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On isoparametric hypersurfaces in complex hyperbolic spaces

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Abstract

We review some results of an ongoing research on isoparametric hypersurfaces and hypersurfaces with constant principal curvatures in the complex hyperbolic space. In order to motivate this topic, we recall first the main steps of its long history.

1 Introduction

The history of isoparametric hypersurfaces traces back (at least) to the work [42] of Somigliana in 1919, where the following problem of Geometric Optics was studied. Consider a solution φ to the wave equation

$$\Delta\varphi = \frac{\partial^2\varphi}{\partial t^2},$$

where Δ is the Laplace operator of $\mathbb{R}^3$ (that is, with respect to the space variables), and t is the time variable. Assume that the connected components of the level surfaces of φ (in other words, the wavefronts of φ) are parallel. Somigliana refers to this condition as Huygens principle. What are then the possible wavefronts? He then showed that these level surfaces must have constant mean curvature, and from this, he deduced that only very particular wavefronts satisfy this kind of Huygens principle, namely: concentric spheres, coaxial cylinders and parallel planes.

The term *isoparametric hypersurface* was probably introduced by Levi-Civita [30] in the year 1937, and it is motivated by a classical terminology that we explain now. Let $f\colon M \to \mathbb{R}$ be a smooth function, where M is a

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Riemannian manifold (in [30], $M = \mathbb{R}^3$). The *first* and the *second differential parameters* of f are, respectively,

$$\Delta_1 f = \|\operatorname{grad} f\|^2 \quad \text{and} \quad \Delta_2 f = \Delta f,$$

where Δ is the Laplace-Beltrami operator and $\operatorname{grad} f$ denotes the gradient of f. When the first and the second differential parameters of a nonconstant function f are constant along the level sets of f, we say that f is an *isoparametric function.* Its regular level sets are then called *isoparametric hypersurfaces*, and the collection of all the level sets of f is called an *isoparametric family of hypersurfaces.* Note that f is isoparametric if and only if there exist real functions F_1 and F_2 of real variable such that

$$\Delta_1 f = F_1(f) \quad \text{and} \quad \Delta_2 f = F_2(f).$$

The constancy of the first differential parameter along the level sets means that the level sets are *parallel* (equidistant), while for the second differential parameter the condition means that these level sets have *constant mean curvature.* In fact, Cartan showed that a hypersurface is isoparametric if and only if it and its sufficiently close parallel hypersurfaces have constant mean curvature [10]. Since this characterization holds in every Riemannian manifold, it is sometimes taken as definition of isoparametric hypersurface.

The study of isoparametric hypersurfaces has today a long history which has revealed many connections with different areas of mathematics, such as Riemannian geometry, but also Lie group theory, algebraic geometry, algebraic topology, differential equations and Hilbert spaces; even some applications in physics have been found (for instance, see [40] for the appearance of isoparametric hypersurfaces in a problem of fluid mechanics). Although in Section 2 we will review some of the most important results on isoparametric hypersurfaces, our exposition here does not attempt to be complete. For a more detailed introduction to this topic and other related subjects (such as isoparametric submanifolds of higher codimension, equifocal submanifolds, Dupin hypersurfaces and polar actions), we refer the reader to the excellent surveys [46], [12] and [47] and to the books [39] and [5].

Our purpose in this work is to introduce the reader to the topic of isoparametric hypersurfaces and related notions (homogeneous hypersurfaces, and hypersurfaces with constant principal curvatures) in complex hyperbolic spaces. We will focus on explaining some results on this subject obtained recently by Díaz-Ramos and the author. More specifically, we will present some restrictions on the extrinsic geometry of isoparametric hypersurfaces in complex hyperbolic spaces that we have obtained in the ongoing research [18], and also a large new family of examples constructed in [16].

This paper is organized as follows. In Section 2 we recall the main classical and modern results on isoparametric hypersurfaces in real space forms. The differences between spaces of constant and nonconstant sectional curvature with respect to the study of isoparametric hypersurfaces are explained in Section 3. In Section 4, we recall some terminology for the study of real hypersurfaces in complex space forms. In Section 5, we explain some recent results on isoparametric hypersurfaces in complex hyperbolic spaces. Finally, we provide a list of open problems on these topics in Section 6.

2 Isoparametric hypersurfaces in real space forms

In this section, we will give an idea of the main aspects of the history of isoparametric hypersurfaces in the ambient manifolds where their study was first developed: in real space forms. Recall that a real space form is a complete simply connected Riemannian manifold with constant sectional curvature, that is, Euclidean spaces $\mathbb{R}^n$, spheres S^n or real hyperbolic spaces $\mathbb{R}H^n$.

2.1 The Euclidean and real hyperbolic cases

In the paper [30] published in 1937, Levi-Civita classified isoparametric hypersurfaces in $\mathbb{R}^3$. He was probably not aware that a similar result had been obtained almost two decades ago by Somigliana [42]. In 1938, Segre [41] explains that one can extend the results of [42] and [30] to Euclidean spaces $\mathbb{R}^n$ of arbitrary dimension. Segre shows that isoparametric hypersurfaces in $\mathbb{R}^n$ have constant principal curvatures (i.e. the eigenvalues of the shape operator are independent of the point in the hypersurface), he proves that there are at most two principal curvatures and from this he derives a complete classification, in which there are again three types of examples: concentric spheres, generalized coaxial cylinders (i.e. tubes around an affine subspace of dimension at least 1) or parallel hyperplanes.

In the late thirties, Cartan also addressed the study of isoparametric hypersurfaces. In [9] he characterized isoparametric hypersurfaces in real space forms by the property of having *constant principal curvatures*. This equivalence turns out to be very helpful in the investigation of isoparametric hypersurfaces, and sometimes even the constancy of the principal curvatures is taken as the definition of isoparametric hypersurface. However, this should only be done in real space forms, since in spaces of nonconstant curvature both notions are different, as we will see in Section 3.

