# Some results on isoparametric hypersurfaces in nonflat complex space forms

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## 1. Preliminaries

Aim: classify isoparametric hypersurfaces in nonflat complex space forms

Nonflat complex space forms: $\mathbb{C}P^n = \frac{SU(n+1)}{S(U(1)U(n))}$, $\mathbb{C}H^n = \frac{SU(1,n)}{S(U(1)U(n))}$

Simply connected complete Kähler manifolds of nonzero constant holomorphic sectional curvature (complex structure $J$)

Isoparametric hypersurface: hypersurface whose sufficiently close parallel hypersurfaces have constant mean curvature [9]

Constant principal curvatures: the eigenvalues of the shape operator are constant along the hypersurface

Homogeneous hypersurface: orbit of a cohomogeneity one action of a closed subgroup of the isometry group of the ambient manifold

Homogeneous hypersurface ⇒
- constant principal curvatures
- isoparametric

Homogeneous hypersurfaces in $\mathbb{C}P^n$ and $\mathbb{C}H^n$ were classified in [8] and [3].

Notation: $M$ hypersurface in $\mathbb{C}P^n$ or $\mathbb{C}H^n$

$\xi$ normal unit vector field

$g$ number of principal curvatures

$J\xi$ is called the Hopf vector field (it is tangent to $M$)

$h$ number of nontrivial projections of the Hopf vector field onto the principal curvature spaces

## 2. The case $h \leq 2$

É. Cartan proved that a hypersurface in a real space form is isoparametric if and only if it has constant principal curvatures. The same holds in nonflat complex space forms if $h \leq 2$, but not if $h$ is arbitrary (see Section 4).

**Theorem.** *Let $M$ be a connected hypersurface in $\mathbb{C}P^n$ or $\mathbb{C}H^n$ with $h \leq 2$. Then, $M$ is isoparametric if and only if $M$ has constant principal curvatures. In this case, $h$ is constant.*

Hypersurfaces with constant principal curvatures in nonflat complex space forms with $h = 1$ (the so called Hopf hypersurfaces) and with $h = 2$ have been classified. See [6] for a survey.

**Theorem.** [1,7] *Let $M$ be a Hopf hypersurface with constant principal curvatures of a nonflat complex space form. We have*

(a) *If $M \subset \mathbb{C}P^n$, then $M$ is an open part of the projection via the Hopf map $S^{2n+1} \to \mathbb{C}P^n$ of a principal orbit of the isotropy representation of a Hermitian symmetric space of rank 2*

(b) *If $M \subset \mathbb{C}H^n$, then $M$ is an open part of*
   - (i) *A tube around a totally geodesic $\mathbb{C}H^k$, for $k \in \{0, \dots, n-1\}$*
   - (ii) *A tube around a totally geodesic $\mathbb{R}H^n$*
   - (iii) *A horosphere*

*In particular, $M$ is an open part of a homogeneous hypersurface.*

**Theorem.** [4] *Let $M$ be a hypersurface with constant principal curvatures of a nonflat complex space form, with $h = 2$. We have*

(a) *The case $M \subset \mathbb{C}P^n$ is impossible.*

(b) *If $M \subset \mathbb{C}H^n$, then $M$ is an open part of:*
   - (i) *a ruled minimal real hypersurface $W^{2n-1} \subset \mathbb{C}H^n$ or one of the equidistant hypersurfaces to $W^{2n-1}$*
   - (ii) *a tube around a ruled minimal Berndt-Brück submanifold with totally real normal bundle $W^{2n-k} \subset \mathbb{C}H^n$, for $k \in \{2, \dots, n-1\}$*

*In particular, $M$ is an open part of a homogeneous hypersurface.*

For the description of the submanifolds $W^{2n-k}$ see Section 4.

