ANTI-DE SITTER SPACETIMES AND ISOPARAMETRIC HYPERSURFACES IN COMPLEX HYPERBOLIC SPACES

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Abstract. By lifting hypersurfaces in complex hyperbolic spaces to anti-De Sitter spacetimes, we prove that an isoparametric hypersurface in the complex hyperbolic space has the same principal curvatures as a homogeneous one.

1. Introduction

One of the final aims of the research of the authors of this paper is guided by the following question: to what extent do the symmetries of an object determine its shape? It is intuitively clear that the existence of symmetries reduces the number of degrees of freedom in the description of a geometric object and imposes constraints on how the different parameters defining it are related. The more symmetries an object has, the more likely it is that this object is uniquely determined. Somewhat more complicated is to address the following converse problem: if the shape of an object A is the same as that of an object B with symmetries, can we assert that object A is the same as B?

In order to make this broad question more concrete, we introduce the mathematical context into which we will tackle the problem. Our area of research is submanifold geometry of Riemannian manifolds. Our "symmetric objects" will be the so-called homogeneous submanifolds. Let $\bar{M}$ be a Riemannian manifold, and M a submanifold of $\bar{M}$. We say that M is extrinsically homogeneous, henceforth simply *homogeneous*, if for any two points p, $q \in M$ there exists an isometry g of $\bar{M}$ such that $g(M) = M$ and $g(p) = q$. Equivalently, M is homogeneous if it is an orbit of a subgroup G of the isometry group of $\bar{M}$, that is, $M = G \cdot p$, for some $p \in \bar{M}$. In this paper we will actually be interested in homogeneous hypersurfaces, that is, homogeneous submanifolds of codimension one.

A homogeneous hypersurface has a great deal of symmetries, namely, the isometries of G. It is thus conceivable that homogeneous hypersurfaces can be classified in a broad class of Riemannian manifolds with large isometry groups. (If isometry groups are small, homogeneous hypersurfaces might not exist at all.) This is true for example in Euclidean spaces [17], spheres [11], real hyperbolic spaces [4], complex projective spaces [18], irreducible symmetric spaces of compact type [12], complex hyperbolic spaces, and the Cayley hyperbolic plane [3]. Remarkably, no such classification is known for quaternionic hyperbolic spaces.

Homogeneous hypersurfaces have two interesting properties related to their shape: they are isoparametric and have constant principal curvatures. A hypersurface is called isoparametric if its nearby parallel hypersurfaces have constant mean curvature. A hypersurface has constant principal curvatures if the eigenvalues of its shape operator are constant. These two concepts are equivalent for spaces of constant curvature. Indeed, this property of their shape is characteristic of homogeneous hypersurfaces in Euclidean and real hyperbolic spaces: Segre, for Euclidean

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spaces [17], and Cartan, for real hyperbolic spaces [4], proved that isoparametric hypersurfaces are homogeneous and derived their classification. Surprisingly, this is not true of spheres, as the examples in [9] show. The classification of isoparametric hypersurfaces in spheres has been the aim of several recent and important works (see for example [5] and [14]).

In complex space forms, isoparametric hypersurfaces do not necessarily have constant principal curvatures. A classification of isoparametric hypersurfaces in complex projective spaces $\mathbb{C}P^n$, $n \neq 15$, has been obtained by the second author in [8] as a consequence of the available classifications in spheres. As a consequence, there exist inhomogeneous isoparametric hypersurfaces in complex projective spaces.

In this paper we are interested in isoparametric hypersurfaces in complex hyperbolic spaces. Their classification has recently been obtained by the authors in [7]:

Theorem 1.1. *Let M be a connected real hypersurface in the complex hyperbolic space $\mathbb{C}H^n$, $n \geq 2$. Then, M is isoparametric if and only if M is congruent to an open part of:*

(i) *a tube around a totally geodesic complex hyperbolic space $\mathbb{C}H^k$, $k \in \{0, \dots, n-1\}$, or*
(ii) *a tube around a totally geodesic real hyperbolic space $\mathbb{R}H^n$, or*
(iii) *a horosphere, or*
(iv) *a ruled homogeneous minimal Lohnherr hypersurface W^{2n-1}, or some of its equidistant hypersurfaces, or*
(v) *a tube around a ruled homogeneous minimal Berndt-Brück submanifold W^{2n-k}_φ, for $k \in \{2, \dots, n-1\}$, $\varphi \in (0, \pi/2]$, where k is even if $\varphi \neq \pi/2$, or*
(vi) *a tube around a ruled homogeneous minimal submanifold $W_\mathfrak{w}$, for some proper real subspace $\mathfrak{w}$ of $\mathfrak{g}_\alpha \cong \mathbb{C}^{n-1}$ such that $\mathfrak{w}^\perp$, the orthogonal complement of $\mathfrak{w}$ in $\mathfrak{g}_\alpha$, has nonconstant Kähler angle.*

