Four-dimensional manifolds with degenerate self-dual Weyl curvature operator

A. Cortés-Ayaso* J. C. Díaz-Ramos* E. García-Río*

January 11, 2014

Abstract

It is shown that any four-dimensional Walker metric of nowhere zero scalar curvature has a natural almost para-Hermitian structure. In contrast to the Goldberg-Sachs theorem, if this structure is self-dual and $*$-Einstein, it is symplectic but not necessarily integrable. This is due to the non-diagonalizability of the self-dual Weyl conformal curvature tensor.

MS Classification. 53C15, 53C50, 53C55, 53B30.

Key words and phrases. Para-Hermitian and para-Kähler structure, self-dual Weyl curvature tensor, Walker metric.

1 Introduction

A widely investigated topic in differential geometry is the influence of the curvature on the underlying structure of the manifold (see, for example [3], [21] and the references therein). Certainly related but less explored, is the existence and properties of additional structures defined by certain curvature operators (see [11] for an example of certain Clifford module structures defined by means of the Jacobi operators). Another example, which is the starting motivation of this work, comes from the generalized Goldberg-Sachs theorem in Riemannian geometry, which asserts that under certain degeneracy condition on the self-dual Weyl conformal curvature tensor, a four-dimensional Einstein metric becomes Hermitian [2].

Generalizations of the Goldberg-Sachs theorem have been carried out for Lorentzian [26], [27] and for indefinite metrics of signature $(--++)$ under natural assumptions on the degeneracy and the diagonalizability of the self-dual Weyl curvature operator W^+. Any such Einstein metric becomes Hermitian (with respect to a negative complex structure) or para-Hermitian according to the causal character of the one-dimensional eigenspace associated with the distinguished eigenvalue of the self-dual Weyl curvature operator [1], [22]. Since

*Supported by projects MTM2006-01432 and PGIDIT06PXIB207054PR (Spain)

the diagonalizability of W^+ plays an essential role in both results, our plan is to exhibit examples showing the necessity of such assumption. If W^+ is non-diagonalizable and $W^+ \neq 0$, then W^+ is necessarily degenerate and the distinguished eigenvalue with multiplicity one has associated timelike eigenspace since the induced metric on $\Lambda^2_\pm M$ is of Lorentzian signature $(--+)$. In such a case, an additional structure can still be constructed on the underlying manifold, namely, an almost para-Hermitian structure. However, the Einstein assumption does not force this structure to be integrable although, if the metric is assumed to be self-dual, the associated two-form is closed.

The paper is organized as follows. In Section 2, we introduce the kind of manifolds under consideration (Walker manifolds), with special attention to their Weyl conformal structure, and we show that the self-dual Weyl curvature operator is always degenerate. The existence of a distinguished eigenvalue of W^+ is considered in §3. We study the associated almost para-Hermitian structure and show that it is isotropic (i.e., $\|\nabla\Omega\|^2 = \|N_J\|^2 = \|d\Omega\|^2 = 0$). Moreover, the existence of such an almost para-Hermitian structure allows us to obtain some topological restrictions on the existence of such Walker metrics. Finally, the influence of additional properties of the curvature is considered in §4, where we show that any $*$-Einstein self-dual Walker metric with nonzero scalar curvature is symplectic.

2 Walker metrics

A *Walker manifold* is a triple $(M, g, \mathcal{D})$, where M is an n-dimensional manifold, g an indefinite metric and $\mathcal{D}$ a parallel isotropic r-dimensional plane. Walker metrics constitute the underlying structure of many strictly pseudo-Riemannian situations with no Riemannian counterpart: indecomposable (but not irreducible) holonomy [4], Einstein hypersurfaces with nilpotent Jacobi operators [23] or some classes of nonsymmetric Osserman metrics [16] are typical examples. Walker metrics have also been considered in general relativity in the study of $\mathfrak{hh}$ spaces [8], [17]. Moreover, the fact that para-Kähler and hypersymplectic metrics are necessarily of Walker type motivates the consideration of such metrics in connection with almost para-Hermitian structures.

