REAL HYPERSURFACES WITH CONSTANT PRINCIPAL CURVATURES IN COMPLEX SPACE FORMS

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Abstract. We present the motivation and current state of the classification problem of real hypersurfaces with constant principal curvatures in complex space forms. In particular, we explain the classification result of real hypersurfaces with constant principal curvatures in nonflat complex space forms and whose Hopf vector field has nontrivial projection onto two eigenspaces of the shape operator. This constitutes the following natural step after Kimura and Berndt's classifications of Hopf real hypersurfaces with constant principal curvatures in complex space forms.

1. Introduction

An isometric action on a Riemannian manifold $\bar{M}$ is called a cohomogeneity one action if its principal (or generic) orbits are hypersurfaces. These hypersurfaces are then called (extrinsically) homogeneous hypersurfaces of $\bar{M}$. The study of cohomogeneity one actions is a topic of current interest because it has shown to be useful in the construction of geometrical structures on manifolds, such as Ricci solitons, Einstein metrics and metrics with special holonomies. The reason is that certain systems of partial differential equations defining those structures can be reduced to ordinary differential equations, which can help to find explicit solutions.

From the point of view of Submanifold Geometry, an important problem is to classify cohomogeneity one actions on a given ambient manifold $\bar{M}$, and also to characterize the outcoming homogeneous hypersurfaces in terms of geometric data. This work focuses on this aim and, in particular, on the geometric property of having constant principal curvatures. It is clear that every homogeneous hypersurface has constant principal curvatures, because the shape operators at two different points are always conjugate by the differential of an element of the group that is acting with cohomogeneity one. It is natural to ask to what extent the constancy of principal curvatures characterizes homogeneous hypersurfaces. In other words, if M is a hypersurface with constant principal curvatures, is then M an open part of a homogeneous hypersurface?

In general, the methods that have been developed to answer this question address directly the problem of the classification of hypersurfaces with constant principal curvatures in a certain ambient manifold $\bar{M}$. But already since the decade of the 30s, when Élie Cartan studied this topic, the problem appears to be far from being trivial, as we will see.

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In this work we will expose the evolution of this topic, focusing on the case where the ambient manifold is a complex space form. The objective of this work is to provide the necessary contextualization, definitions and notation required to understand the following classification theorem, which constitutes the main result of the work [11], developed by José Carlos Díaz Ramos and the author.

Main Theorem. *We have:*

(a) *There are no real hypersurfaces with constant principal curvatures in the complex projective space* $\mathbb{C}P^n$*,* $n \geq 2$*, whose Hopf vector field has* $h = 2$ *nontrivial projections onto the principal curvature spaces.*

(b) *Let* M *be a connected real hypersurface in the complex hyperbolic space* $\mathbb{C}H^n$*,* $n \geq 2$*, with constant principal curvatures and whose Hopf vector field has* $h = 2$ *nontrivial projections onto the principal curvature spaces of* M*. Then,* M *has* $g \in \{3, 4\}$ *principal curvatures and is holomorphically congruent to an open part of:*
   (i) *a ruled minimal real hypersurface* $W^{2n-1} \subset \mathbb{C}H^n$ *or one of the equidistant hypersurfaces to* W^{2n-1}*, or*
   (ii) *a tube around a ruled minimal Berndt-Brück submanifold with totally real normal bundle* $W^{2n-k} \subset \mathbb{C}H^n$*, for some* $k \in \{2, \dots, n-1\}$*.*

   In particular, M *is an open part of a homogeneous real hypersurface of* $\mathbb{C}H^n$*.*

The structure of this text is as follows. In Section 2 we give a historical approach to the problem of hypersurfaces with constant principal curvatures in space forms. Then, in Section 3, we move on to the case when the ambient manifold is a complex space form, introducing some important notions and exposing the results known so far. The examples that appear in our classification result are explained in Section 4. Finally, in Section 5, we state some open problems of interest in this topic.

2. The problem in space forms

Given a Riemannian manifold $\bar{M}$, a hypersurface M is called isoparametric if it and its nearby equidistant hypersurfaces have constant mean curvature. This terminology was first introduced by Levi-Civita [17], motivated by a problem in Geometric Optics (see [27] for more details). Cartan [9] proved that, if the ambient manifold is a space form, a hypersurface is isoparametric if and only if it has constant principal curvatures. This is the reason why it is very common to refer to hypersurfaces with constant principal curvatures in real space forms as isoparametric hypersurfaces. However, it is important to notice that this equivalence is not true in general, as some examples in the complex projective space show [28].