We give a brief description of the examples (iv) through (vi); see [6] or [7] for more details. Let $\mathfrak{g}$ denote the Lie algebra of $SU(1,n)$, the isometry group of $\mathbb{C}H^n$, and let $\mathfrak{g} = \mathfrak{g}_{-2\alpha} \oplus \mathfrak{g}_{-\alpha} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$ be a restricted root space decomposition of $\mathfrak{g}$ with respect to some point $o \in \mathbb{C}H^n$ and some point at infinity $x \in \mathbb{C}H^n(\infty)$. The point x determines a maximal flat $\mathfrak{a} \subset \mathfrak{g}_0$. It turns out that $\mathfrak{a}$ and $\mathfrak{g}_{2\alpha}$ are 1-dimensional, and $\mathfrak{g}_\alpha$ is a complex vector space of complex dimension $n-1$, whose complex structure we denote by J. If $\mathfrak{w}$ is a real subspace of $\mathfrak{g}_\alpha$, we define $\mathfrak{s}_\mathfrak{w} = \mathfrak{a} \oplus \mathfrak{w} \oplus \mathfrak{g}_{2\alpha}$. Then, $\mathfrak{s}_\mathfrak{w}$ is a Lie subalgebra of $\mathfrak{g}$, and the connected subgroup $S_\mathfrak{w}$ of $SU(1,n)$ whose Lie algebra is $\mathfrak{s}_\mathfrak{w}$ acts isometrically on $\mathbb{C}H^n$. We define $W_\mathfrak{w} = S_\mathfrak{w} \cdot o$. Then, the tubes around $W_\mathfrak{w}$ are isoparametric hypersurfaces of $\mathbb{C}H^n$. If $\mathfrak{w}$ is a hyperplane, then $W_\mathfrak{w}$ is denoted by W^{2n-1} and we obtain the examples in (iv) (see also [13]). If $\mathfrak{w}^\perp$, the orthogonal complement of $\mathfrak{w}$ in $\mathfrak{g}_\alpha$, has constant Kähler angle $\varphi \in (0, \pi/2]$ and codimension k, then $W_\mathfrak{w}$ is denoted by W^{2n-k}_φ and we get (v) (see [1]). Recall that $\mathfrak{w}^\perp$ has constant Kähler angle φ if for any nonzero $\xi \in \mathfrak{w}^\perp$ the angle between $J\xi$ and $\mathfrak{w}^\perp$ is always φ. If $\mathfrak{w}^\perp$ does not have constant Kähler angle, then we obtain the examples in (vi).

Theorem 1.1 implies the classification of homogeneous hypersurfaces in complex hyperbolic spaces. In fact,

Corollary 1.2. [3, 7] *A real hypersurface in $\mathbb{C}H^n$ is homogeneous if and only if it belongs to one of the families* (i)-(v) *in* Theorem 1.1.

Thus, if $n \geq 3$, there are uncountably many families of inhomogeneous isoparametric hypersurfaces in complex hyperbolic spaces.

The aim of this paper is to prove the following result.

Theorem 1.3. *Let M be an isoparametric hypersurface in $\mathbb{C}H^n$. Then, the principal curvatures of M are pointwise the same as the principal curvatures of a homogeneous hypersurface of $\mathbb{C}H^n$.*

It is clear that, by working out the principal curvatures of the examples appearing in Theorem 1.1, the conclusion of Theorem 1.3 follows from the classification of isoparametric hypersurfaces in $\mathbb{C}H^n$. The purpose of this paper is to prove Theorem 1.3 by a more direct approach, which avoids several intricate arguments needed for the proof of Theorem 1.1.

The complex hyperbolic space $\mathbb{C}H^n$ is the quotient of the anti-De Sitter spacetime $H^{2n+1}_1 \subset \mathbb{C}^{1,n}$ by S^1. Let us call $\pi\colon H^{2n+1}_1 \to \mathbb{C}H^n$ the projection map, the so-called Hopf map. See Section 2.

Recall that $\mathbb{C}H^n$ is a Kähler manifold of constant holomorphic sectional curvature. We denote its Kähler structure by J. The anti-De Sitter space is, in turn, a Lorentzian manifold of constant negative curvature. It can be shown that, if a hypersurface M of $\mathbb{C}H^n$ is isoparametric, then $\pi^{-1}(M)$ has constant principal curvatures. A generalization of a result by Cartan [4] implies that the number of real principal curvatures of $\pi^{-1}(M)$ is bounded by two. This allows us to deduce many interesting properties of M just by using the fundamental equations of a submersion and some algebraic calculations. It is remarkable, for example, that the principal curvatures of M at a point coincide with the principal curvatures of some homogeneous hypersurface in $\mathbb{C}H^n$. In particular, if $g(p)$ denotes the number of principal curvatures of M at p, and $h(p)$ denotes the number of nontrivial projections of $J\xi_p$ onto the principal curvature spaces, where ξ_p is a normal vector of M at p, we have

Proposition 1.4. *If M is an isoparametric hypersurface of $\mathbb{C}H^n$, then $h \leq 3$ and $g \leq 5$.*

The results obtained by the authors in this paper precede those of [7]. Although the principal curvatures of isoparametric hypersurfaces in $\mathbb{C}H^n$ are the same as in the homogeneous examples, we found inhomogeneous examples of isoparametric hypersurfaces in $\mathbb{C}H^n$. These examples, corresponding to case (vi) of Theorem 1.1, were first constructed in [6]. It was surprising at the moment to notice that, pointwise, the principal curvatures of these examples are the same as those of a homogeneous hypersurface. Nonetheless, the principal curvatures of these examples are nonconstant, so they are not homogeneous.

A major disadvantage of working with $\pi^{-1}(M)$ instead of M is that the shape operator of the former is not necessarily diagonalizable. There are exactly four different types of Jordan canonical forms for this shape operator, described in Section 3. Using the algebraic approach that we describe in this paper we can get Theorem 1.3. However, we will only deal with Type III points. There are two reasons for this. Firstly, Type III is considerably more involved than the other types. Once Type III is sorted out, the other types can be handled with similar arguments. Secondly, Types I, II and IV are tackled in [7] by this very same method. The arguments in [7] diverge considerably from our approach here for Type III points. In this paper we get a weaker result, but the argument is much shorter. The core of this paper is the proof of Theorem 3.4 from where Theorem 1.3 and Proposition 1.4 follow.