For any Walker metric a canonical form was obtained in [28], where it is shown the existence of suitable coordinates (x_1, x_2, x_3, x_4) for which the metric is expressed as

$$g = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & a & c \\ 0 & 1 & c & b \end{pmatrix} \tag{1}$$

for some functions a, b and c depending on the variables (x_1, x_2, x_3, x_4).

Let us denote by ∇ the Levi-Civita connection of g and by R its curvature tensor, with the sign convention $R(X,Y) = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y]$. The curvature tensor can be understood as an endomorphism of the bundle of 2-forms, R :

$\Lambda^2 M \to \Lambda^2 M$, by setting $R(dx^i \wedge dx^j)(\partial_k, \partial_l) = R(\partial_i, \partial_j, \partial_k, \partial_l)$. Since the metric has neutral signature, we have the following $O(2,2)$-decomposition

$$R = \frac{\tau}{12}\mathrm{Id}_{\Lambda^2} + \rho^0 + W,$$

where ρ^0 denotes the traceless Ricci tensor, τ is the scalar curvature and W denotes the Weyl conformal curvature tensor.

The Hodge star operator $\star : \Lambda^2 M \to \Lambda^2 M$ associated with g induces a further splitting, $\Lambda^2 M = \Lambda^2_+ M \oplus \Lambda^2_- M$, where $\Lambda^2_\pm M$ denote the ± 1-eigenspaces of the Hodge star operator, that is, $\Lambda^2_\pm M = \{\alpha \in \Lambda^2 M : \star\alpha = \pm\alpha\}$. Since W and $\star$ commute, the curvature tensor further decomposes as

$$R = \frac{\tau}{12}\,\mathrm{Id}_{\Lambda^2} + \rho^0 + W^+ + W^-,$$

where $W^\pm = (W \pm \star W)/2$. Recall that a pseudo-Riemannian 4-dimensional manifold is called *self-dual* (resp. *anti-self-dual*) if $W^- = 0$ (resp. $W^+ = 0$).

Let $\{e_1, e_2, e_3, e_4\}$ be an orthonormal basis with e_1 and e_2 spacelike vectors and e_3 and e_4 timelike vectors. Local bases of the spaces of self-dual and anti-self-dual two-forms may be constructed as $\Lambda^2_\pm M = \mathrm{span}\left\{E_1^\pm, E_2^\pm, E_3^\pm\right\}$, where

$$E_1^\pm = \frac{e^1 \wedge e^2 \pm e^3 \wedge e^4}{\sqrt{2}}, \quad E_2^\pm = \frac{e^1 \wedge e^3 \pm e^2 \wedge e^4}{\sqrt{2}}, \quad E_3^\pm = \frac{e^1 \wedge e^4 \mp e^2 \wedge e^3}{\sqrt{2}}.$$

Here, $\{e^1, e^2, e^3, e^4\}$ is the dual basis of $\{e_1, e_2, e_3, e_4\}$.

So far, we have just recalled general results for pseudo-Riemannian manifolds of signature $(++--)$. Next, we specialize the above results for Walker manifolds by considering the following orthonormal basis:

$$\begin{aligned} e_1 &= \frac{1}{2}(1-a)\partial_1 + \partial_3, & e_2 &= -c\partial_1 + \frac{1}{2}(1-b)\partial_2 + \partial_4, \\ e_3 &= -\frac{1}{2}(1+a)\partial_1 + \partial_3, & e_4 &= -c\partial_1 - \frac{1}{2}(1+b)\partial_2 + \partial_4. \end{aligned} \tag{2}$$

Observe that $\left\{E_1^\pm, E_2^\pm, E_3^\pm\right\}$ is an orthonormal basis of $\Lambda^2_\pm M$ with $\langle E_1^\pm, E_1^\pm\rangle = 1$, $\langle E_2^\pm, E_2^\pm\rangle = -1$, $\langle E_3^\pm, E_3^\pm\rangle = -1$.