From now on in this section, we will denote by g the number of distinct constant principal curvatures of an isoparametric hypersurface, $\lambda_1, \dots, \lambda_g$ will be the values of the principal curvatures, and $m_1, \dots, m_g$ their corresponding multiplicities. Cartan derived the following *fundamental formula* of an isoparametric hypersurface in a real space form of constant sectional curvature κ:

$$\sum_{j=1,\,\lambda_j \neq \lambda_i}^{g} m_j \frac{\kappa + \lambda_i \lambda_j}{\lambda_i - \lambda_j} = 0, \quad \text{for each } i = 1, \dots, g.$$

From this relation, it is easy to show that if $\kappa \leq 0$, then $g \in \{1, 2\}$. Using this fact, Cartan was able to classify isoparametric hypersurfaces in real hyperbolic spaces $\mathbb{R}H^n$. The examples that appear in this classification are: geodesic spheres, totally geodesic real hyperbolic hyperspaces $\mathbb{R}H^{n-1}$ and their equidistant hypersurfaces, tubes around totally geodesic real hyperbolic subspaces $\mathbb{R}H^k$ $(1 \leq k \leq n-2)$ and horospheres. Since it will appear frequently along the exposition, we recall here the definition of tube and equidistant hypersurface. Given an embedded submanifold M of an ambient Riemannian manifold $\bar{M}$, a *tube* of radius r around M is the set of points $\{\exp_p(r\xi) : p \in M,\, \xi \in \nu_p M,\, \|\xi\| = 1\}$, where exp is the Riemannian exponential map of $\bar{M}$ and where $\nu_p M$ denotes the normal space of M at $p \in M$. Locally and for r sufficiently small, the tube of radius r around M is an embedded hypersurface in $\bar{M}$. When M is a hypersurface, tubes around M are normally called *equidistant* or *parallel hypersurfaces* to M, in case they are embedded hypersurfaces.

2.2 The problem in spheres

Cartan also investigated isoparametric hypersurfaces in spheres. In this case, since $\kappa > 0$, the fundamental formula does not provide much information. In fact, the problem in spheres is much more involved and rich. Cartan was able to classify isoparametric hypersurfaces in spheres S^n with $g \in \{1, 2, 3\}$. The examples with $g = 1$ are just geodesic spheres, while those with $g = 2$ are tubes around totally geodesic submanifolds S^k of S^n with $1 \leq k \leq n-2$. For $g = 3$, Cartan showed that all three multiplicities m_i are equal, and one has $m = m_1 = m_2 = m_3 \in \{1, 2, 4, 8\}$. He also proved that the corresponding isoparametric hypersurfaces are tubes around the standard embedding of the projective plane $\mathbb{F}P^2$ in S^{3m+1}, where $\mathbb{F}$ is the division algebra $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ (quaternions) or $\mathbb{O}$ (octonions), for $m = 1, 2, 4, 8$, respectively. Cartan also found two examples of isoparametric hypersurfaces with four principal curvatures in S^5 and in S^9, but he could get neither a classification for $g \geq 4$, nor an upper bound on g (as for $\mathbb{R}^n$ and $\mathbb{R}H^n$).

However, Cartan noticed that all the isoparametric hypersurfaces he constructed were homogeneous. We say that a connected submanifold M in an ambient manifold $\bar{M}$ is *(extrinsically) homogeneous* if it is an orbit of an isometric action on $\bar{M}$, i.e. if there exists a (in general, closed) subgroup G of the isometry group of $\bar{M}$ that acts on $\bar{M}$ having M as one of its orbits. When the minimal codimension of the orbits of an isometric action is one, we have a *cohomogeneity one action* and these orbits of codimension one are called *homogeneous hypersurfaces*. It is clear that every homogeneous hypersurface (or open part of it) has constant principal curvatures, since the shape operators at any two points are conjugate by the differential of an element of the group G that acts on the ambient manifold with cohomogeneity one. Moreover, the codimension one orbits of a cohomogeneity one action are equidistant, since the action is isometric. Since each one of these codimension one orbits has constant principal curvatures (in particular, constant mean curvature) and are equidistant, they are always isoparametric hypersurfaces. Therefore, the set of orbits of a cohomogeneity one action in a Riemannian manifold $\bar{M}$ forms an isoparametric family of hypersurfaces with constant principal curvatures. As all isoparametric hypersurfaces known to Cartan were homogeneous (those in spheres, but also those in $\mathbb{R}^n$ and $\mathbb{R}H^n$), he asked the question whether every isoparametric hypersurface is extrinsically homogeneous. A surprising negative answer would only come several decades later.

The study of isoparametric hypersurfaces was taken up again in the early seventies. Nomizu [36] shows that the focal manifolds of an isoparametric family of hypersurfaces in a sphere are minimal; the *focal manifolds* of an isoparametric family are those elements of the family with codimension greater than one. About that time and based on the work [27] of Hsiang and Lawson, Takagi and Takahashi gave the classification of homogeneous (isoparametric) hypersurfaces in spheres [45]. According to this result, every homogeneous hypersurface in a sphere is a principal orbit (or, equivalently in this case, an orbit of maximal dimension) of the isotropy representation of a Riemannian symmetric space of rank two. Let us briefly recall what this means. A Riemannian *symmetric space* is a connected Riemannian manifold M for which the geodesic reflection $\exp_o(tv) \mapsto \exp_o(-tv)$ (where $o \in M$, v runs through T_oM and t runs through $\mathbb{R}$) is a well-defined isometry of M. This implies that M is a complete homogeneous space. Then it admits a representation as homogeneous space G/K, where G is the identity connected component of the isometry group of M, and $K = G_o$ is the isotropy group at some point o. G and K are Lie groups, and K is a compact Lie subgroup of G. The group K acts on M and fixes o, and hence induces an action on T_oM, which is called the *isotropy representation* of the symmetric space $M = G/K$. The *rank* of G/K is defined as the maximal dimension of a totally geodesic flat submanifold of G/K. Riemannian symmetric spaces have been classified (also by Cartan); a classical reference on this topic is [26] (tables can be found in [26, pp. 515–520]).

A consequence of Takagi and Takahashi's work is that the number of principal curvatures g of a homogeneous hypersurface in a sphere satisfies $g \in \{1, 2, 3, 4, 6\}$. In two remarkable articles [33], [34] (that were written around 1973, but were published in 1980-1981), Münzner was able to prove that the same restriction on g holds for every (not necessarily homogeneous) isoparametric hypersurface in a sphere. Münzner's papers contain a deep analysis of the structure of isoparametric families of hypersurfaces in spheres, using both geometric and topological methods. Apart from the restriction on g, we emphasize other two consequences of Münzner's work. The first one is that, if $\lambda_1 < \cdots < \lambda_g$ are the principal curvatures of an isoparametric hypersurface in a sphere, and $m_1, \ldots, m_g$ their corresponding multiplicities, then $m_i = m_{i+2}$ (indices modulo g); in particular, if g is odd, all the multiplicities coincide, and if g is even, there are at most two different multiplicities. The second result is the algebraic character of isoparametric hypersurfaces in spheres. More precisely, a hypersurface M in S^n is isoparametric if and only if $M \subset F^{-1}(c) \cap S^n$, where F is a homogeneous polynomial of degree g on $\mathbb{R}^{n+1}$ satisfying the differential equations

$$\begin{aligned} &\| \operatorname{grad} F(x)\|^2 = g^2\|x\|^{2g-2}, \\ &\Delta F(x) = \frac{1}{2}(m_2 - m_1)g^2\|x\|^{g-2}, \qquad x \in \mathbb{R}^{n+1}. \end{aligned}$$