## 3. The hyperbolic case

For the case of complex hyperbolic spaces we have obtained the following

**Theorem.** *Let $M$ be an isoparametric hypersurface in $\mathbb{C}H^n$ and $p \in M$. Then the principal curvatures of $M$ at $p$ and their multiplicities coincide with those of the homogeneous hypersurfaces in $\mathbb{C}H^n$. In particular, we have that $h(p) \in \{1, 2, 3\}$ and $g(p) \in \{2, 3, 4, 5\}$.*

In general, the functions $h$ and $g$ could be nonconstant.

## 4. New isoparametric examples

In this section we construct examples of isoparametric hypersurfaces in $\mathbb{C}H^n$ that, in general, are not homogeneous [5]. These inhomogeneous examples do not have constant principal curvatures and the functions $h$ and $g$ can be nonconstant on these hypersurfaces.

$G = SU(1,n)$ acts transitively on $\mathbb{C}H^n$

Fix $o \in \mathbb{C}H^n$ $\quad K = G_o \cong S(U(1)U(n))$ (isotropy group of $G$ at $o$)

Cartan decomposition (with respect to $o$): $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$

$\mathfrak{a} \subset \mathfrak{p}$ maximal abelian subspace

Restricted root space decomposition: $\mathfrak{g} = \mathfrak{g}_{-2\alpha} \oplus \mathfrak{g}_{-\alpha} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$

$$\mathfrak{g}_\lambda = \{ X \in \mathfrak{g} : \mathrm{ad}(H)X = \lambda(H)X, \ \forall H \in \mathfrak{a} \}$$

$\mathfrak{n} = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha} \cong$ Heisenberg algebra

Iwasawa decomposition: $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ (Lie algebra level)

$G = KAN$ (Lie group level)

$\mathbb{C}H^n$ is isometric to $AN$ (semi-direct product of Lie groups)

$\mathfrak{w}$ : proper subspace of $\mathfrak{g}_\alpha$

$\mathfrak{w}^\perp = \mathfrak{g}_\alpha \ominus \mathfrak{w}$ (orthogonal complement of $\mathfrak{w}$ in $\mathfrak{g}_\alpha$) $\quad k = \dim \mathfrak{w}^\perp$

$\mathfrak{s}_\mathfrak{w} = \mathfrak{a} \oplus \mathfrak{w} \oplus \mathfrak{g}_{2\alpha}$ (Lie subalgebra of $\mathfrak{a} \oplus \mathfrak{n}$)

$S_\mathfrak{w}$: connected subgroup of $AN$ with Lie algebra $\mathfrak{s}_\mathfrak{w}$

$$W_\mathfrak{w} = S_\mathfrak{w} \cdot o$$ (minimal homogeneous submanifold of $\mathbb{C}H^n$)

**Theorem.** *The tubes around $W_\mathfrak{w}$ are isoparametric hypersurfaces. Moreover, these tubes are homogeneous hypersurfaces if and only if $\mathfrak{w}^\perp$ has constant Kähler angle.*

By definition, $\mathfrak{w}^\perp$ has constant Kähler angle if the angle between $Jv$ and $\mathfrak{w}^\perp$ is the same for all $v \in \mathfrak{w}^\perp$.

If $\mathfrak{w}^\perp$ has constant Kähler angle $\varphi$, one gets the Berndt-Brück submanifolds $W^{2n-k}_\varphi = W_\mathfrak{w}$, and $W^{2n-k} = W^{2n-k}_{\pi/2}$ [2].

However, for a generic choice of $\mathfrak{w}^\perp$, tubes around $W_\mathfrak{w}$ are *inhomogeneous hypersurfaces with nonconstant principal curvatures.*

The tubes around $W_\mathfrak{w}$ have constant principal curvatures if and only if they are homogeneous, that is, when they are precisely the tubes around the Berndt-Brück submanifolds.

For the inhomogeneous examples, the functions $h$ and $g$ may be nonconstant, and we can have $h \in \{1, 2, 3\}$ and $g \in \{3, 4, 5\}$.

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