2. Anti-De Sitter spacetime and complex hyperbolic space

In $\mathbb{C}^{n+1}$ we define the flat semi-Riemannian metric

$$\langle z, w\rangle = \operatorname{Re}\Big(-z_0\bar{w}_0 + \sum_{k=1}^{n} z_k\bar{w}_k\Big).$$

It is customary to denote by $\mathbb{C}^{1,n}$ the vector space $\mathbb{C}^{n+1}$ endowed with the previous scalar product. The anti-De Sitter spacetime of radius $r > 0$ is defined as

$$H_1^{2n+1}(r) = \left\{ z \in \mathbb{C}^{1,n} : \langle z, z\rangle = -r^2 \right\}.$$

This hypersurface of $\mathbb{C}^{1,n}$ is a Lorentzian manifold of constant negative curvature $c = -4/r^2$ and dimension $2n+1$. The map $S^1 \times H_1^{2n+1}(r) \to H_1^{2n+1}(r)$, $(\lambda, z) \mapsto \lambda z$ defines an S^1-action on $H_1^{2n+1}(r)$. The quotient space $\mathbb{C}H^n(c) = H_1^{2n+1}(r)/S^1$ turns out to be a Kähler manifold of real dimension $2n$ with constant holomorphic sectional curvature c. The natural projection map $\pi\colon H_1^{2n+1}(r) \to \mathbb{C}H^n(c)$ is called the Hopf map. The complex hyperbolic space $\mathbb{C}H^n(c)$ inherits its metric by requiring the Hopf map to be a semi-Riemannian submersion with timelike totally geodesic fibers. We denote by $\tilde{\nabla}$ and $\bar{\nabla}$ the Levi-Civita connections of $H_1^{2n+1}(r)$ and $\mathbb{C}H^n(c)$, respectively. From now on, we will drop r and c in the notations of the anti-De Sitter spacetime and the complex hyperbolic space.

Let V denote the vector field on H_1^{2n+1} defined by $V_z = i\sqrt{-c}\,z/2$ for each $z \in H_1^{2n+1}$. This is a unit timelike vector field that is tangent to the S^1-flow. Now, we have the isomorphism

$$T_z H_1^{2n+1} \cong T_{\pi(z)}\mathbb{C}H^n \oplus \mathbb{R}V_z,$$

and $\ker \pi_{*z} = \mathbb{R}V_z$. Vectors in $\ker \pi_*$ are called vertical, and vectors orthogonal to $\ker \pi_*$ are called horizontal. The subspace of horizontal vectors of $T_z H_1^{2n+1}$ is a spacelike complex vector subspace of $\mathbb{C}^{1,n}$; this induces a Kähler structure on $\mathbb{C}H^n$ which we denote by J. Note that, for a vector field $X \in \Gamma(T\mathbb{C}H^n)$ there is a unique horizontal vector field $X^L \in \Gamma(TH_1^{2n+1})$, the horizontal lift of X, such that $\pi_* X^L = X$.

Since π is a semi-Riemannian submersion, the fundamental equations of a semi-Riemannian submersion [15] relate the Levi-Civita connections of H_1^{2n+1} and $\mathbb{C}H^n$ as

$$\begin{aligned} \tilde{\nabla}_{X^L} Y^L &= (\bar{\nabla}_X Y)^L + \frac{\sqrt{-c}}{2}\langle JX^L, Y^L\rangle V, \\ \tilde{\nabla}_V X^L = \tilde{\nabla}_{X^L} V &= \frac{\sqrt{-c}}{2}(JX)^L = \frac{\sqrt{-c}}{2} JX^L, \end{aligned} \tag{1}$$

for all $X,\, Y \in \Gamma(T\mathbb{C}H^n)$.

Now let M be a real hypersurface in $\mathbb{C}H^n$ and denote by ξ a (local) unit normal vector field to M. Then, $\tilde{M} = \pi^{-1}(M)$ is a hypersurface in H_1^{2n+1} that is invariant under the S^1-action, and ξ^L is a (local) spacelike normal unit vector field to $\tilde{M}$. We denote by ∇ the Levi-Civita connection on M or $\tilde{M}$, as there will not be a chance for confusion. We also denote by S and $\tilde{S}$ the shape operators of M and $\tilde{M}$, respectively. Since M is Riemannian, S_p is diagonalizable for each p. The eigenvalues of S_p are called the principal curvatures of M at p. The sum of the eigenvalues of S_p, which is also the trace of S_p, is called the mean curvature at p. We denote by $g(p)$ the number of principal curvatures of M at p.

Recall that M is said to be *Hopf* at $p \in M$ if $J\xi_p$ is an eigenvector of S_p; M is said to be Hopf, if it is Hopf at all points $p \in M$. We also denote by $h(p)$ the number of nontrivial projections of $J\xi_p$ onto the principal curvature spaces. Thus, M is Hopf at p if and only if $h(p) = 1$.

The Gauss and Weingarten formulas for $\tilde{M}$ are

$$\tilde{\nabla}_X Y = \nabla_X Y + \langle \tilde{S}X, Y\rangle \xi^L, \qquad \tilde{\nabla}_X \xi^L = -\tilde{S}X.$$

Thus, (1) implies

$$\tilde{S}X^L = (SX)^L + \frac{\sqrt{-c}}{2}\langle J\xi^L, X^L\rangle V, \qquad \tilde{S}V = -\frac{\sqrt{-c}}{2} J\xi^L. \tag{2}$$

Hence, if $\lambda_1, \dots, \lambda_{2n-1}$ are the principal curvatures of M, then (2) implies that, with respect to a suitable basis of TH_1^{2n+1}, the endomorphism $\tilde{S}$ can be represented by the matrix

$$\begin{pmatrix} \lambda_1 & & 0 & -\frac{b_1\sqrt{-c}}{2} \\ & \ddots & & \vdots \\ 0 & & \lambda_{2n-1} & -\frac{b_{2n-1}\sqrt{-c}}{2} \\ \frac{b_1\sqrt{-c}}{2} & \dots & \frac{b_{2n-1}\sqrt{-c}}{2} & 0 \end{pmatrix}, \tag{3}$$

where $b_i = \langle J\xi, X_i\rangle \circ \pi$, $i \in \{1, \dots, 2n-1\}$, are S^1-invariant functions on (an open set of) $\tilde{M}$. In particular, it follows that M and $\tilde{M}$ have the same mean curvature.