In the rest of this work, g will denote the number of principal curvatures of a hypersurface with constant principal curvatures.

The classification of isoparametric hypersurfaces in Euclidean spaces $\mathbb{R}^n$ is usually attributed to Levi-Civita [17] for $n = 3$, and to Segre [22] for the general case. The examples that appear in this classification are affine hyperplanes, spheres and products of spheres by affine subspaces. When the ambient manifold is a real hyperbolic space $\mathbb{R}H^n$, the analogous result is due to Cartan [9], who proved that, in this case, the examples can be geodesic hyperspheres, horospheres, totally geodesic real hyperbolic hyperspaces and their equidistant hypersurfaces, and tubes around totally geodesic real hyperbolic subspaces of codimension greater than one. A consequence of these results is that every isoparametric hypersurface in a space form of nonpositive curvature satisfies $g \in \{1, 2\}$ and is an open part of a homogeneous hypersurface, so the constancy of principal curvatures characterizes homogeneous hypersurfaces for these ambient manifolds.

Nevertheless, the problem in spheres turns out to be much more involved. In a series of papers at the end of the 30s, Cartan classified hypersurfaces with $g \in \{1, 2, 3\}$ constant principal curvatures in spheres, but could not solve the general case. Then, the problem stayed abandoned for about thirty years, until Hsiang and Lawson Jr. [13] classified cohomogeneity one actions on spheres and, hence, homogeneous hypersurfaces in spheres. For these homogeneous examples $g \in \{1, 2, 3, 4, 6\}$ holds. Using Algebraic Topology methods, Münzner proved that this restriction on g is also valid for every isoparametric hypersurface in a sphere, what could lead to think that, again, every isoparametric hypersurface is homogeneous. However, this is not the case, as follows from the paper [14], where Ferus, Karcher and Münzner, using representations of Clifford algebras, found a family of inhomogeneous isoparametric hypersurfaces in spheres, with $g = 4$. This made the problem much more difficult and interesting. Recently, some important advances have been made towards a final classification, which is still not known. We emphasize the works of Cecil, Chi and Jensen [10] and Immervoll [15] who proved, using quite different methods, that, with a few possible exceptions, hypersurfaces with $g = 4$ constant principal curvatures are among the known homogeneous and inhomogeneous examples. Some progress has also been made in the case $g = 6$ (Abresch [1], Dorfmeister and Neher [12]), but the problem in this case seems to be open, as well.

For a more detailed exposition on the history of isoparametric hypersurfaces in space forms and related problems, see the survey article [27].

3. The problem in complex space forms

In this work, by a complex space form we will understand a simply connected complete Kähler manifold with constant holomorphic sectional curvature c. These spaces are classified in three families according to the curvature: even-dimensional Euclidean spaces $\mathbb{C}^n$ ($c = 0$), complex hyperbolic spaces $\mathbb{C}H^n(c)$ with the Bergman metric if $c < 0$, or complex projective spaces $\mathbb{C}P^n(c)$ with the Fubini-Study metric if $c > 0$. As $\mathbb{C}^n$ is isometric to $\mathbb{R}^{2n}$, we will restrict to the nonflat case and, since $\mathbb{C}P^1(c)$ is isometric to a sphere, and $\mathbb{C}H^1(c)$ is isometric to the real hyperbolic plane $\mathbb{R}H^2$, we also restrict to the case $n \geq 2$. When we are not concerned about the value of c, we will just write $\mathbb{C}P^n$ or $\mathbb{C}H^n$.

When an ambient manifold is Kähler, we have two different notions of a hypersurface: either a submanifold with real codimension one, or a complex submanifold with complex codimension one. As we are interested in the relation with cohomogeneity one actions, we will focus on the first case, that is, on real hypersurfaces.

From now on, M will denote a real hypersurface in a nonflat complex space form $\bar{M}$ (that is, $\mathbb{C}P^n$ or $\mathbb{C}H^n$) with complex structure J. Let ξ be a unit normal vector field to the hypersurface M (maybe defined just locally).

