About the geodesic flow of Hilbert metrics

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Hilbert geometries

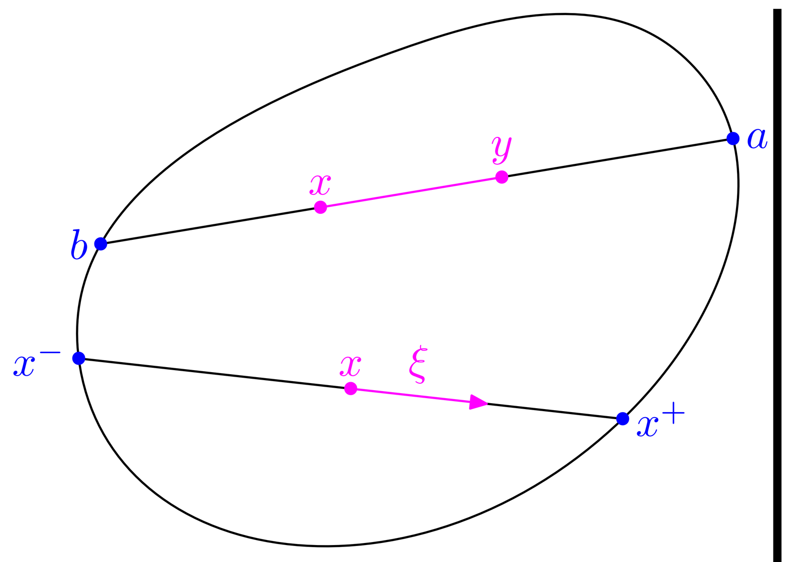
are metric spaces (Ω, d_Ω), where Ω is a bounded convex open set of $\mathbb{R}^n$ and d_Ω is defined by

$$d_\Omega(x,y) = \frac{1}{2}\log([a,b,x,y]);$$

$[a,b,x,y] = \frac{ax/bx}{ay/by}$ is the usual cross-ratio.

d_Ω is a Finsler metric, whose corresponding norm is given by

$$F(x,\xi) = \frac{|\xi|}{2}\left(\frac{1}{xx^+} + \frac{1}{xx^-}\right),\ x \in \Omega,\ \xi \in T_x\Omega.$$



Lines are always geodesics and if Ω is not strictly convex, there may be other geodesics. The group of isometries contains the subgroup P_Ω of projective transformations which preserve Ω ; if Ω is strictly convex, then $Isom(\Omega, d_\Omega) = P_\Omega$. Hilbert geometries can behave in different ways, depending on Ω :

- when Ω is a polytope, (Ω, d_Ω) is bi-Lipschitz equivalent to the Euclidean space ;
- when Ω is an ellipsoid, (Ω, d_Ω) is the Beltrami-Klein model of the hyperbolic space.

The other geometries should have a behaviour "in between". We can expect Hilbert geometries defined by strictly convex sets with C^1 boundary to exhibit some hyperbolic behaviour, from the geometrical and dynamical points of view.

In what follows, Ω is a **strictly convex** *subset of $\mathbb{R}^n$ with* $\mathbf{C}^1$ **boundary**, *$\Gamma < Isom(\Omega, d)$ a discrete group without torsion and $M = \Omega/\Gamma$ the quotient manifold. The geodesic flow φ^t is defined on the unitary tangent bundle SM.*

1. Compact quotients

For compact quotients, the situation is well understood and summarized by the following

Theorem 1 ([Ben04], [Cra09]). *Assume $M = \Omega/\Gamma$ is compact. Then the geodesic flow is Anosov, with topological entropy $h_{top} \leq n-1$. Moreover, the following are equivalent :*

- *Ω is an ellipsoid (M is hyperbolic) ;*
- *the geodesic admits an absolutely continuous measure ;*
- *$h_{top} = n-1$.*

Thus, the global hyperbolic behaviour of the flow is well preserved. But some dynamical properties can detect the Riemannian nature of the metric.

2. Noncompact quotients (joint work with L. Marquis)

Even if the situation here is more flexible, most of the definitions and results of hyperbolic geometry can be extended. The key result is the following lemma about parabolic subgroups, which says that cusps are indeed hyperbolic.

Lemma 2 ([CM]). *If $\mathcal{P}$ is a parabolic group acting on Ω, then it is conjugated to a parabolic subgroup of $Isom(\mathcal{H}^n)$.*

Geometrically finite quotients can be defined as in hyperbolic geometry. Some work has to be done for that, but the usual picture of geometrically finite Kleinian groups is the one to have in mind (*work in progress [CM]*).