3. Isoparametric hypersurfaces

We say that a hypersurface M of $\mathbb{C}H^n$ is *isoparametric* if all sufficiently close parallel hypersurfaces have constant mean curvature. We take $\tilde{M} = \pi^{-1}(M)$, which is a Lorentzian hypersurface of anti-De Sitter spacetime, and note that, since it is a semi-Riemannian submersion, π maps parallel hypersurfaces of $\tilde{M}$ to parallel hypersurfaces of M. As we have seen above, a hypersurface and its lift have the same mean curvature, and thus, if M is isoparametric, parallel hypersurfaces to $\tilde{M}$ have constant mean curvature. It follows from the work of Hahn [10] that $\tilde{M}$ has constant principal curvatures with constant algebraic multiplicities. However, it is important to point out that M does not necessarily have constant principal curvatures. Even more, the functions g and h do not have to be constant.

The rest of this paper is devoted to proving Theorem 1.3.

The shape operator $\tilde{S}_q$ at a point $q \in \tilde{M}$ is a self-adjoint endomorphism of $T_q\tilde{M}$. Since $\tilde{M}$ is Lorentzian, $\tilde{S}$ is not necessarily diagonalizable, but it is known to have one of the following Jordan canonical forms (see for example [16, Chapter 9]):

$$\text{I.} \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_{2n} \end{pmatrix} \qquad \text{II.} \begin{pmatrix} \lambda_1 & 0 & & & \\ \varepsilon & \lambda_1 & & & \\ & & \lambda_2 & & \\ & & & \ddots & \\ & & & & \lambda_{2n-1} \end{pmatrix}, \ \varepsilon = \pm 1$$

$$\text{III.} \begin{pmatrix} \lambda_1 & 0 & 1 & & & \\ 0 & \lambda_1 & 0 & & & \\ 0 & 1 & \lambda_1 & & & \\ & & & \lambda_2 & & \\ & & & & \ddots & \\ & & & & & \lambda_{2n-2} \end{pmatrix} \qquad \text{IV.} \begin{pmatrix} a & -b & & & \\ b & a & & & \\ & & \lambda_3 & & \\ & & & \ddots & \\ & & & & \lambda_{2n} \end{pmatrix}$$

The eigenvalues $\lambda_i \in \mathbb{R}$ can be repeated and, in case IV we have $\lambda_1 = a + ib, \lambda_2 = a - ib$ $(b \neq 0)$. In cases I and IV, the basis with respect to which $\tilde{S}_q$ is represented is orthonormal, and the first vector of this basis is timelike. In cases II and III the basis is semi-null. A basis $\{u, v, e_1, \dots, e_{m-2}\}$ is semi-null if all inner products are zero except $\langle u, v\rangle = \langle e_i, e_i\rangle = 1$, for each $i \in \{1, \dots, m-2\}$. A point $q \in \tilde{M}$ is said to be of type I, II, III or IV according to the type of the Jordan canonical form of $\tilde{S}_q$.

In his work on isoparametric hypersurfaces in spaces of constant curvature [4], Cartan proved a fundamental formula relating the curvature of the ambient manifold and the principal curvatures. A similar argument works for the anti-De Sitter spacetime. In particular, the following consequence can be derived from this fundamental formula [7, Lemma 3.4]:

Lemma 3.1. *Let $q \in \tilde{M}$ be a point of type I, II or III. Then the number $\tilde{g}(q)$ of constant principal curvatures at q satisfies $\tilde{g}(q) \in \{1, 2\}$. Moreover, if $\tilde{g}(q) = 2$ and the principal curvatures are λ and μ, then $c + 4\lambda\mu = 0$.*

The objective of this paper is to analyze the eigenvalue structure of the shape operator of an isoparametric hypersurface in $\mathbb{C}H^n$ and obtain, as a consequence of this study, Theorem 1.3. As a corollary, we derive a bound for h and g (Proposition 1.4). The proof of these facts will be mostly algebraic, and is carried out by analyzing the possible Jordan canonical forms for the shape operator of $\tilde{M}$ at a point q as described above.

As stated in the introduction, we only deal with Type III points. For points of Types I, II and IV the proof is very similar and can be found in [7, Section 3].

Proposition 3.2. *Let $\tilde{M}$ be the lift of an isoparametric hypersurface in $\mathbb{C}H^n$ to the anti-De Sitter spacetime, and let $q \in \tilde{M}$ and $p = \pi(q)$. Then:*

(i) *If q is of type I, then M is Hopf at p, and $g(p) \in \{2, 3\}$. The principal curvatures of M at p are:*

$$\lambda \in \Big(-\frac{\sqrt{-c}}{2}, \frac{\sqrt{-c}}{2}\Big), \lambda \neq 0, \qquad \mu = -\frac{c}{4\lambda} \in \Big(-\infty, -\frac{\sqrt{-c}}{2}\Big) \cup \Big(\frac{\sqrt{-c}}{2}, \infty\Big), \qquad \lambda + \mu.$$

The last principal curvature has multiplicity one and corresponds to the Hopf vector.

(ii) *If q is of type II, then M is Hopf at p, and $g(p) = 2$. Moreover, $\tilde{M}$ has one principal curvature $\lambda = \pm\sqrt{-c}/2$, and the principal curvatures of M at p are λ and 2λ. The second one has multiplicity one and corresponds to the Hopf vector.*

(iii) *If q is of type IV, then M is Hopf at p. Let λ and $\mu = -c/(4\lambda)$ be the real principal curvatures of $\tilde{M}$ at q (μ might not exist). Then the principal curvatures of M at p are*

$$\lambda, \ \mu, \quad \text{and} \quad 2a = \frac{4c\lambda}{c - 4\lambda^2} \in \big(-\sqrt{-c}, \sqrt{-c}\big),$$

where $2a$ is the principal curvature associated with the Hopf vector.

Remark 3.3. Proposition 3.2 implies Theorem 1.3 for types I, II and IV.