An important definition for the study of real hypersurfaces in complex space forms is the following. The *Hopf vector field* (also *Reeb vector field*) of the hypersurface M is the tangent vector field $J\xi$. When this vector field is a principal curvature vector field, that is, $J\xi$ is an eigenvector field of the shape operator of the hypersurface, then M is called a *Hopf hypersurface*. We will denote by h the number of principal curvature spaces of M onto where the Hopf vector field has nontrivial projection. Then h is an integer-valued map defined on the hypersurface M, and h is pointwise less or equal to the number of principal curvatures of M, that is, $h \leq g$. With this notation, a real hypersurface is Hopf if and only if $h = 1$.

In the study of real hypersurfaces, it is very common to impose geometric conditions which usually imply the property of being Hopf, and this often simplifies considerably some calculations. However, in this work we are mainly concerned with the study of non-Hopf real hypersurfaces, as will become clearer soon.

In the rest of this section, we will summarize the evolution and current state of the classification problem for real hypersurfaces with constant principal curvatures in complex space forms. Another survey on this topic is [3]. For a schematic view of the possible real hypersurfaces with constant principal curvatures in $\mathbb{C}P^n$ and $\mathbb{C}H^n$ for different values of h and g, we refer to the tables placed two pages later.

In 1963, Tashiro and Tachibana proved that there are no umbilical real hypersurfaces in nonflat complex space forms. In particular, there is no real hypersurface with $g = 1$ constant principal curvature. Years later, in 1973, Takagi achieved the classification of homogeneous real hypersurfaces in the complex projective space [23]. He used the classification of homogeneous hypersurfaces in spheres due to Hsiang and Lawson, to prove that the only homogeneous hypersurfaces in spheres which preserve the S^1-fiber of the Hopf fibration $S^{2n+1} \to \mathbb{C}P^n$ are the principal orbits of isotropy representations of rank 2 Hermitian symmetric spaces. Hence, every homogeneous hypersurface in $\mathbb{C}P^n$ is the projection by the Hopf map of one of these principal orbits (see Table 1 for the description of the examples). From this classification it follows that the homogeneous examples satisfy $g \in \{2, 3, 5\}$. A remarkable feature of homogeneous hypersurfaces in $\mathbb{C}P^n$ is that they are Hopf.

Subsequently, Takagi classified real hypersurfaces with $g \in \{2, 3\}$ constant principal curvatures in $\mathbb{C}P^n$ [24], [25], with the exception of the case $n = 2$, $g = 3$, which was solved by Wang [29]. All the examples classified in these results are Hopf and open parts of homogeneous hypersurfaces. In 1986, Kimura [16] classified Hopf real hypersurfaces with constant principal curvatures in $\mathbb{C}P^n$ and showed that these are open parts of homogeneous ones. No examples are known of real hypersurfaces with constant principal curvatures in $\mathbb{C}P^n$ with $h > 1$.

The situation is more interesting in the complex hyperbolic space $\mathbb{C}H^n$, where, surprisingly, there are non-Hopf homogeneous real hypersurfaces. Such real hypersurfaces were constructed by Berndt and Brück [4]. Indeed, with $h = 3$ there are uncountably many non-congruent homogeneous real hypersurfaces with $g \in \{4, 5\}$. In Section 4 and in [7] one can find the definition of these non-Hopf examples and some of their remarkable properties. Recently, Berndt and Tamaru obtained in [8] the classification of cohomogeneity one actions on $\mathbb{C}H^n$. The number of principal curvatures of the resulting homogeneous hypersurfaces is $g \in \{2, 3, 4, 5\}$ and the number of nontrivial projections of the Hopf vector field onto the principal curvature spaces is $h \in \{1, 2, 3\}$. In 1985, Montiel [19] had classified real hypersurfaces with $g = 2$ constant principal curvatures in $\mathbb{C}H^n$ ($n \geq 3$), proving that they are Hopf. In 1989, Hopf real hypersurfaces with constant principal curvatures in $\mathbb{C}H^n$ were classified by Berndt [2]. All these hypersurfaces are open parts of homogeneous ones. Berndt and Díaz-Ramos solved the cases $g = 3$, and $g = 2$, $n = 2$ in [5] and [6]. It follows from these results that $h = 1$ when $g = 2$ and that $h \leq 2$ if $g = 3$. To our knowledge, the first classifications of this kind involving non-Hopf real hypersurfaces are [5] and [6]. Nothing is known about h if $g \geq 4$.