The intersection of S^n with the level sets of such an F form an isoparametric family of hypersurfaces in S^n. From this result, it also follows that every isoparametric hypersurface in S^n is an open part of a complete isoparametric hypersurface in S^n (this happened for $\mathbb{R}^n$ and $\mathbb{R}H^n$ as well). A polynomial F like the one above is called a *Cartan-Münzner polynomial*. Notice that, according to this result, the classification problem of isoparametric hypersurfaces in spheres is reduced to a problem of algebraic geometry, but a very difficult one!

Since the restriction on g obtained by Münzner coincides with the one for homogeneous hypersurfaces, Cartan's question on the homogeneity of isoparametric hypersurfaces became even more attractive. However, in 1975

Ozeki and Takeuchi gave a negative answer to this question [37]. They constructed some Cartan-Münzner polynomials that give rise to isoparametric hypersurfaces with $g = 4$ that are not homogeneous, because their multiplicities do not coincide with the possible multiplicities of the homogeneous examples (remember that these had been classified).

Some years later, Ferus, Karcher and Münzner [22] found a much larger family of inhomogeneous examples that included the ones given by Ozeki and Takeuchi. For each representation of a Clifford algebra they constructed a Cartan-Münzner polynomial that yields an isoparametric family of hypersurfaces with $g = 4$. We call these examples *of FKM-type* or *of Clifford type*. Their inhomogeneity was proved in [22] in a direct way, without using the classification of homogeneous hypersurfaces. As a consequence of this result, one gets the existence of an infinite countable collection of non-congruent inhomogeneous isoparametric families in spheres. This made the study of isoparametric hypersurfaces in spheres a much more appealing and interesting topic of research.

Even today, all known isoparametric hypersurfaces in spheres are either homogeneous or of FKM-type; and all those hypersurfaces with $g = 4$ are of FKM-type, with the exception of two homogeneous families of hypersurfaces with multiplicities $(2, 2)$ and $(4, 5)$. A first step towards a classification would be to determine the possible triples (g, m_1, m_2) that an isoparametric hypersurface with $g = 4$ or $g = 6$ can take. Several authors have contributed to this question (we just mention some of them, and refer to the surveys [46] and [12] for further references). In [33] and [34], Münzner already found some restrictions, which were improved by Abresch [1]. In particular, Abresch showed that the only possible triples with $g = 6$ are $(6, 1, 1)$ and $(6, 2, 2)$; moreover, there exist homogeneous examples in both cases. The determination of all possible triples with $g = 4$ was established by Stolz in 1999 [43]. He proved that every isoparametric hypersurface with $g = 4$ constant principal curvatures in a sphere has the multiplicities of one of the known homogeneous or inhomogeneous examples; in other words, the possible triples $(4, m_1, m_2)$ are $(4, 2, 2)$, $(4, 4, 5)$ and the ones of FKM-type hypersurfaces (which can be consulted in [22]).

As we mentioned before, isoparametric hypersurfaces in spheres with $g \in \{1, 2, 3\}$ had been classified by Cartan. In 1976, Takagi [44] showed that if $g = 4$ and one of the multiplicities is one, then the hypersurface is homogeneous and of FKM-type. Ozeki and Takeuchi [38] proved that those isoparametric hypersurfaces with $g = 4$ and one multiplicity equal to 2 are homogeneous and, except for the case of multiplicities $(2, 2)$ (which corresponds to the homogeneous example in S^9 obtained by Cartan), also of FKM-type. In 1985, Dorfmeister and Neher [20] proved the uniqueness of the hypersurface with triple $(6,1,1)$, which is hence homogeneous. Quite recently, in 2007-2008, Cecil, Chi and Jensen [11], and independently Immervoll [28], proved that, with a few possible exceptions, every isoparametric hypersurface with $g = 4$ is one of the known examples. More precisely, if the multiplicities (m_1, m_2) of an isoparametric hypersurface with $g = 4$ in a sphere satisfy $m_2 \geq 2m_1 - 1$, then such hypersurface must be of FKM-type. Together with other known results, this one gives a classification of the case $g = 4$ with the exception of the pairs of multiplicities $(3,4)$, $(4,5)$, $(6,9)$ and $(7,8)$. The methods used in both articles are different: while Cecil, Chi and Jensen make use of the theory of moving frames and commutative algebra, Immervoll uses the tool of isoparametric triple systems developed by Dorfmeister and Neher [20]. In the last years, on the one hand, after Chi's investigation of the exceptional cases with $g = 4$, the classification in this case seems not to be very far; see [13] for more details. On the other hand, the approach used by Miyaoka in [32] to reprove the homogeneity of the hypersurface with triple $(6,1,1)$ might be used to solve the case $(6,2,2)$, although this question seems to be open as well. A solution to these two particular cases would give a complete classification of isoparametric hypersurfaces in spheres, and hence, a solution to Problem 34 in Yau's list of important problems in geometry [50].

3 Isoparametric hypersurfaces vs. hypersurfaces with constant principal curvatures

As shown by Cartan, in a space of constant curvature, an isoparametric hypersurface is the same as a hypersurface with constant principal curvatures. However, this equivalence does not hold in general, as we will comment on in this section.

The first counterexamples were found by Wang [48], who constructed some isoparametric hypersurfaces with nonconstant principal curvatures in the complex projective space $\mathbb{C}P^n$, by projecting some of the inhomogeneous isoparametric hypersurfaces of FKM-type in odd-dimensional spheres S^{2n+1} to $\mathbb{C}P^n$ via the Hopf map. Other inhomogeneous isoparametric hypersurfaces with nonconstant principal curvatures in complex projective spaces were constructed by Xiao [49] (cf. [23]); these examples are again related to the isoparametric hypersurfaces in spheres.