Indeed, the values given in part (i) correspond to the principal curvatures of a tube of radius r around a totally geodesic $\mathbb{C}H^k$ in $\mathbb{C}H^n$, where

$$\lambda = \frac{\sqrt{-c}}{2}\tanh\Big(\frac{r\sqrt{-c}}{2}\Big), \tag{4}$$

and $2(n-k)$ is the multiplicity of μ.

The values obtained in (ii) correspond to the principal curvatures of a horosphere in $\mathbb{C}H^n$.

Finally, the values obtained in (iii) correspond to the principal curvatures of a tube of radius r around a totally geodesic $\mathbb{R}H^n$ in $\mathbb{C}H^n$, where r is given by the same formula as in (4).

In the rest of the paper we deal with Type III points. The arguments that follow are not contained in [7]. Thus, let M be an isoparametric hypersurface of $\mathbb{C}H^n$, whose lift to the anti-De Sitter spacetime is denoted by $\tilde{M}$. We fix a point $q \in \tilde{M}$ and assume that q is of Type III. We analyze the possible principal curvatures of M at the point $p = \pi(q)$.

Theorem 3.4. *Let λ be the principal curvature of $\tilde{M}$ at q whose algebraic and geometric multiplicities do not coincide. Then $h(p) \in \{2,3\}$ and $\lambda \in \big(-\sqrt{-c}/2, \sqrt{-c}/2\big)$.*

There exists a number $\varphi \in (0, \pi/2]$ such that the zeroes of the polynomial

$$f_{\lambda,\varphi}(x) = -x^3 + \Big(-\frac{c}{4\lambda} + 3\lambda\Big)x^2 + \frac{1}{2}\left(c - 6\lambda^2\right)x + \frac{-c^2 - 16c\lambda^2 + 16\lambda^4 + (c+4\lambda^2)^2\cos(2\varphi)}{32\lambda},$$

are principal curvatures of M at p. If $\varphi = \pi/2$, then $h(p) = 2$ and $g(p) \in \{2,3,4\}$. Moreover, we have the following possibilities:

(i) *If $\varphi = \pi/2$ and $g = 4$, then $0 \neq \lambda \neq \pm\sqrt{-c}/(2\sqrt{3})$, and the principal curvatures of M at p are:*

$$\frac{1}{2}\left(3\lambda \pm \sqrt{-c-3\lambda^2}\right), \qquad \lambda, \qquad \mu = -\frac{c}{4\lambda}.$$

The principal curvature spaces corresponding to the first two principal curvatures are one dimensional and the Hopf vector has nontrivial projection onto both of them.

(ii) *If $\varphi = \pi/2$ and $g \in \{2,3\}$ then we have two cases:*

(a) *If $\lambda = \pm\sqrt{-c}/(2\sqrt{3})$ then the principal curvatures of M at p are*

$$0, \qquad \mu = -\frac{c}{4\lambda} = \pm\frac{\sqrt{-3c}}{2}, \qquad \lambda = \pm\frac{\sqrt{-c}}{2\sqrt{3}}.$$

The principal curvature space associated with 0 is one dimensional, and the Hopf vector has nontrivial projection onto the principal curvature spaces corresponding to the first two principal curvatures. The value λ might not appear as a principal curvature.

(b) *If $0 \neq \lambda \neq \pm\sqrt{-c}/(2\sqrt{3})$, then the principal curvatures of M at p are*

$$\frac{1}{2}\left(3\lambda \pm \sqrt{-c-3\lambda^2}\right), \qquad \lambda \text{ or } \mu = -\frac{c}{4\lambda}.$$

The principal curvature spaces corresponding to the first two principal curvatures are one dimensional and the Hopf vector has nontrivial projection onto both of them.

(iii) *If $\varphi \in \big(0, \pi/2\big)$, then $\lambda \neq 0$ and the three zeros of the polynomial $f_{\lambda,\varphi}$ are different, and also different from λ and $-c/(4\lambda)$. Therefore, M has $g(p) \in \{3,4,5\}$ principal curvatures at p:*

$$\text{the zeroes of } f_{\lambda,\varphi}, \qquad \lambda, \qquad \mu = -\frac{c}{4\lambda}.$$

The principal curvature spaces corresponding to the first three principal curvatures are one dimensional and the Hopf vector has nontrivial projection onto all of them. The values λ and/or μ might not appear as principal curvatures.

Proof. For the sake of readability we will shorten the notation and write $v = V_q$. We also write $J\xi$ instead of $J\xi_p$, $\tilde{S}$ instead of $\tilde{S}_q$ and so on.

Assume that the shape operator $\tilde{S}$ has a type III matrix expression at q with respect to a semi-null basis $\{e_1, e_2, e_3 \dots, e_{2n}\}$, where

$$\begin{gathered}\langle e_1, e_1\rangle = \langle e_2, e_2\rangle = \langle e_1, e_3\rangle = \langle e_2, e_3\rangle = 0, \qquad \langle e_1, e_2\rangle = \langle e_3, e_3\rangle = 1, \\ \tilde{S}e_1 = \lambda e_1, \qquad \tilde{S}e_2 = \lambda e_2 + e_3, \qquad \tilde{S}e_3 = e_1 + \lambda e_3.\end{gathered} \tag{5}$$

We denote by $T_\lambda(q)$ and $T_\mu(q)$ the eigenspaces of λ and μ at q. Then $T_\lambda(q) \ominus \mathbb{R}e_2$ and $T_\mu(q)$ are spacelike. As a matter of caution, $e_2 \notin T_\lambda(q)$, and $T_\lambda(q) \ominus \mathbb{R}e_2$ denotes the vectors of $T_\lambda(q)$ that are orthogonal to e_2. For example, $e_1 \notin T_\lambda(q) \ominus \mathbb{R}e_2$ because e_1 and e_2 are not orthogonal.