The Main Theorem stated in Section 1, which is proved in [11], addresses the next natural step after Kimura and Berndt's classifications of Hopf real hypersurfaces with constant principal curvatures in $\mathbb{C}P^n$ and $\mathbb{C}H^n$ [16], [2]. Thus, it provides the classification of real hypersurfaces with constant principal curvatures and $h = 2$ nontrivial projections of the Hopf vector field onto the principal curvature spaces in $\mathbb{C}P^n$ and $\mathbb{C}H^n$. In the projective case, there are no such hypersurfaces, and, in the hyperbolic case, all the examples are homogeneous and have $g \in \{3, 4\}$ principal curvatures. The construction of these hypersurfaces is the aim of the next section.

4. Non-Hopf homogeneous hypersurfaces

For some time it was believed that, as in the case of the complex projective space, every homogeneous hypersurface in the complex hyperbolic space was Hopf. However, in 1998 Lohnherr [18] constructed a counterexample: the minimal ruled hypersurface W^{2n-1} in $\mathbb{C}H^n$. Later, in [4], Berndt and Brück generalized this construction to the minimal ruled submanifolds W^{2n-k} and W^{2n-k}_φ. As a consequence of the classification of cohomogeneity one actions on $\mathbb{C}H^n$ [8], tubes around these submanifolds constitute the only nonclassical (and non-Hopf) examples of homogeneous hypersurfaces in the complex hyperbolic space.

The aim of this section is to construct the submanifolds W^{2n-k} and W^{2n-k}_φ, which we are going to call *Berndt-Brück submanifolds*, and explain some of the properties of the non-Hopf real hypersurfaces that they give rise to. For a more detailed description of these, see [4] and [7].

In order to define the Berndt-Brück submanifolds, we will have to recall some definitions and results about the structure of the complex hyperbolic space as a symmetric space of noncompact type.

The expression of $\mathbb{C}H^n$ as a symmetric space is G/K where $G = SU(1, n)$ is a connected simple Lie group that acts isometrically and transitively on $\mathbb{C}H^n$, and $K = S(U(1)U(n))$ is the isotropy group of G at a point $o \in \mathbb{C}H^n$. Write $\mathfrak{g}$ for the Lie algebra of G and $\mathfrak{k}$ for the Lie algebra of K. Let B be the Killing form of $\mathfrak{g}$, which is nondegenerate due to Cartan's criterion for semisimple Lie algebras. Then we have the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ with respect to o, where $\mathfrak{p}$ is the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to B. This means that $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}$, $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$, $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$ and B is negative definite on $\mathfrak{k}$ and positive definite on $\mathfrak{p}$. The Cartan involution θ corresponding to the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is the automorphism of $\mathfrak{g}$ defined by $\theta(X) = X$ for all $X \in \mathfrak{k}$ and $\theta(X) = -X$ for all $X \in \mathfrak{p}$. Then we can define a positive definite inner product B_θ on $\mathfrak{g}$ by $B_\theta(X, Y) = -B(X, \theta Y)$, for all $X, Y \in \mathfrak{g}$.

Now fix a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$. It can be shown that $\dim \mathfrak{a} = 1$, which means by definition that the rank of the symmetric space G/K is one. The set $\{\operatorname{ad}(H) : H \in \mathfrak{a}\}$ is a family of commuting B_θ-selfadjoint endomorphisms of $\mathfrak{g}$, hence simultaneously diagonalizable. Their common eigenspaces are the root spaces of the semisimple Lie algebra $\mathfrak{g}$. In other words, if for each $\lambda \in \mathfrak{a}^*$ we define

$$\mathfrak{g}_\lambda = \{X \in \mathfrak{g} : \operatorname{ad}(H)X = \lambda(H)X \text{ for all } H \in \mathfrak{a}\},$$

then the root space decomposition of $\mathfrak{g}$ with respect to $\mathfrak{a}$ adopts the form $\mathfrak{g} = \mathfrak{g}_{-2\alpha} \oplus \mathfrak{g}_{-\alpha} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$, for a certain covector $\alpha \in \mathfrak{a}^*$. In addition, $[\mathfrak{g}_\lambda, \mathfrak{g}_\mu] \subset \mathfrak{g}_{\lambda+\mu}$. Now assume that α is a positive root. Then, due to the properties of the root space decomposition, $\mathfrak{n} = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$ is a 2-step nilpotent subalgebra of $\mathfrak{g}$, which is in fact