Another large set of examples is given by small geodesic spheres in the non-symmetric *Damek-Ricci spaces*. These are certain solvable Lie groups endowed with a left-invariant metric which are harmonic as Riemannian manifolds; they were constructed by Damek and Ricci [14] in 1992. One characterization of harmonicity is that sufficiently small geodesic spheres have constant mean curvature, and hence, are isoparametric. The family of Damek-Ricci spaces includes the Riemannian symmetric spaces of noncompact type and rank one as particular cases (these are precisely real, complex and quaternionic hyperbolic spaces $\mathbb{R}H^n$, $\mathbb{C}H^n$ and $\mathbb{H}H^n$, and the Cayley hyperbolic plane $\mathbb{O}H^2$). However, for those non-symmetric Damek-Ricci spaces, the small geodesic spheres have nonconstant principal curvatures, in spite of being isoparametric (see [8, §4.5]).

Recently, Díaz-Ramos and the author have constructed many inhomogeneous isoparametric hypersurfaces with nonconstant principal curvatures in the complex hyperbolic space [16] and, more generally, in Damek-Ricci spaces [17]. In Section 5 we will recall the construction of the examples in complex hyperbolic spaces.

In view of all these examples, two different properties of hypersurfaces in ambient manifolds of nonconstant curvature generalize in a natural way the property of being an isoparametric hypersurface in a real space form: the original notion of isoparametric hypersurface, but also the notion of hypersurface with constant principal curvatures. The study of hypersurfaces with constant principal curvatures, particularly in complex space forms, has been a fruitful area of research in the last decades. We refer the reader to the surveys [3] and [19] for further information on this topic. In this work we will focus on the study of isoparametric hypersurfaces in complex hyperbolic spaces.

4 Real hypersurfaces in complex space forms

In this section we present some basic definitions and notation for the study of real hypersurfaces in complex space forms. A thorough introduction to this topic can be found in [35].

First of all, recall that a complex space form is a simply connected complete Kähler manifold with constant holomorphic sectional curvature. These manifolds are classified in three families according to the value of their constant holomorphic sectional curvature c: complex projective spaces $\mathbb{C}P^n$ if $c > 0$, complex Euclidean spaces $\mathbb{C}^n$ if $c = 0$ and complex hyperbolic spaces $\mathbb{C}H^n$ if $c < 0$. We will denote by J the almost complex structure of a complex space form. In what follows we disregard the flat case, as well as $\mathbb{C}P^1$ (which is isometric to a 2-sphere) and $\mathbb{C}H^1$ (which is isometric to $\mathbb{R}H^2$).

The metric on $\mathbb{C}P^n$ (resp. on $\mathbb{C}H^n$) can be obtained by requiring that the Hopf map $S^{2n+1} \to \mathbb{C}P^n$ (resp. $\mathrm{AdS}^{2n+1} \to \mathbb{C}H^n$) is a Riemannian submersion (resp. a semi-Riemannian submersion). By AdS^m we denote the m-dimensional anti De Sitter space, i.e. the Lorentzian space form of constant negative sectional curvature. The metrics on $\mathbb{C}P^n$ and $\mathbb{C}H^n$ are called the Fubini-Study and the Bergman metric, respectively. The curvature tensor of these spaces can be obtained easily and admit a rather simple expression (see [35] for the complete construction).

As for any other Kähler manifolds, one would have in principle two natural notions of hypersurface in a complex space form: either a complex hypersurface or a real hypersurface. We are interested in the latter, which refers to a submanifold with real codimension one (and not complex codimension one).

Let M be a real hypersurface in a complex space form $\bar{M}$, and let ξ be a unit normal vector field on (an open part of) M. Then the vector field $J\xi$ is tangent to M, and is called the *Hopf vector field* of the hypersurface M (also the *Reeb vector field* or the *structure vector field* of M). We will denote by $g(p)$ the number of principal curvatures of the hypersurface M at the point $p \in M$; since we are not assuming that M has constant principal curvatures, g may vary from point to point, whence the notation $g(p)$.

Another notation that will be relevant later is the following. For each point $p \in M$ we will write $h(p)$ for the number of nontrivial projections of the Hopf vector field $J\xi$ onto the distinct principal curvature spaces at p (that is, onto the distinct eigenspaces of the shape operator of M at p). Again, as well as g, h is an integer-valued function on M. Note that, obviously, $h(p) \leq g(p)$ for each $p \in M$. When $h = 1$ along M, that is, when $J\xi$ is an eigenvector of the shape operator at every point, we say that M is a *Hopf hypersurface.*

A related notion is that of *curvature-adapted* hypersurface, which refers to a hypersurface whose shape operator and normal Jacobi operator commute. Recall that the normal Jacobi operator of a hypersurface M in an ambient manifold $\bar{M}$ is the self-adjoint (local) $(1,1)$-tensor field on M defined by $\bar{R}(\cdot,\xi)\xi$, where $\bar{R}$ is the $(1,3)$-curvature tensor field of $\bar{M}$. In real space forms, every hypersurface is curvature-adapted, but in spaces of nonconstant curvature, the curvature-adaptedness imposes restrictions on the geometry of a hypersurface. For instance, in nonflat complex space forms, a hypersurface is curvature-adapted if and only if it is Hopf.

The curvature-adaptedness is quite a common condition in the research on hypersurfaces in spaces of nonconstant curvature, because it simplifies the Gauss and Codazzi equations of the hypersurface and, however, it still allows to obtain some interesting examples. See for example [29] and [2], where Kimura and Berndt classify Hopf real hypersurfaces with constant principal curvatures in $\mathbb{C}P^n$ and $\mathbb{C}H^n$, respectively. When one does not assume curvature-adaptedness, calculations get harder, but one still obtains examples. See for instance the recent work [15] by Díaz-Ramos and the author, where we classified non-Hopf real hypersurfaces with constant principal curvatures satisfying $h = 2$ in $\mathbb{C}P^n$ and $\mathbb{C}H^n$. This result can be seen as the next natural step after Kimura's and Berndt's classifications of the case $h = 1$. While in $\mathbb{C}P^n$ no new examples appear, the classification in $\mathbb{C}H^n$ includes some non-classical examples constructed by Berndt and Brück [4]. We will come back to these results and examples in the next section.

5 Isoparametric hypersurfaces in complex hyperbolic spaces

Our purpose in this section is to explain some recent advances in the study of isoparametric real hypersurfaces in complex hyperbolic spaces. We will state some partial results on the extrinsic geometry and classification of these objects [18], and explain the construction of new examples [16].

