Affability of Euclidean tilings*

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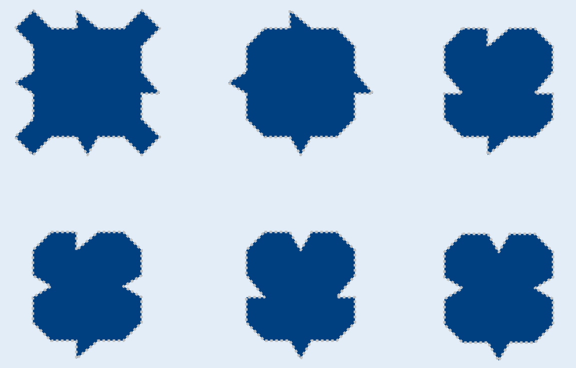
Introduction

An *Euclidean tiling* is a partition of $\mathbb{R}^m$ into *tiles*, which are polyhedra touching face to face, obtained by translation from a finite set of *prototiles*. A tiling is said to be *aperiodic* if it has no translation symmetries. It is said to be *repetitive* if for any patch M, there exists $R > 0$ such that any ball of radius R contains a translated copy of M.

Let $\mathfrak{T}(\mathcal{P})$ be the space of tilings $\mathcal{T}$ obtained from a finite set of prototiles $\mathcal{P}$, endowed with *the Gromov-Hausdorff topology* [2,3]. Thus $\mathfrak{T}(\mathcal{P})$ becomes a compact metrizable space which is foliated by the orbits $L_{\mathcal{T}}$ of the natural $\mathbb{R}^m$-action. For any aperiodic and repetitive Euclidean tiling $\mathcal{T}$, $\mathfrak{X} = \overline{L_{\mathcal{T}}}$ is a minimal saturated set without holonomy, the *continuous hull* of $\mathcal{T}$. This is transversely modeled on the Cantor set $X = \{\mathcal{T} \in \mathfrak{X} / \ 0 \in D_{\mathcal{T}}\}$, where $D_{\mathcal{T}}$ is the Delone set associated to the choice of base points in the (proto)tiles . Therefore the foliation $\mathcal{F} = \{L_{\mathcal{T}}\}_{\mathcal{T} \in \mathfrak{X}}$ may be identified with the *étale equivalence relation* (EER) $\mathcal{R} = \{(\mathcal{T}, \mathcal{T} - v) \in X \times X / \ v \in D_{\mathcal{T}}\}$ induced on X.

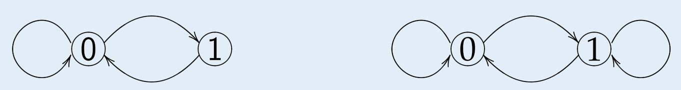
Penrose tilings by kites and darts are one of the most celebrated examples of aperiodic tilings. According to an idea by R. M. Robinson [5], any Penrose tiling may be encoded by a sequence $\{x_n\}_{n \in \mathbb{N}}$ of 0's and 1's such that $x_n = 1 \Rightarrow x_{n+1} = 0$.

The Penrose kite and dart tiles



The six Robinson tiles

Another interesting aperiodic set has been described by Robinson in [7]. In this case, any repetitive Robinson tiling is encoded by a sequence of pairs 00, 01, 11 and 10. In both cases, if we replace the first terms in a sequence, the origin is translated from a tile to another tile. In other words, Penrose and Robinson tilings have the same dynamics that the Fibonacci automaton and the 2-adic odometer respectively.



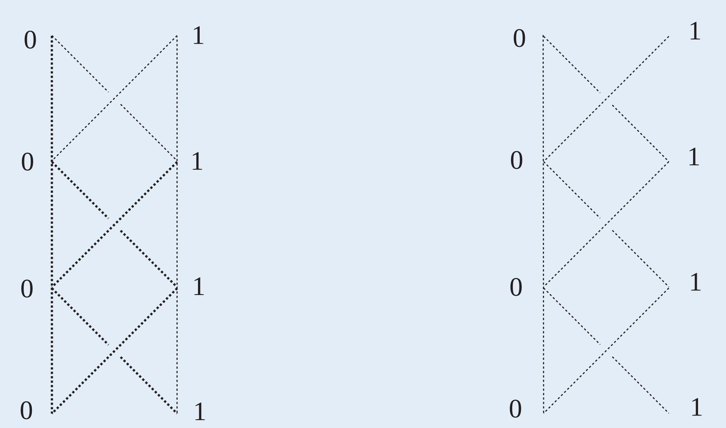
2-adic odometer   Fibonacci automaton

Affability

Let $\mathcal{R}$ be an EER on a locally compact space X. We say that $\mathcal{R}$ is a *compact étale equivalence relation* (CEER) if $\mathcal{R} - \Delta$ is a compact subset of $X \times X$. An equivalence relation $\mathcal{R}$ on a totally disconnected compact space X is *affable* [4] if there exists an increasing sequence of CEERs $\mathcal{R}_n$ such that $\mathcal{R} = \bigcup_{n \in \mathbb{N}} \mathcal{R}_n$. The inductive limit topology turns $\mathcal{R}$ into an EER and we say that $\mathcal{R} = \varinjlim \mathcal{R}_n$ is *approximately finite* or AF.