TABLE 1. Real hypersurfaces with constant principal curvatures in $\mathbb{C}P^n(c)$

	h = 1 Kimura [16]	Hermitian symmetric space of rank 2	**h = 2** Main Theorem [11]	**h ≥ 3**
g = 1 Tachibana, Tashiro [26]	Impossible			
g = 2 Takagi [24]	Geodesic hypersphere	$\mathbb{C}P^1 \times \mathbb{C}P^n$	Impossible	
g = 3 Takagi [25] Wang [29]	Tube around a totally geodesic $\mathbb{C}P^k$, $1 \leq k \leq n-2$	$\mathbb{C}P^{k+1} \times \mathbb{C}P^{n-k}$	Impossible	?
	Tube around the complex quadric $\left\{[z] \in \mathbb{C}P^n : z_0^2 + \cdots + z_n^2 = 0\right\}$	$G_2^+(\mathbb{R}^{n+3})$		
g = 4	Impossible		Impossible	?
g = 5	Tube around the Segre embedding of $\mathbb{C}P^1 \times \mathbb{C}P^k$ in $\mathbb{C}P^{2k+1}$, $k \geq 2$	$G_2(\mathbb{C}^{k+3})$	Impossible	?
	Tube around the Plücker embedding of the complex Grassmannian $G_2(\mathbb{C}^5)$ in $\mathbb{C}P^9$	$SO(10)/U(5)$		
	Tube around the half spin embedding of $SO(10)/U(5)$ in $\mathbb{C}P^{15}$	$E_6/\left(U(1) \times Spin(10)\right)$		
g ≥ 6	Impossible		Impossible	?

(1) In Tables 1 and 2 all known classification results and examples of real hypersurfaces with constant principal curvatures in $\mathbb{C}P^n$ and $\mathbb{C}H^n$ are shown, up to holomorphic congruence.
(2) A shaded cell just means that $h > g$ is impossible.
(3) When for a particular case of g and h there are some known examples, but a classification is missing, we write *Not yet classified*. If neither any example nor a classification is known, we write a question mark ?.
(4) For each homogeneous hypersurface in $\mathbb{C}P^n$, we indicate the associated Hermitian symmetric space of rank 2 whose isotropy representation give rise to that homogeneous hypersurface, via the projection of a principal orbit by the Hopf fibration.

TABLE 2. Real hypersurfaces with constant principal curvatures in $\mathbb{C}H^n(c)$

	$\mathbf{h=1}$ Berndt [2]	$\mathbf{h=2}$ Main Theorem [11]	$\mathbf{h=3}$	$\mathbf{h\geq 4}$
$\mathbf{g=1}$ Tachibana, Tashiro [26]	Impossible			
$\mathbf{g=2}$ Montiel [19] Berndt, Díaz-Ramos [6]	Horosphere Geodesic hypersphere Tube around a totally geodesic $\mathbb{C}H^{n-1}$ Tube of radius $\frac{1}{\sqrt{-c}}\ln\left(2+\sqrt{3}\right)$ around a totally geodesic $\mathbb{R}H^n$	Impossible		
$\mathbf{g=3}$ Berndt, Díaz-Ramos [5], [6]	Tube around a totally geodesic $\mathbb{C}H^k$, $1\leq k\leq n-2$ Tube of radius $r\neq\frac{1}{\sqrt{-c}}\ln\left(2+\sqrt{3}\right)$ around a totally geodesic $\mathbb{R}H^n$	Hypersurface W^{2n-1} and its equidistant hypersurfaces Tube of radius $\frac{1}{\sqrt{-c}}\ln\left(2+\sqrt{3}\right)$ around a submanifold W^{2n-k}, $2\leq k\leq n-1$	Impossible	
$\mathbf{g=4}$	Impossible	Tube of radius $r\neq\frac{1}{\sqrt{-c}}\ln\left(2+\sqrt{3}\right)$ around a submanifold W^{2n-k}, $2\leq k\leq n-1$	Tube around a submanifold W_{φ}^{2n-2}, $0<\varphi<\frac{\pi}{2}$ *Not yet classified*	?
$\mathbf{g=5}$	Impossible	Impossible	Tube around a submanifold W_{φ}^{2n-k}, $0<\varphi<\frac{\pi}{2}$, k even, $4\leq k\leq n-1$ *Not yet classified*	?
$\mathbf{g\geq 6}$	Impossible	Impossible	?	?