5.1 On the extrinsic geometry of isoparametric hypersurfaces in $\mathbb{C}H^n$

The nice behaviour of isoparametric hypersurfaces with respect to Riemannian submersions under certain conditions is quite a well-known fact; see [25, §3], which deals with the more general case of isoparametric submanifolds. For the Hopf map $\pi\colon \mathrm{AdS}^{2n+1} \to \mathbb{C}H^n$, which is a semi-Riemannian submersion, one can easily show that a hypersurface M in $\mathbb{C}H^n$ is isoparametric if and only if its lift $\pi^{-1}M$ is isoparametric in the anti De Sitter space AdS^{2n+1}. Note that $\pi^{-1}M$ is a Lorentzian hypersurface in AdS^{2n+1}, since it is foliated by Hopf circles, which are timelike. Although we have defined isoparametric hypersurfaces only in the Riemannian context, it makes sense [24] to consider the analogous notion in the semi-Riemannian setting, just restricting oneself to nondegenerate hypersurfaces (that is, hypersurfaces whose induced metric is nondegenerate). Since the anti De Sitter space has constant sectional curvature, a hypersurface in AdS^{2n+1} is isoparametric if and only if it has constant principal curvatures (see [24] again). Therefore, a hypersurface M in $\mathbb{C}H^n$ is isoparametric if and only if its lift $\pi^{-1}M$ to AdS^{2n+1} has constant principal curvatures.

This fact is fundamental for the proof of the following result. Recall that g denotes the number of principal curvatures of a hypersurface M in $\mathbb{C}H^n$, and h the number of nontrivial projections of the Hopf vector field onto the different eigenspaces of the shape operator of M.

Theorem 5.1. [18] *Let M be an isoparametric hypersurface in the complex hyperbolic space and $p \in M$. Then, the principal curvatures of M at p and their multiplicities coincide pointwise with those of the homogeneous hypersurfaces in complex hyperbolic spaces.*

In particular, $h(p) \in \{1, 2, 3\}$, $g(p) \in \{2, 3, 4, 5\}$ and:

- *If $h(p) = 1$ then $g(p) \in \{2, 3\}$.*
- *If $h(p) = 2$ then $g(p) \in \{2, 3, 4\}$.*
- *If $h(p) = 3$ then $g(p) \in \{3, 4, 5\}$.*

Several observations are in order.

Remark 5.1. *Theorem 5.1 is not a local result, but a pointwise result. The principal curvatures and the values of g and h may vary from point to point.*

Remark 5.2. *The cases $h(p) = g(p) = 2$ and $h(p) = g(p) = 3$ do not arise in the known isoparametric examples, but so far we could not prove that these cases are not possible.*

Remark 5.3. *Homogeneous hypersurfaces in complex hyperbolic spaces have been classified by Berndt and Tamaru [7]; the values of their principal curvatures and multiplicities can be found in [6]. Apart from the "classical" examples that appeared already in the classification of Hopf real hypersurfaces with constant principal curvatures in $\mathbb{C}H^n$ (namely, geodesic spheres, tubes around totally geodesic complex hyperbolic subspaces $\mathbb{C}H^k$ with $k \in \{1, \dots, n-1\}$, tubes around totally geodesic real hyperbolic subspaces $\mathbb{R}H^n$, and horospheres), there are many more non-Hopf homogeneous hypersurfaces. These "non-classical" examples were introduced by Lohnherr in his thesis (see [31]) and generalized by Berndt and Brück in [4], and will arise as particular cases of a more general construction that will be explained later in this section.*

According to Theorem 5.1 the value of h is always less or equal than 3 for every isoparametric hypersurface in $\mathbb{C}H^n$. Based on [2] and [15], we were able to show the following classification result for isoparametric hypersurfaces with $h \leq 2$.

Theorem 5.2. [18] *Let M be a connected isoparametric hypersurface in $\mathbb{C}H^n$ with $h \leq 2$ nontrivial projections of the Hopf vector field onto the principal curvature spaces. Then h is constant and M is an open part of a homogeneous hypersurface in $\mathbb{C}H^n$. Moreover:*

- *If $h = 1$, then M is an open part of one of the following hypersurfaces:*
  - *A tube around a totally geodesic $\mathbb{C}H^k$, $0 \leq k \leq n-1$.*
  - *A tube around a totally geodesic $\mathbb{R}H^n$.*
  - *A horosphere.*
- *If $h = 2$, then M is an open part of one of the following hypersurfaces:*
  - *A Lohnherr hypersurface W^{2n-1}.*
  - *An equidistant hypersurface to W^{2n-1}.*
  - *A tube around a Berndt-Brück submanifold W^{2n-k}, $2 \leq k \leq n-1$.*

The only homogeneous hypersurfaces of $\mathbb{C}H^n$ not appearing in this partial classification are tubes around the Berndt-Brück submanifolds W_φ^{2n-k}, where $k \in \{2, \dots, n-1\}$ is even and $\varphi \in (0, \pi/2)$. These hypersurfaces have $h = 3$. We will describe these examples below in this section.

In view of Theorems 5.1 and 5.2 it would seem reasonable to suspect that (complete) isoparametric hypersurfaces in $\mathbb{C}H^n$ are homogeneous, as happened in $\mathbb{R}H^n$. However, this is not true. In the rest of this section we aim to present a large collection of counterexamples: inhomogeneous isoparametric hypersurfaces with nonconstant principal curvatures in complex hyperbolic spaces $\mathbb{C}H^n$, $n \geq 3$.

5.2 $\mathbb{C}H^n$ as a solvable Lie group

In order to introduce the new examples, we will recall first the construction of a model of the complex hyperbolic space $\mathbb{C}H^n$ as a solvable Lie group AN equipped with a left-invariant metric. This model is not exclusive to $\mathbb{C}H^n$: every symmetric space of noncompact type is a solvable Lie group and its metric is left-invariant with respect to the Lie group structure. The proof of this general fact, which is based on the Iwasawa decomposition of the noncompact symmetric space, follows along the same lines as for $\mathbb{C}H^n$. We will use certain basic notions of semisimple Lie groups and symmetric spaces; good references on these subjects are [21, Ch. 1–2] and [26, Ch. III–VI].