Assume first that there are two distinct principal curvatures λ, μ. By Lemma 3.1 we have $c + 4\lambda\mu = 0$ and thus, $\lambda, \mu \neq 0$. We can write $v = r_1 e_1 + r_2 e_2 + r_3 e_3 + u + w$, where $u \in T_\lambda \ominus \mathbb{R}e_2$, and $w \in T_\mu(q)$. Changing the orientation of $\{e_1, e_2, e_3\}$ if necessary, we can also assume $r_2 \geq 0$. We have

$$-1 = \langle v, v\rangle = 2r_1 r_2 + r_3^2 + \langle u, u\rangle + \langle w, w\rangle.$$

Thus, $r_2 > 0$ and $r_1 < 0$. If $u \neq 0$ we define

$$e_1' = e_1, \qquad e_2' = -\frac{\langle u, u\rangle}{2r_2^2} e_1 + e_2 + \frac{1}{r_2} u, \qquad e_3' = e_3.$$

Then, the vectors in $\{e_1', e_2', e_3'\}$ satisfy the same equations as in (5), and $v = (r_1 + \langle u, u\rangle/(2r_2))e_1' + r_2 e_2' + r_3 e_3' + w$. This shows that we can assume, swapping to $\{e_1', e_2', e_3'\}$ if necessary, that $u = 0$.

Thus, we have

$$\begin{aligned}-1 &= \langle v, v\rangle = 2r_1 r_2 + r_3^2 + \langle w, w\rangle, \\ \tilde{S}v &= (r_1\lambda + r_3)e_1 + r_2\lambda e_2 + (r_2 + r_3\lambda)e_3 + \mu w.\end{aligned}$$

Using (2) we get

$$J\xi^L = -\frac{2}{\sqrt{-c}}\tilde{S}v = -\frac{2}{\sqrt{-c}}\Big((r_1\lambda + r_3)e_1 + r_2\lambda e_2 + (r_2 + r_3\lambda)e_3 + \mu w\Big),$$

and since $2r_1 r_2 = -1 - r_3^2 - \langle w, w\rangle$ we obtain

$$\begin{aligned}1 = \langle J\xi^L, J\xi^L\rangle &= -\frac{4}{c}\left(2r_1 r_2\lambda^2 + 4r_2 r_3\lambda + r_2^2 + r_3^2\lambda^2 + \langle w, w\rangle\mu^2\right) \\ &= -\frac{4}{c}\left(4r_2 r_3\lambda + r_2^2 - \lambda^2 + (\mu^2 - \lambda^2)\langle w, w\rangle\right), \\ 0 = \langle \tilde{S}v, v\rangle &= 2r_1 r_2\lambda + 2r_2 r_3 + r_3^2\lambda + \mu\langle w, w\rangle = 2r_2 r_3 - \lambda + (\mu - \lambda)\langle w, w\rangle.\end{aligned}$$

Hence, we get

$$r_2^2 + (\mu - \lambda)^2\langle w, w\rangle = -\frac{c}{4} - \lambda^2, \text{ or equivalently, } \left(\frac{2r_2}{\sqrt{-c - 4\lambda^2}}\right)^2 + \left(\frac{2(\mu - \lambda)\|w\|}{\sqrt{-c - 4\lambda^2}}\right)^2 = 1.$$

Since $r_2 > 0$ we obtain $\lambda \in \big(-\sqrt{-c}/2, \sqrt{-c}/2\big) \setminus \{0\}$. Note that, since $c + 4\lambda\mu = 0$ we have $-c - 4\lambda^2 = 4\lambda(\mu - \lambda)$. Solving the previous equations yields

$$r_2 = \sin(\varphi)\frac{\sqrt{-c - 4\lambda^2}}{2}, \qquad \|w\| = \cos(\varphi)\frac{2\lambda}{\sqrt{-c - 4\lambda^2}}, \qquad r_3 = \frac{\lambda}{\sqrt{-c - 4\lambda^2}}\sin(\varphi), \tag{6}$$

for a suitable $\varphi \in \big(0, \pi/2\big]$. The proof now diverges from the one that can be found in [7].

Assume $\varphi \neq \pi/2$, that is, $w \neq 0$. We have that the vectors in $T_\lambda(q) \ominus \mathbb{R}e_2$ and in $T_\mu(q) \ominus \mathbb{R}w$ are orthogonal to v and $J\xi^L$. These vectors project bijectively, via the Hopf map π_{*q}, to eigenvectors of the principal curvatures λ and μ respectively, and they are all orthogonal to $J\xi$. Let $L = T_q\tilde{M} \ominus \big((T_\lambda(q) \ominus \mathbb{R}e_2) \oplus (T_\mu(q) \ominus \mathbb{R}w) \oplus \mathbb{R}v\big)$. Then, L is a 3-dimensional space, and thus, $h(p) \leq 3$. Furthermore, by (3) we see that $h(p) \neq 1$; otherwise $\tilde{S}$ would contain at most a 2×2 nondiagonal block, and so q would not be of type III. In fact, L is spanned by the following basis: $l_1 = r_1 e_1 - r_2 e_2$, $l_2 = r_3 e_1 - r_2 e_3$ and $l_3 = -\langle w, w\rangle e_1 + r_2 w$. We have $\operatorname{span}\{e_1, e_2, e_3, w\} = L \oplus \mathbb{R}v$.