The cofinal or tail equivalence relation on the infinite path space of a *Bratteli diagram* is an example of affable equivalence relation. They are actually the only examples of AF equivalence relations:

Theorem 1. *([4]) Let $\mathcal{R}$ be an AF equivalence relation on a compact space X. There exists a standard Bratteli diagram (V, E) such that the tail equivalence relation on the infinite path space $X_{(V,E)}$ is isomorphic to $\mathcal{R}$.*



Bratteli diagrams for the 2-adic odometer and the Fibonacci automaton

Main theorem

Theorem ([1]). *Any equivalence relation $\mathcal{R}$ on a Cantor set X arising from the continous hull $\mathfrak{X}$ of an aperiodic and repetitive Euclidean tiling is affable*

Sketch of the proof

S1. Construction of an affable equivalence subrelation $\mathcal{R}_\infty \subset \mathcal{R}$

S2. $\mathcal{R}_\infty$ is minimal and every $\mathcal{R}$-class split into a finite number of $\mathcal{R}_\infty$-classes

S3. Its boundary $\partial\mathcal{R}_\infty$ is $\mathcal{R}$-*thin*, i.e. $\mu(\partial\mathcal{R}_\infty) = 0$ for every $\mathcal{R}$-invariant probability measure μ and $\mathcal{R}|_{\partial\mathcal{R}_\infty}$ is a countable union of graph of partial transformations $\phi_i : A_i \longrightarrow B_i$ between clopen disjoint subsets of $\partial\mathcal{R}_\infty$.

Inflation

To construct $\mathcal{R}_\infty$ we use the *inflation process* developed in [2]. By definition, $\mathfrak{X}$ admits a *flow box decomposition* $\mathcal{B}$, i.e. a family of closed flow boxes $\varphi_i : B_i \to P_i \times X_i$ such that $\mathfrak{X} = \bigcup_{i=1}^k B_i$ and $\mathring{B}_i \cap \mathring{B}_j = \emptyset$ if $i \neq j$. The set $\partial_v B_i = \varphi_i^{-1}(\partial P_i \times X_i)$ is the *vertical boundary* of B_i. A flow box decomposition is *well adapted to* $\mathcal{P}$ if P_i is a $\mathcal{P}$-patch and $\bigsqcup_{i=1}^k X_i$ is a clopen subset of X.

Theorem 2. *([2]) For any flow box decomposition $\mathcal{B}$, there exists another one $\mathcal{B}'$* inflated *of $\mathcal{B}$, that is*
i) for each tiling $\mathcal{T}$ in a box $B \in \mathcal{B}$ and in a box $B' \in \mathcal{B}'$, the transversal of B' through $\mathcal{T}$ is contained in the corresponding transversal of B;
ii) the vertical boundary of boxes of $\mathcal{B}'$ is contained in the same one of $\mathcal{B}$;
iii) for each box $B' \in \mathcal{B}'$, there exists a box $B \in \mathcal{B}$ such that $B \cap B' \neq \emptyset$ and $B \cap \partial_v B' = \emptyset$.

Corollary 1. *For any flow box decomposition $\mathcal{B}$ well adapted to $\mathcal{P}$, there exists a sequence of flow box decompositions $\mathcal{B}^{(n)}$ verifying:*
i) $\mathcal{B}^{(0)} = \mathcal{B}$;
ii) $\mathcal{B}^{(n+1)}$ is inflated of $\mathcal{B}^{(n)}$;
iii) $\mathcal{B}^{(n+1)}$ defines a finite set of $\mathcal{P}^{(n)}$-patches $\mathcal{P}^{(n+1)}$ (which contain at least a $\mathcal{P}^{(n)}$-tile in its interior) and a tiling in $\mathfrak{T}(\mathcal{P}^{(n+1)})$ of each leaf of $\mathfrak{X}$.

Now we obtain a increasing sequence of CEERs $\mathcal{R}_n$ on X:

$\mathcal{R}_n[\mathcal{T}]$ is the *discrete plaque* $\check{P} = P \cap X$

where P is the plaque of $\mathcal{B}^{(n)}$ through $\mathcal{T}$.