The complex hyperbolic space $\mathbb{C}H^n$ is a rank one Hermitian symmetric space of noncompact type and, as such, admits the representation as a coset space G/K, where $G = SU(1,n)$ is the identity connected component of the isometry group of $\mathbb{C}H^n$, and $K = S(U(1)U(n))$ is the isotropy group at some point $o \in \mathbb{C}H^n$ (i.e. the elements of K are those in G that fix o). Denote by $\mathfrak{g} = \mathfrak{su}(1,n)$ and $\mathfrak{k} = \mathfrak{s}(\mathfrak{u}(1)\oplus\mathfrak{u}(n))$ the Lie algebras of G and K, respectively, and denote by $\mathrm{ad}\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ the adjoint map, where $\mathrm{ad}(X) = [X, \cdot\,]$ for $X \in \mathfrak{g}$. Let $\mathcal{B}$ be the Killing form of $\mathfrak{g}$, that is, $\mathcal{B}\colon (X,Y) \in \mathfrak{g}\times\mathfrak{g} \mapsto \mathcal{B}(X,Y) = \mathrm{tr}(\mathrm{ad}(X)\circ\mathrm{ad}(Y)) \in \mathbb{R}$, which is a nondegenerate bilinear form by virtue of Cartan's criterion for semisimple Lie algebras ($\mathfrak{g}$ is actually a simple Lie algebra). Then $\mathfrak{g} = \mathfrak{k}\oplus\mathfrak{p}$ is the Cartan decomposition of $\mathfrak{g}$ with respect to $o \in \mathbb{C}H^n$, where $\mathfrak{p}$ is the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to $\mathcal{B}$. This means that we have the bracket relations $[\mathfrak{k},\mathfrak{k}] \subset \mathfrak{k}$, $[\mathfrak{k},\mathfrak{p}] \subset \mathfrak{p}$ and $[\mathfrak{p},\mathfrak{p}] \subset \mathfrak{k}$, and $\mathcal{B}$ is negative definite on $\mathfrak{k}$ and positive definite on $\mathfrak{p}$. The Cartan involution θ corresponding to the Cartan decomposition above is the automorphism of the Lie algebra $\mathfrak{g}$ defined by $\theta(X) = X$ for all $X \in \mathfrak{k}$ and $\theta(X) = -X$ for all $X \in \mathfrak{p}$. Then the bilinear form $\mathcal{B}_\theta$, defined by $\mathcal{B}_\theta(X,Y) = -\mathcal{B}(X,\theta Y)$ for all $X, Y \in \mathfrak{g}$, is a positive definite inner product on $\mathfrak{g}$.

We take now a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$. It can be easily proved that the dimension of $\mathfrak{a}$ is 1, which means that the rank of the symmetric space $G/K = \mathbb{C}H^n$ (or of the real simple Lie algebra $\mathfrak{g}$) is precisely one. The set $\{\mathrm{ad}(H) : H \in \mathfrak{a}\}$ is a family of commuting self-adjoint (with respect to $\mathcal{B}_\theta$) endomorphisms of $\mathfrak{g}$, and hence simultaneously diagonalizable. By definition, their common eigenspaces are the root spaces of the simple Lie algebra $\mathfrak{g}$, and their nonzero eigenvalues (which do depend on $H \in \mathfrak{a}$) are the roots of $\mathfrak{g}$. Denoting by $\mathfrak{a}^*$ the dual vector space of $\mathfrak{a}$, if we define for each $\lambda \in \mathfrak{a}^*$

$$\mathfrak{g}_\lambda = \{X \in \mathfrak{g} : [H,X] = \lambda(H)X, \text{ for all } H \in \mathfrak{a}\},$$

then the root space decomposition of $\mathfrak{g}$ with respect to $\mathfrak{a}$ has the form

$$\mathfrak{g} = \mathfrak{g}_{-2\alpha}\oplus\mathfrak{g}_\alpha\oplus\mathfrak{g}_0\oplus\mathfrak{g}_\alpha\oplus\mathfrak{g}_{2\alpha}$$

for a certain covector $\alpha \in \mathfrak{a}^*$. These five mutually $\mathcal{B}_\theta$-orthogonal subspaces are precisely the root spaces, while -2α, $-\alpha$, α and 2α are the roots of $\mathfrak{g}$. Moreover, $\mathfrak{a} \subset \mathfrak{g}_0$, and for every $\lambda, \mu \in \mathfrak{a}^*$, we have that $[\mathfrak{g}_\lambda, \mathfrak{g}_\mu] \subset \mathfrak{g}_{\lambda+\mu}$. If one writes down the matrices of $\mathfrak{g} = \mathfrak{su}(1,n)$ that belong to each root space, one can show that $\dim\mathfrak{g}_{2\alpha} = \dim\mathfrak{g}_{-2\alpha} = \dim\mathfrak{a} = 1$ and $\dim\mathfrak{g}_\alpha = \dim\mathfrak{g}_{-\alpha} = 2(n-1)$.

Now we fix a criterion of positivity in the set of roots; in our case, let us say that α is a positive root. Define $\mathfrak{n} = \mathfrak{g}_\alpha\oplus\mathfrak{g}_{2\alpha}$ as the sum of the root spaces corresponding to all positive roots. Due to the properties of the root space decomposition, $\mathfrak{n}$ is a nilpotent Lie subalgebra of $\mathfrak{g}$ with center $\mathfrak{g}_{2\alpha}$; in fact $\mathfrak{n}$ is isomorphic to the $(2n-1)$-dimensional Heisenberg algebra (see [8, Ch. 3]). Then $\mathfrak{a}\oplus\mathfrak{n}$ is a solvable Lie subalgebra of $\mathfrak{g}$, since $[\mathfrak{a}\oplus\mathfrak{n},\mathfrak{a}\oplus\mathfrak{n}]=\mathfrak{n}$, which is nilpotent.

The direct sum decomposition $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{a}\oplus\mathfrak{n}$ is called the Iwasawa decomposition of the semisimple Lie algebra $\mathfrak{g}$. It is important to mention that, even though $\mathfrak{k}$, $\mathfrak{a}$ and $\mathfrak{n}$ are Lie subalgebras of $\mathfrak{g}$, the previous decomposition of $\mathfrak{g}$ is just a decomposition in a direct sum of vector subspaces, but neither an orthogonal decomposition, nor a direct sum of Lie algebras.

Let A, N and AN be the connected subgroups of G with Lie algebras $\mathfrak{a}$, $\mathfrak{n}$ and $\mathfrak{a}\oplus\mathfrak{n}$, respectively. The Iwasawa decomposition theorem at the Lie group level ensures that the product map $(k,a,m)\in K\times A\times N\mapsto kam\in G$ is a diffeomorphism. Again, we just mean that G and $K\times A\times N$ are diffeomorphic as manifolds, but not that G is isomorphic to the direct product of the groups K, A and N.