isomorphic to the $(2n-1)$-dimensional Heisenberg algebra. Moreover, $\mathfrak{a}\oplus\mathfrak{n}$ is a solvable Lie algebra. The direct sum decomposition $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{a}\oplus\mathfrak{n}$ is called the Iwasawa decomposition of the Lie algebra $\mathfrak{g}$ with respect to $\mathfrak{a}$ and the choice of α as a positive root. We emphasize that this is only a direct sum of vector spaces and not a decomposition at the Lie algebra level.

Let A, N and AN be the connected subgroups of G with Lie algebras $\mathfrak{a}$, $\mathfrak{n}$ and $\mathfrak{a}\oplus\mathfrak{n}$, respectively. These three groups are simply connected and the Iwasawa decomposition of $\mathfrak{g}$ induces an Iwasawa decomposition of G, as the Cartesian product $K\times A\times N$. Again, by this we just mean that G is diffeomorphic to that Cartesian product, and not that G is isomorphic to the direct product of the groups K, A and N. From this Iwasawa decomposition it follows that the simply connected solvable Lie group AN acts simply transitively on $\mathbb{C}H^n$. Thus, we can identify $\mathfrak{a}\oplus\mathfrak{n}$ with the tangent space $T_o\mathbb{C}H^n$, and the group AN with the complex hyperbolic space $\mathbb{C}H^n$. The Riemannian metric of $\mathbb{C}H^n$ induces a left-invariant metric on AN which makes AN isometric to $\mathbb{C}H^n$. Similarly, the complex structure J on $\mathbb{C}H^n$ induces a complex structure on $\mathfrak{a}\oplus\mathfrak{n}$ and on AN. We will denote these structures also by J. It is then possible to prove that $\mathfrak{g}_\alpha$ is J-invariant and $J\mathfrak{a}=\mathfrak{g}_{2\alpha}$.

Altogether, we got a model for the complex hyperbolic space as a solvable Lie group AN with a left-invariant metric, which turns out to be very useful for the study of some properties of $\mathbb{C}H^n$. In our case, this model and the constructions explained above will allow us to define the Berndt-Brück submanifolds, as we are going to see in the rest of this section.

Let $\mathfrak{w}$ be a vector subspace of the root space $\mathfrak{g}_\alpha$, such that its orthogonal complement $\mathfrak{w}^\perp=\mathfrak{g}_\alpha\ominus\mathfrak{w}$ in $\mathfrak{g}_\alpha$ has constant Kähler angle $\varphi\in(0,\pi/2]$. This means that, for all nonzero $v\in\mathfrak{w}^\perp$, the angle between Jv and $\mathfrak{w}^\perp$ is φ or, equivalently, the projection of Jv onto $\mathfrak{w}^\perp$ has length $\cos(\varphi)\,\|v\|$. A particular case is when $\varphi=\pi/2$, which means that $\mathfrak{w}^\perp$ is real, that is, $J\mathfrak{w}^\perp$ is orthogonal to $\mathfrak{w}^\perp$.

Now define $\mathfrak{s}=\mathfrak{a}\oplus\mathfrak{w}\oplus\mathfrak{g}_{2\alpha}$. According to the properties of the root space decomposition, $\mathfrak{s}$ is a Lie subalgebra of $\mathfrak{a}\oplus\mathfrak{n}$. Let us denote by S the connected subgroup of AN with Lie algebra $\mathfrak{s}$, and set $k=\dim\mathfrak{w}^\perp$. The group S is a simply connected closed subgroup of AN of dimension $2n-k$. We define the Berndt-Brück submanifolds as the orbits through the point o of the isometric action of S on $\mathbb{C}H^n$:

$$W^{2n-k}_\varphi=S\cdot o\qquad\text{and}\qquad W^{2n-k}=W^{2n-k}_{\pi/2}.$$

These are $(2n-k)$-dimensional homogeneous submanifolds of $\mathbb{C}H^n$ and their normal bundles have constant Kähler angle $\varphi\in(0,\pi/2]$. In particular, if $\varphi=\pi/2$, one gets the submanifold W^{2n-k}, which has totally real normal bundle. One can also give a geometric construction of Berndt-Brück submanifolds [7]. In particular, for the case $\varphi=\pi/2$, the construction is as follows. Fix a horosphere $\mathcal{H}$ in a totally geodesic real hyperbolic subspace $\mathbb{R}H^{k+1}\subset\mathbb{C}H^n$. Attach at each point $p\in\mathcal{H}$ the totally geodesic $\mathbb{C}H^{n-k}$ which is tangent to the orthogonal complement of the complex span of the tangent space of $\mathcal{H}$ at p. The resulting submanifold is congruent to W^{2n-k}. Moreover, the submanifolds W^{2n-k}_φ are minimal and ruled by the totally geodesic complex hyperbolic subspaces determined by their maximal holomorphic tangent distribution.

The Berndt-Brück submanifolds are orbits of cohomogeneity one actions on $\mathbb{C}H^n$. This was proved in [4]. Although the proof is not elementary, we can sketch an idea. Let $N^0_K(S)$ be the connected component of the identity transformation of the normalizer of S in K, $N_K(S) = \{k \in K : kSk^{-1} \subset S\}$, which consists of all the elements of K that fix $S \cdot o$. Therefore, $S \cdot o$ is an orbit of the action of $N_K^0(S)S$ on $\mathbb{C}H^n$. It follows that $N_K^0(S)$ leaves invariant the unit sphere of the normal bundle of $S \cdot o$, because the tangent bundle is also invariant under $N_K^0(S)$. To conclude that the action of $N_K^0(S)S$ on $\mathbb{C}H^n$ is of cohomogeneity one, it remains to see that $N_K^0(S)$ acts transitively on the unit sphere of the normal bundle of $S \cdot o$. This can be consulted in [4]. We conclude that $W_\varphi^{2n-k} = N_K^0(S)S \cdot o = S \cdot o$ is the orbit through o of the cohomogeneity one action of $N_K^0(S)S$ on $\mathbb{C}H^n$. In particular, when $k = 1$, then $\varphi = \pi/2$, and the orbits of this action generate a codimension one homogeneous foliation.

As a result of the previous argument, tubes around the submanifolds W_φ^{2n-k}, $k \in \{2, \dots, n-1\}$, and equidistant hypersurfaces to the hypersurface W^{2n-1} are principal orbits of cohomogeneity one actions and, hence, homogeneous hypersurfaces. All these hypersurfaces are non-Hopf real hypersurfaces with constant principal curvatures in $\mathbb{C}H^n$. If $\varphi = \pi/2$, these hypersurfaces have $g \in \{3, 4\}$ principal curvatures and their Hopf vector field has nontrivial projection onto $h = 2$ principal curvature spaces. If $\varphi \in (0, \pi/2)$, tubes around W_φ^{2n-k} have $g \in \{4, 5\}$ principal curvatures and $h = 3$ nontrivial projections of the Hopf vector field onto the principal curvature spaces. The proof of these facts can be found in [7].

By means of the Main Theorem stated in the first section of this work, tubes around the Berndt-Brück submanifolds W^{2n-k} with totally real normal bundle (equidistant hypersurfaces if $k = 1$) exhaust all the examples of real hypersurfaces with constant principal curvatures in nonflat complex space forms satisfying $h = 2$.

5. Open problems

To conclude, we state some open problems on homogeneous hypersurfaces and hypersurfaces with constant principal curvatures in complex space forms.

- Are there inhomogeneous real hypersurfaces with constant principal curvatures in complex space forms? If the answer is yes, these examples would satisfy $g \geq 4$ due to the results of Montiel, Berndt and Díaz-Ramos, and $h \geq 3$, due to the classifications of Kimura, Berndt and the Main Theorem.
- Find a bound on the number of principal curvatures g of a hypersurface with constant principal curvatures in $\mathbb{C}P^n$ or $\mathbb{C}H^n$.
- Find a bound on h for a hypersurface with constant principal curvatures in $\mathbb{C}P^n$ or $\mathbb{C}H^n$. This seems to be even better than a bound of g, with the aim to achieve a final classification.
- Classify real hypersurfaces with constant principal curvatures in complex space forms.

We refer to [21] for a more extensive list of problems on real hypersurfaces in complex space forms.

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