S1. $\mathcal{R}_\infty = \varinjlim \mathcal{R}_n$ *is an AF open equivalence subrelation of* $\mathcal{R}$

Boundary

We define the *boundary of* $\mathcal{R}_n$ as

$$\partial\mathcal{R}_n = \bigcup_{B \in \mathcal{B}^{(n)}} \partial_v \check{B} = \bigcup_{\mathcal{T} \in X^{(n)}} \partial\mathcal{R}_n[\mathcal{T}]$$

where $\partial\mathcal{R}_n[\mathcal{T}] = \{\mathcal{T}' = \mathcal{T} - v / \ v$ is the base point of a tile meeting $\partial P\}$. Then the boundary of $\mathcal{R}_\infty$ is the meager closed subset of X

$$\partial\mathcal{R}_\infty = \bigcap_{n \in \mathbb{N}} \partial\mathcal{R}_n$$

For each $\mathcal{T} \in \partial\mathcal{R}_\infty$, $\mathcal{R}[\mathcal{T}]$ split into several $\mathcal{R}_\infty$-classes and each one contains an increasing sequence of discrete plaques associated to $\mathcal{P}^{(n)}$-tiles. In the Euclidean case, this fact implies that:

S2. *$\mathcal{R}_\infty$ is minimal and every $\mathcal{R}$-class split at most into a finite number of $\mathcal{R}_\infty$-classes*

A Rohlin lemma

In [8], C. Series has proved that every foliation having polynomial growth is hyperfinite. We will use the same method (analogous to one of the standard proofs of the Rohlin lemma) in order to prove that:

S3. $\mu(\partial\mathcal{R}_\infty) = 0$ *for every $\mathcal{R}$-invariant probability measure μ*

Now all the boxes considered shall be discretized. We will say that $\check{B}$ is a *flow box of axis C and width* $2n$ if $\check{B}$ contains a transversal C such that each plaque P through $\mathcal{T} \in C$ is equal to $\overline{B}_{\mathcal{R}[\mathcal{T}]}(\mathcal{T}, n) = \{\mathcal{T}' \in \mathcal{R}[\mathcal{T}] / d(\mathcal{T}, \mathcal{T}') \leq n\}$. The union $\check{B} = \bigsqcup_{i=1}^k \check{B}_i$ of a family of disjoint flow boxes $\check{B}_1, \ldots, \check{B}_k$ of axis $C_1, \ldots, C_k$ and width $2n$ will be called a *stack of axis $C = \bigsqcup_{i=1}^k C_i$ and width* $2n$. Here $d(\mathcal{T}, \mathcal{T}')$ is the length of the shortest path of $\mathcal{P}$-tiles from the origin of $\mathcal{T}$ to the origin of $\mathcal{T}'$, $v_{\mathcal{T}}(n) = \#\overline{B}_{\mathcal{R}[\mathcal{T}]}(\mathcal{T}, n)$ and $\partial^r P = \{\mathcal{T}' \in P / \exists \mathcal{T}'' \notin P : d(\mathcal{T}', \mathcal{T}'') \leq r\}$.

The properties of $\mathfrak{X}$ allows us to exhibit a global version of Jenkins's result used in [8]:

Lemma 1. *There are an increasing sequence of locally constant functions $n_q : X \to \mathbb{N}$ and a constant $M > 0$ such that $v_{\mathcal{T}}(2n_q(\mathcal{T})) \leq M v_{\mathcal{T}}(n_q(\mathcal{T}))$ and $\lim_{q \to \infty} v_{\mathcal{T}}(n_q(\mathcal{T}) - r)/v_{\mathcal{T}}(n_q(\mathcal{T})) = 1$ for each $\mathcal{T} \in X$ and each $r \geq 1$.*

And this allows us to adapt the Series proof to obtain the following result:

Lemma 2. *For each $r \in \mathbb{N}$ and each $\varepsilon > 0$, there are $n \geq 0$ and a stack $\check{B}$ of width $\leq 2n$ such that $\mu(\check{B}) > 1 - \varepsilon$ and $\mu(\partial^r \check{B}) < \varepsilon\mu(\check{B})$.*

Thus, we obtain an increasing sequences of CEERs $\mathcal{R}_n$ and positive integers k_n such that

$\mathcal{R}_n[\mathcal{T}] \subset \overline{B}_{\mathcal{R}[\mathcal{T}]}(\mathcal{T}, k_n)$ *and* $\mu(\partial\mathcal{R}_n) < 1/2^n$ *for each* $n \geq 0$

Conclusion

Finally, we can remove $\partial\mathcal{R}_\infty$ by applying inductively Theorem 4.18 of [4]:

Theorem 3. *([4]) If $\mathcal{R}_\infty$ is a minimal affable equivalence relation and A_i is $\mathcal{R}$-thin, then $\mathcal{R}_\infty \vee Graph(\phi_i)$ is minimal affable.*

In fact, our proof aplies to the broader class of *tilable laminations* [2]. So we have the following result (which extends the main theorem of [5]):

Corollary.- *Any free minimal action of $\mathbb{Z}^m$ on the Cantor set is affable*

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