Let us show that the group AN acts simply transitively on $\mathbb{C}H^n$. Recall first that the elements in G are isometries of $\mathbb{C}H^n$. Let $p\in\mathbb{C}H^n$ be arbitrary, and let $k\in K$ and $h\in AN$ be such that $kh(p)=o$ (they exist since $G=KAN$ acts transitively on $\mathbb{C}H^n$), but since k^{-1} fixes o, then $h(p)=o$. This implies that AN acts transitively on $\mathbb{C}H^n$. Let now $h\in AN$ be such that $h(o)=o$; then $h\in K\cap AN$, and the Iwasawa decomposition implies that h is the identity element of G. Thus AN also acts freely on $\mathbb{C}H^n$.

Consider now the differentiable map $\phi\colon h\in G\mapsto h(o)\in\mathbb{C}H^n$. Since AN acts simply transitively on $\mathbb{C}H^n$, the map $\phi|_{AN}\colon AN\to\mathbb{C}H^n$ is a diffeomorphism, and one can identify $\mathfrak{a}\oplus\mathfrak{n}$ with the tangent space $T_o\mathbb{C}H^n$. The Bergman metric g of the complex hyperbolic space $\mathbb{C}H^n$ induces a metric ϕ^*g on AN. The Riemannian manifolds (AN,ϕ^*g) and $(\mathbb{C}H^n,g)$ are then trivially isometric. Let us denote by L_h the left translation in G by the element $h\in G$. As the metric g on $\mathbb{C}H^n$ is invariant under isometries (and then under elements of G), it follows that

$$L_h^*(\phi^*g)=L_h^*\phi^*(h^{-1})^*g=(h^{-1}\circ\phi\circ L_h)^*g=\phi^*g,\quad \text{for all } h\in G,$$

because $(h^{-1}\circ\phi\circ L_h)(h')=h^{-1}(hh'(o))=h'(o)=\phi(h')$ for all $h'\in G$. Therefore the metric ϕ^*g on AN is left-invariant. Thus, we have obtained that $\mathbb{C}H^n$ can be seen as a solvable Lie group AN endowed with a left-invariant metric.

By means of $\phi|_{AN}$ we can also equip AN with the Kähler structure induced by the one in $\mathbb{C}H^n$, and we obtain the corresponding almost complex structure J on AN, and also on $\mathfrak{a}\oplus\mathfrak{n}$. Some calculations with matrices would show that the almost complex structure J on $\mathfrak{a}\oplus\mathfrak{n}$ leaves $\mathfrak{g}_\alpha$ invariant and $J\mathfrak{a}=\mathfrak{g}_{2\alpha}$.

Thus, we have obtained a model for the complex hyperbolic space $\mathbb{C}H^n$ as a solvable Lie group AN with left-invariant Riemannian metric whose Lie algebra $\mathfrak{a}\oplus\mathfrak{n}=\mathfrak{a}\oplus\mathfrak{g}_\alpha\oplus\mathfrak{g}_{2\alpha}$ can be identified with the tangent space $T_o\mathbb{C}H^n$, and such that $\mathfrak{g}_\alpha$ can be seen as a complex vector space $\mathbb{C}^{n-1}$. This model will be fundamental in the construction we describe below.

5.3 The new examples

We proceed now with the construction of the inhomogeneous isoparametric hypersurfaces in $\mathbb{C}H^n$.

Take any proper subspace $\mathfrak{w}$ of $\mathfrak{g}_\alpha$ and denote by $\mathfrak{w}^\perp$ the orthogonal complement of $\mathfrak{w}$ in $\mathfrak{g}_\alpha$. Define $\mathfrak{s}_\mathfrak{w}=\mathfrak{a}\oplus\mathfrak{w}\oplus\mathfrak{g}_{2\alpha}$. Using the properties of the root space decomposition of $\mathfrak{g}$, it is easy to check that $\mathfrak{s}_\mathfrak{w}$ is a solvable Lie subalgebra of $\mathfrak{a}\oplus\mathfrak{n}$. Let $S_\mathfrak{w}$ be the connected subgroup of AN with Lie algebra $\mathfrak{s}_\mathfrak{w}$, and let $W_\mathfrak{w}=S_\mathfrak{w}\cdot o$ be the orbit of the action of $S_\mathfrak{w}$ on $\mathbb{C}H^n$ through the base point o. Then $W_\mathfrak{w}$ is a homogeneous submanifold of $\mathbb{C}H^n$. $W_\mathfrak{w}$ turns out to be also a minimal submanifold of $\mathbb{C}H^n$.

The new isoparametric hypersurfaces are the tubes around these homogeneous minimal submanifolds $W_\mathfrak{w}$ of $\mathbb{C}H^n$. That these tubes are indeed isoparametric is guaranteed by the following result.

Theorem 5.3. **[16]** *Let $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$ be the Cartan decomposition of the Lie algebra of the isometry group $G=SU(1,n)$ of $\mathbb{C}H^n$ with respect to a point $o\in\mathbb{C}H^n$. Assume $\mathfrak{a}\subset\mathfrak{p}$ is a maximal abelian subspace and let $\mathfrak{g}=\mathfrak{g}_{-2\alpha}\oplus\mathfrak{g}_{-\alpha}\oplus\mathfrak{g}_0\oplus\mathfrak{g}_\alpha\oplus\mathfrak{g}_{2\alpha}$ be the root space decomposition with respect to $\mathfrak{a}$. Let $W_\mathfrak{w}$ be the orbit through o of the connected subgroup $S_\mathfrak{w}$ of G whose Lie algebra is $\mathfrak{s}_\mathfrak{w}=\mathfrak{a}\oplus\mathfrak{w}\oplus\mathfrak{g}_{2\alpha}$, where $\mathfrak{w}$ is any proper subspace of $\mathfrak{g}_\alpha$.*

Then, the tubes around the submanifold $W_\mathfrak{w}$ are isoparametric hypersurfaces of $\mathbb{C}H^n$. Moreover, the following conditions are equivalent:

- *The tubes around $W_\mathfrak{w}$ are homogeneous.*
- *The tubes around $W_\mathfrak{w}$ have constant principal curvatures.*
- *$\mathfrak{w}^\perp$ has constant Kähler angle as a subspace of $\mathfrak{g}_\alpha$.*

Therefore, the submanifold $W_\mathfrak{w}$ and the tubes around it form an isoparametric family of hypersurfaces in $\mathbb{C}H^n$. Theorem 5.3 also gives us a characterization of those isoparametric hypersurfaces in our construction that have constant principal curvatures by means of the notion of constant Kähler angle. We recall what this concept means.