The limit set of Γ is defined as $\Lambda_\Gamma = \overline{\Gamma o} \setminus \Gamma o$ for any point $o \in \Omega$. A point $p \in \Lambda_\Gamma$ is said to be

- a **conical** point if there exists a geodesic $c(t)$ ending at p and $C > 0$ such that $d_\Omega(c(t), \Gamma.o) < C$ for any $t \geq 0$;
- a **bounded parabolic** point if $Stab_\Gamma(p)$ is a parabolic group acting cocompactly on $\partial\Omega$.

Definition 3. *The group Γ (or the quotient $M = \Omega/\Gamma$) is said to be* **geometrically finite** *if Λ_Γ consists only of conical and bounded parabolic limit points.*

Remarks : If M is geometrically finite, then

- M has finite volume, with respect to the Hausdorff measure of d_Ω, if and only if $\Lambda_\Gamma = \partial\Omega$;
- if $\Lambda_\Gamma \neq \partial\Omega$, then Γ acts on many convex sets Ω. The convex hull $C(\Lambda_\Gamma)$ of Λ_Γ, which is not in general strictly convex, is the *minimal* set on which Γ acts.

The **nonwandering set** $N \subset SM$ of the flow is $N = \tilde{N}/\Gamma$ where

$$\tilde{N} = \{w = (x,\xi) \in S\Omega,\ x^+, x^- \in \Lambda_\Gamma\}.$$

This is the subset of SM on which the dynamics of the flow is recurrent. The geometry of a cusp being *almost* hyperbolic, we can control the behaviour of the flow in the cusps, and we obtain the

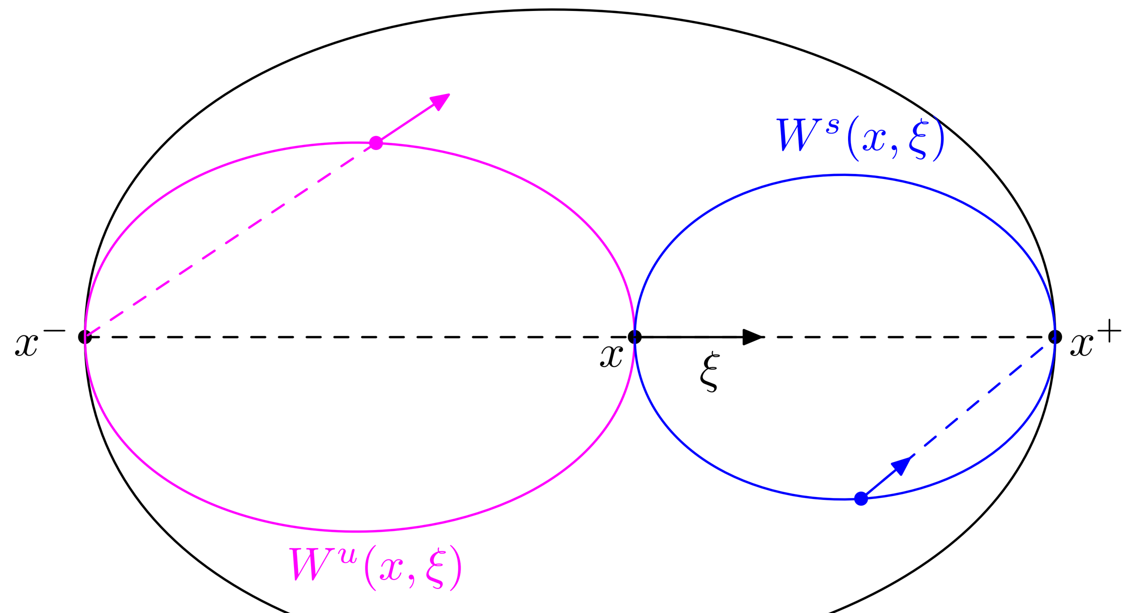
Theorem 4 ([CM]). *Assume $M = \Omega/\Gamma$ is geometrically finite. Then the geodesic flow is Anosov on the nonwandering set N : there exist a decomposition*

$$TN = \mathbb{R}.X \oplus E^s \oplus E^u,$$

a continuous Finsler metric $|.|$ and constants $C, \alpha, \beta > 0$ such that for any $t \geq 0$,

$$|d\varphi^t(Z^s)| \leq Ce^{-\alpha t}|Z^s|,\ Z^s \in E^s,$$

$$|d\varphi^{-t}(Z^u)| \leq Ce^{-\beta t}|Z^u|,\ Z^u \in E^u.$$



Stable and unstable manifolds

3. Entropy and critical exponent

For a dynamical system $f : (X,d) \mapsto (X,d)$ with X compact, the **variational principle** asserts that $h_{top} = \sup_m h_m$, where the $\sup$ is taken with respect to all f-invariant probability measures. When X is not compact, this can be taken as a definition of h_{top}.

