborhood $U' \subset U$ of 0 such that for all $v \in U'$:

$$S_v = P_{\gamma_v}^{-1} \cdot R^{\nabla}_{\gamma_v(1)}, \tag{5}$$

Outline

Sketch of the Proof

Thanks to the auxiliary results preceeding the Main Theorem, we can provide a short sketch of its Proof:

- Let $g\colon U' \longrightarrow G$ be the map such that $\theta = g\mathrm{d}v$, and let $\Gamma \in \Omega^1(U', \mathfrak{g})$ be defined by:

$$\Gamma := \mathrm{Ad}(g^{-1}) \circ \omega + g^{-1}\mathrm{d}g.$$

- This $\mathfrak{g}$-valued 1-Form satisfies $\Gamma \wedge \mathrm{d}v = 0$ (ii) in Proposition 2.2). This implies that, for any i, j, k,

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- Γ thus defines a torsion-free connection on U' by the formula

$$\nabla_X s \coloneqq \mathrm{d}s(X) + \Gamma(X)s.$$

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That is:

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Corollary

In the context of Theorem 1, one notices that

$$(\mathrm{d}S + \rho_*(\omega) \wedge S)(g^{-1}) = g \cdot \nabla R^{\nabla}.$$

In other words: The map $(\mathrm{d}S + \rho_*(\omega) \wedge S)(g^{-1})\colon U' \longrightarrow V^* \otimes K(\mathfrak{g})$ actually takes values in the subspace $K^1(\mathfrak{g})$.

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Holonomy of torsion-free connections

Theorem 1 is well-suited for the study of holonomy theory. Indeed one has the following:

Theorem

Let $S\colon U \longrightarrow K(\mathfrak{g})$, $\theta \in \Omega^1(U, V)$, $\omega \in \Omega^1(U, \mathfrak{g})$ as in Theorem 1. Let ∇ be the torsion-free connection on TU' which satisfies $S_v = P_{\gamma_v}^{-1} \cdot R^{\nabla}_{\gamma_v(1)}$ on U'. It holds:

$$\mathfrak{hol}_0(\nabla) = \mathrm{span}\left\{ S_v(x, y) \mid v \in U', x, y \in V \right\}. \tag{6}$$

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Holonomy of torsion-free connections

An immediate consequence of the last Theorem is the following:

Corollary

In the situation of Theorem 1. The holonomy algebra of the torsion-free connection induced by the analytic map S is contained in the Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$ if, and only if, the map S takes values in the subspace $K(\mathfrak{h})$.

Summary

- The main takeaway is the fact solutions to the singular IVP

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  are key in the problem of prescribing torsion-free connections.

- In contrast to the classical Ambrose-Singer Holonomy Theorem, Theorem 2 offers a significant simplification of that statement.
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For Further Reading I

E. Basurto-Arzate.
Torsion-free Connections with prescribed Curvature.
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