Given the subspace $\mathfrak{w}^\perp$ of $\mathfrak{g}_\alpha$ and a nonzero vector $\xi \in \mathfrak{w}^\perp$, we say that the *Kähler angle* of $\mathfrak{w}^\perp$ with respect to ξ is $\varphi \in [0, \pi/2]$ if the angle between $J\xi$ and $\mathfrak{w}^\perp$ is φ (remember that $\mathfrak{g}_\alpha$ is a complex vector space with complex structure J and with a certain inner product); this is equivalent to saying that the projection of $J\xi$ onto $\mathfrak{w}^\perp$ has length $\|\xi\| \cos(\varphi)$. When the Kähler angles of $\mathfrak{w}^\perp$ with respect to any nonzero vector $\xi \in \mathfrak{w}^\perp$ coincide, we say that $\mathfrak{w}^\perp$ has *constant Kähler angle*. For example, totally real subspaces of a complex vector space are exactly those subspaces with constant Kähler angle $\pi/2$, whereas complex subspaces are those with constant Kähler angle 0. Subspaces with constant Kähler angle of a complex vector space have been classified (see [4, Prop. 7]): a subspace V of $\mathbb{C}^m$ has constant Kähler angle $\varphi \in (0, \pi/2)$ if and only if there exist $2k$ $\mathbb{C}$-orthonormal vectors $e_1, \dots, e_{2k}$ in $\mathbb{C}^m$ such that

$$e_1, \cos(\varphi) J e_1 + \sin(\varphi) J e_2, \dots, e_{2k-1}, \cos(\varphi) J e_{2k-1} + \sin(\varphi) J e_{2k}$$

is an orthonormal basis of V. In particular, for each $\varphi \in (0, \pi/2)$ there exist subspaces of $\mathbb{C}^m$ with constant Kähler angle φ, whenever $m \geq 2$.

However, a generic subspace of $\mathbb{C}^m$ does not need to have constant Kähler angle. Sums of $\mathbb{C}$-orthogonal subspaces with different constant Kähler angles have nonconstant Kähler angle; for example, if V_φ and V_ψ are $\mathbb{C}$-orthogonal subspaces of $\mathbb{C}^m$ (i.e. $V_\varphi \perp V_\psi$ and $V_\varphi \perp J V_\psi$) with constant Kähler angles φ and ψ respectively, then $V_\varphi \oplus V_\psi$ is a subspace of $\mathbb{C}^m$ whose Kähler angles vary from φ to ψ.

Theorem 5.3 asserts that tubes around the submanifold $W_\mathfrak{w}$ are orbits of a cohomogeneity one isometric action on $\mathbb{C}H^n$ (that is, are homogeneous hypersurfaces) if and only if $\mathfrak{w}^\perp$ has constant Kähler angle. In this situation, define $k = \dim \mathfrak{w}^\perp$, which coincides with the codimension of $W_\mathfrak{w}$, let φ be the constant Kähler angle of $\mathfrak{w}^\perp$, and set $W^{2n-k}_\varphi = W_\mathfrak{w}$. If $\varphi = \pi/2$ we write W^{2n-k} instead of W^{2n-k}_φ. The submanifolds W^{2n-k}_φ are the Berndt-Brück submanifolds that we mentioned in Remark 5.3 and Theorem 5.2, and W^{2n-1} is the Lohnherr hypersurface. Tubes around the submanifolds W^{2n-k} have $g \in \{3, 4\}$ constant principal curvatures and $h = 2$ nontrivial projections of their Hopf vector field onto the principal curvature spaces. Tubes around the submanifolds W^{2n-k}_φ with $\varphi \in (0, \pi/2)$ have $g \in \{4, 5\}$ and $h = 3$. The proof of these facts can be consulted in [6].

If the subspace $\mathfrak{w}^\perp$ of $\mathfrak{g}_\alpha$ does not have constant Kähler angle, tubes around $W_\mathfrak{w}$ are *inhomogeneous isoparametric hypersurfaces with noncon-*

stant principal curvatures. For these inhomogeneous examples, the functions g and h may be nonconstant as well, and they satisfy $h(p) \in \{1, 2, 3\}$ and $g(p) \in \{3, 4, 5\}$ for each point p in the hypersurface (see [16] for details). Since for $n = 2$, every subspace of $\mathfrak{g}_\alpha \cong \mathbb{C}$ has constant Kähler angle, our construction does not provide any inhomogeneous isoparametric hypersurface in the complex hyperbolic plane $\mathbb{C}H^2$. However, for every $n \geq 3$, the complex hyperbolic space $\mathbb{C}H^n$ admits inhomogeneous isoparametric hypersurfaces with nonconstant principal curvatures. Notice that our construction provides *uncountably many* noncongruent inhomogeneous isoparametric families of hypersurfaces in complex hyperbolic spaces; this should be compared with what happens in spheres, where the hypersurfaces of FKM-type are infinite, but only countably many.

6 Open problems

We conclude this work proposing some open problems around the topic of isoparametric hypersurfaces and hypersurfaces with constant principal curvatures in complex space forms.

(1) Classify isoparametric hypersurfaces in complex projective spaces $\mathbb{C}P^n$. Since the property of being an isoparametric hypersurface is preserved under the Hopf map $S^{2n+1} \to \mathbb{C}P^n$, this classification would in principle follow from a classification of isoparametric hypersurfaces in spheres (which is still open).

(2) Classify isoparametric hypersurfaces in complex hyperbolic spaces. It would also be interesting to answer the following questions, thus providing some partial results. Does any inhomogeneous isoparametric hypersurface in $\mathbb{C}H^n$ have nonconstant principal curvatures? (The answer is known to be yes in the projective case, see [48]). Are there isoparametric hypersurfaces M in $\mathbb{C}H^n$ that satisfy $h(p) = g(p) \in \{2, 3\}$ at some point $p \in M$? Is every isoparametric hypersurface in $\mathbb{C}H^2$ homogeneous?

(3) Classify hypersurfaces with constant principal curvatures in nonflat complex space forms $\mathbb{C}P^n$ and $\mathbb{C}H^n$. A bound on g, and even better on h, would be very interesting. A classification of the case $h = 3$ might already give insight into the main difficulties of the problem. The existence of an inhomogeneous non-isoparametric hypersurface with constant principal curvatures (even in any other Riemannian symmetric space) would point out a very strange but appealing phenomenon.

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