When $M = \Omega/\Gamma$ is compact, there is a unique probability measure on SM achieving this $\sup$, called the Bowen-Margulis measure. This picture was extended in [OP04] for noncompact manifolds of pinched negative curvature. This can also be done here.

Patterson-Sullivan measures. As for Kleinian groups, we can define *conformal* densities of dimension $\alpha > 0$, that is families (μ_x) of measures such that

$$\frac{d\mu_x}{d\mu_y}(\xi) = e^{-\alpha b_\xi(x,y)},$$

where $b_\xi(x,y) = \lim_{p\to\xi} d(x,p) - d(y,p)$ are the usual Busemann functions.

Patterson and Sullivan explained how to construct conformal densities from the Poincaré series $g_\Gamma(x,s) = \sum_{\gamma\in\Gamma} e^{-sd(x,\gamma o)}$. The critical exponent δ_Γ of this series is

$$\delta_\Gamma = \limsup_{R\to+\infty} \sharp\{\gamma \in \Gamma,\ d(o,\gamma o) < R\},$$

such that the series diverges for $s < \delta_\Gamma$ and converges for $s > \delta_\Gamma$.

Theorem 5 (Patterson, Sullivan). *Let δ_Γ be the critical exponent of the Poincaré series of Γ. Then there exists a Γ-invariant conformal density $(\mu_x)_{x\in\Omega}$ of dimension δ_Γ.*

To this family of measures on $\partial\Omega$ with support in Λ_Γ, we associate a φ^t-invariant measure μ on SM with support in N, which coincide with the Bowen-Margulis measure when M is compact. The following proposition and its corollary were proved by Sullivan for hyperbolic manifolds.

Proposition 6. *If $M = \Omega/\Gamma$ is geometrically finite, then μ is finite.*

Corollary 7 (Sullivan, Kaimanovich, Roblin). *If $M = \Omega/\Gamma$ is geometrically finite, then the Poincaré series diverges at δ_Γ and μ is ergodic.*

The method developed in [DPPS09] is very simple and efficient in our context to prove the

Proposition 8. *If $M = \Omega/\Gamma$ has finite volume, then $h_{vol} = \delta_\Gamma$.*

Here $h_{vol} = \limsup_{R\to+\infty} \frac{1}{R}\log vol(B(o,R))$ denotes the volume entropy.

As a counterpart, for geometrically finite manifolds, it is quite reasonable to expect that

$$\delta_\Gamma = h_{vol}(C(\Lambda_\Gamma), d_{C(\Lambda_\Gamma)}).$$

Variational principle and entropy rigidity Based on the work of [LY85] and [OP04], we can prove the following variational principle for geometrically finite quotients. This is true for more general quotients, under the hypothesis that the flow has no zero Lyapunov exponent, which can be expected to be true for any group.

Theorem 9. *Assume $M = \Omega/\Gamma$ is geometrically finite. Then*

$$h_{top} = h_\mu = \delta_\Gamma,$$

and μ is the unique measure to achieve the equality in $h_{top} := \sup_m h_m$.

As a corollary, we have a counterpart of the main theorem of [Cra09] for geometrically finite manifolds.

Theorem 10. *If $M = \Omega/\Gamma$ is geometrically finite, then $h_{top} \leq n-1$, with equality if and only if M has finite volume and is hyperbolic.*

Among all the structures with finite volume on M, the topological entropy thus detects hyperbolic ones. For general geometrically finite quotients, we can expect to prove that the following propositions are equivalent :

- $\mu = \mu_{SRB}$;
- Γ is conjugated to a Kleinian group.

Here the SRB measure would be a counterpart of the equilibrium measure of the potential $f = -\frac{d}{dt}|_{t=0}\log\det d\varphi^t|_{E^u}$, that appears for Anosov flows on Riemannian manifolds [BR75].

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