After some long calculations, and using (2) and $\pi_{*q}v = 0$, we get that the matrix expression of the shape operator of M at p restricted to $\pi_{*q}L$, with respect to the basis $\{\pi_* l_1, \pi_* l_2, \pi_* l_3\}$ is

$$\begin{pmatrix} \lambda + r_2 r_3 & r_2^2 & r_2(\lambda-\mu)\langle w, w\rangle \\ 1 + r_3^2 & \lambda + r_2 r_3 & r_3(\lambda-\mu)\langle w, w\rangle \\ -r_3 & -r_2 & \mu - (\lambda-\mu)\langle w, w\rangle \end{pmatrix}.$$

Using the expressions we got for r_2, r_3, and $\langle w, w\rangle$, together with $4\lambda\mu + c = 0$, we can calculate the characteristic polynomial of the previous matrix. This polynomial turns out to be precisely $f_{\lambda,\varphi}$, as defined in the statement of Theorem 3.4. This is the same characteristic polynomial as that of the nontrivial part of the shape operator of a tube around the submanifolds $W_{\mathfrak{w}}$ in Theorem 1.1 (vi) (see also [6]). We have

$$f_{\lambda,\varphi}(\lambda) = -\frac{(c+4\lambda^2)^2 \sin^2(\varphi)}{16\lambda} > 0, \qquad f_{\lambda,\varphi}(\mu) = \frac{(c+4\lambda^2)^2 \cos^2(\varphi)}{16\lambda} > 0.$$

Therefore, neither λ nor μ are eigenvalues of the matrix above. Moreover, the same argument as in [2, p. 146] proves that the three zeroes of $f_{\lambda,\varphi}$ are different. Hence, if $\varphi \in (0, \pi/2)$, M has $g(p) \in \{3, 4, 5\}$ principal curvatures at p: the zeroes of $f_{\lambda,\varphi}$, possibly λ, and possibly μ. Indeed, $g(p) = 3$ if $T_\lambda(q) \ominus \mathbb{R}e_2 = T_\mu(q) \ominus \mathbb{R}w = 0$, $g(p) = 4$ if either $T_\lambda(q) = \mathbb{R}e_1$ or $T_\mu(q) = \mathbb{R}w$, and $g(p) = 5$ otherwise.

We now prove that, in this case ($\varphi \neq \pi/2$), we have $h(p) = 3$. The characteristic polynomial of the shape operator $\tilde{S}$ restricted to $L \oplus \mathbb{R}v$ is $(x-\lambda)^3(x-\mu)$. Define x_1, x_2, x_3 to be unit eigenvectors of S_p whose corresponding eigenvalues are the three different zeroes λ_1, λ_2, λ_3 of the polynomial $f_{\lambda,\varphi}$, respectively. Set $b_i = \langle J\xi, x_i\rangle$, for $i = 1, 2, 3$. Then, according to (3), the shape operator $\tilde{S}$ of $\tilde{M}$ at q restricted to $L \oplus \mathbb{R}v$ with respect to the basis $\{x_1^L, x_2^L, x_3^L, v\}$ is given by

$$\begin{pmatrix} \lambda_1 & 0 & 0 & -b_1\frac{\sqrt{-c}}{2} \\ 0 & \lambda_2 & 0 & -b_2\frac{\sqrt{-c}}{2} \\ 0 & 0 & \lambda_3 & -b_3\frac{\sqrt{-c}}{2} \\ b_1\frac{\sqrt{-c}}{2} & b_2\frac{\sqrt{-c}}{2} & b_3\frac{\sqrt{-c}}{2} & 0 \end{pmatrix}.$$

Using $b_1^2 + b_2^2 + b_3^2 = 1$, we get the characteristic polynomial of this matrix:

$$\begin{aligned} & x^4 + (-\lambda_1 - \lambda_2 - \lambda_3)\, x^3 \\ & + \frac{1}{4}(-c + 4\lambda_1\lambda_2 + 4\lambda_1\lambda_3 + 4\lambda_2\lambda_3)\, x^2 \\ & + \frac{1}{4}\left(b_1^2 c\lambda_2 + b_1^2 c\lambda_3 + b_2^2 c\lambda_1 + b_3^2 c\lambda_1 + b_3^2 c\lambda_2 + b_2^2 c\lambda_3 - 4\lambda_1\lambda_2\lambda_3\right) x \\ & - \frac{c}{4}\left(b_1^2\lambda_2\lambda_3 + b_3^2\lambda_1\lambda_2 + b_2^2\lambda_1\lambda_3\right). \end{aligned}$$

Both the previous polynomial and $(x-\lambda)^3(x-\mu)$ must coincide, as they come from the same endomorphism of $L \oplus \mathbb{R}v$. Thus, by comparing the linear and independent terms of these polynomials, we obtain the following linear system in the variables b_1^2, b_2^2, b_3^2:

$$\begin{aligned} \frac{c}{4}\big((\lambda_2+\lambda_3)b_1^2 + (\lambda_1+\lambda_3)b_2^2 + (\lambda_1+\lambda_2)b_3^2\big) - \lambda_1\lambda_2\lambda_3 &= -\lambda^2(\lambda + 3\mu), \\ -\frac{c}{4}\left(b_1^2\lambda_2\lambda_3 + b_3^2\lambda_1\lambda_2 + b_2^2\lambda_1\lambda_3\right) &= \lambda^3\mu, \\ b_1^2 + b_2^2 + b_3^2 &= 1. \end{aligned}$$

The determinant of the matrix of this linear system is $c^2(\lambda_1-\lambda_2)(\lambda_3-\lambda_1)(\lambda_2-\lambda_3)/16 \neq 0$, so the system has a unique solution. Using the relations among λ, μ, λ_1, λ_2 and λ_3 that the equality of the characteristic polynomials imposes for the quadratic and cubic terms, namely,

$$\begin{aligned} \lambda_1 + \lambda_2 + \lambda_3 &= 3\lambda + \mu, \\ -\frac{c}{4} + \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 &= 3\lambda(\lambda + \mu), \end{aligned}$$

one can check, after some elementary but long calculations, that the solution to the linear system above is given by:

$$b_i^2 = -\frac{4(\lambda-\lambda_i)^3(\lambda_i-\mu)}{c(\lambda_{i+1}-\lambda_i)(\lambda_i-\lambda_{i+2})}, \qquad i=1,\,2,\,3, \quad \text{(indices modulo 3)}.$$

Since μ and λ are different from any λ_i, $i \in \{1,2,3\}$, we conclude that $b_i \neq 0$ for all $i \in \{1,2,3\}$, whence $h(p)=3$. This finishes the proof of Theorem 3.4 (iii).

Now assume $\varphi = \pi/2$, that is, $w=0$ (recall that we are still assuming that $\tilde{S}$ has two distinct eigenvalues $\lambda, \mu \neq 0$ at q). In this case, (6) yields

$$r_2 = \frac{\sqrt{-c-4\lambda^2}}{2}, \qquad w=0, \qquad r_3 = \frac{\lambda}{\sqrt{-c-4\lambda^2}}.$$

Then, the vectors of $T_\lambda(q) \ominus \mathbb{R}e_2$ and $T_\mu(q)$ are orthogonal to v and $J\xi^L$, project via π_{*q} onto the principal curvature spaces of λ and μ respectively, and these projections are orthogonal to $J\xi$. So, in this case, we have $h(p)=2$. Defining l_1 and l_2 as above, the shape operator of M at p restricted to $\operatorname{span}\{\pi_* l_1, \pi_* l_2\}$, with respect to the basis $\{\pi_* l_1, \pi_* l_2\}$, turns out to be

$$\begin{pmatrix} \lambda + r_2 r_3 & r_2^2 \\ 1 + r_3^2 & \lambda + r_2 r_3 \end{pmatrix} = \begin{pmatrix} \frac{3\lambda}{2} & -\frac{c}{4} - \lambda^2 \\ \frac{c+3\lambda^2}{c+4\lambda^2} & \frac{3\lambda}{2} \end{pmatrix}.$$

Thus, the eigenvalues of the shape operator of M at p restricted to $\operatorname{span}\{\pi_* l_1, \pi_* l_2\}$ are

$$\frac{1}{2}\left(3\lambda \pm \sqrt{-c-3\lambda^2}\right).$$

These eigenvalues are different and also different from λ.

For $\lambda = \sqrt{-c}/(2\sqrt{3})$ we have $\mu = \big(3\lambda + \sqrt{-c-3\lambda^2}\big)/2 = \sqrt{-3c}/2$, hence $g(p) \in \{2,3\}$, and the principal curvatures are 0 (with multiplicity one), $\sqrt{-3c}/2$, and possibly $\sqrt{-c}/(2\sqrt{3})$. The possibility $g(p)=2$ arises if $T_\lambda(q) = \mathbb{R}e_1$, and in this case λ is not a principal curvature of M at p. We have $g(p)=3$ otherwise. The Hopf vector has nontrivial projections onto the principal curvature spaces corresponding to 0 and μ. This corresponds to Theorem 3.4 (iia).

For $\lambda \neq \sqrt{-c}/(2\sqrt{3})$ we get $g(p) \in \{3,4\}$. We have $g(p)=3$ if $T_\lambda(q) = \mathbb{R}e_1$, that is, if λ is not a principal curvature of M at p, and $g(p)=4$ otherwise. The principal curvatures $(3\lambda \pm \sqrt{-c-3\lambda^2})/2$ have both multiplicity one, and the Hopf vector has nontrivial projection onto their corresponding principal curvature spaces. This corresponds to case (i) if $g(p)=4$ and to case (iib) if $g(p)=3$. This finishes the proof if $\tilde{S}$ has two distinct principal curvatures λ and μ.

Finally, assume that $\tilde{M}$ has just one principal curvature $\lambda \geq 0$ at q. In this case, calculations are very similar to what we have just obtained if $w=0$. Thus, we get

$$\lambda \in \left(-\frac{\sqrt{-c}}{2}, \frac{\sqrt{-c}}{2}\right), \qquad r_2 = \frac{\sqrt{-c-4\lambda^2}}{2}, \qquad r_3 = \frac{\lambda}{\sqrt{-c-4\lambda^2}}.$$

Arguing as in the case $\varphi = \pi/2$ above, we obtain $h(p)=2$ and $g(p)=3$ (for dimension reasons $T_\lambda(q) = \mathbb{R}e_1$ cannot happen now). The principal curvatures of M at p are $\big(3\lambda \pm \sqrt{-c-3\lambda^2}\big)/2$ and λ. The first two have multiplicity one and the Hopf vector has nontrivial projection onto their corresponding principal curvature spaces. Now we can have $\lambda=0$, and then, the other principal curvatures would be $\pm\sqrt{-c}/2$. If $\lambda \neq \sqrt{-c}/(2\sqrt{3})$, this corresponds to case (iib) again. If $\lambda = \sqrt{-c}/(2\sqrt{3})$, then we also get case (iia), although now $\sqrt{-3c}/2$ has multiplicity one and $\lambda = \sqrt{-c}/(2\sqrt{3})$ is definitely a principal curvature of M at p. □

Remark 3.5. Theorem 3.4 implies that, for points of Type III, the principal curvatures of isoparametric hypersurfaces in $\mathbb{C}H^n$ and their multiplicities must coincide (at that precise point) with those of the homogeneous examples in cases (iv) and (v) in Theorem 1.1, except for some particular cases which we would like to point out here (we assume the notation given in the proof of Theorem 3.4):

A. Theorem 3.4(iia) for $g(p)=h(p)=2$: this happens if $T_\lambda(p) = \mathbb{R}e_1$.
B. Theorem 3.4(iib) if λ is not a principal curvature of M at p, that is, if $T_\lambda(q) = \mathbb{R}e_1$.

C. Theorem 3.4(iii) for $g(p) = h(p) = 3$ (this happens whenever $T_\lambda(q) \ominus \mathbb{R}e_2 = T_\mu(q) \ominus \mathbb{R}w = 0$) or if $g(p) = 4$ and λ is not a principal curvature of M at p (equivalently, if $T_\lambda(q) = \mathbb{R}e_1$).

The three cases in Remark 3.5 are ruled out by a different method in [7]. Here we content ourselves with Theorem 3.4 which, together with Proposition 3.2, implies Theorem 1.3. The multiplicities of these principal curvatures are the same as in the homogeneous examples except for the three possibilities above. This stronger result is a consequence of Theorem 1.1 but cannot be proved with the method used in this paper. Proposition 1.4 is also obtained from Theorem 3.4.

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