A long but straightforward calculation (see for example, [16]) implies that the self-dual Weyl operator W^+ is given by

$$W^+ = \begin{pmatrix} W_{11}^+ & W_{12}^+ & W_{11}^+ + \tau/12 \\ -W_{12}^+ & \tau/6 & -W_{12}^+ \\ -W_{11}^+ - \tau/12 & -W_{12}^+ & -W_{11}^+ - \tau/6 \end{pmatrix}, \tag{3}$$

where W_{11}^+ and W_{12}^+ are given by

$$
\begin{aligned}
W_{11}^+ &= \frac{1}{12}\Big(6ca_1b_2 - 6a_1b_3 - 2c_{12} + 12a_1c_4 - 6ca_2b_1 + 6a_2b_4 + 6ba_2c_1 \\
&\quad + 6a_3b_1 - 6a_4b_2 - 12a_4c_1 + 6ab_1c_2 - 6ab_2c_1 + 12b_2c_3 - 12b_3c_2 \\
&\quad - 12c^2a_{11} - 12bca_{12} + 24ca_{14} - 3b^2a_{22} + 12ba_{24} - 12a_{44} \\
&\quad - 3a^2b_{11} + 12ab_{13} + 12acc_{11} + 6abc_{12} - 24cc_{13} - 12ac_{14} \\
&\quad - 12bc_{23} + 24c_{34} - a_{11} - b_{22} - 12b_{33}\Big), \\
W_{12}^+ &= \frac{1}{4}\big(-6ba_1c_2 - 2ca_{11} - ba_{12} + 2a_{14} \\
&\quad + ab_{12} - 2b_{23} + ac_{11} - 2cc_{12} - 2c_{13} - bc_{22} + 2c_{24}\big).
\end{aligned}
$$

(Hereafter, for any function h we denote by $h_{i_1 \dots i_k}$, the partial derivatives $\partial^k h/\partial x_{i_1} \dots \partial x_{i_k}$ with respect to the coordinate functions).

Remark 1 It follows from (3) that W^+ has eigenvalues $\tau/6$ and $-\tau/12$, the later with multiplicity two, and hence *the self-dual Weyl conformal curvature operator W^+ of any Walker metric is degenerate. Moreover, if $\tau \neq 0$ then W^+ is diagonalizable if and only if (see* [16]*)*

$$\tau^2 + 12\tau W_{11}^+ + 48\left(W_{12}^+\right)^2 = 0.$$

and $W^+ \neq 0$ if the scalar curvature is non-zero.

Remark 2 Reversing the orientation of the Walker manifold, one could work with the anti-self-dual Weyl conformal operator W^-. However the situation in this case is much more difficult since the eigenvalue structure of W^- is more complicated, allowing complex eigenvalues. (See [9] for more information on anti-self-dual Walker metrics).

3 Almost para-Hermitian Walker structure

A *para-Kähler manifold* is a symplectic manifold admitting two transversal Lagrangian foliations (see [12], [22]). Such a structure induces a decomposition of the tangent bundle TM into the Whitney sum of Lagrangian subbundles L and L', that is, $TM = L \oplus L'$. By generalizing this definition, an *almost para-Hermitian manifold* is an almost symplectic manifold (M, Ω) whose tangent bundle splits into the Whitney sum of Lagrangian subbundles. This definition implies that the $(1,1)$-tensor field J defined by $J = \pi_L - \pi_{L'}$ is an *almost para-complex structure*, that is $J^2 = id$ on M, such that $\Omega(JX, JY) = -\Omega(X, Y)$ for all $X, Y \in \Gamma TM$, where π_L and π'_L are the projections of TM onto L and L', respectively. The 2-form Ω induces a nondegenerate $(0,2)$-tensor field g on M defined by $g(X, Y) = \Omega(X, JY)$, where $X, Y \in \Gamma TM$. It follows now that the two-form Ω defines a timelike section of Λ^2_+, and conversely, any timelike section of Λ^2_+ defines an almost para-Hermitian structure. See [18] for a discussion of almost para-Hermitian geometry from the point of view of general relativity.

Hence, assuming $\tau \neq 0$, the self-dual Weyl curvature operator W^+ has a distinguished eigenvalue $\tau/6$ whose associated eigenspace is timelike for the induced metric on Λ^2_+. After replacing M by a two-fold covering if necessary, it follows that the $\tau/6$-eigenspace of W^+ is generated by a globally defined self-dual two-form. We take as generator the two-form

$$\Omega = -\frac{8W^+_{12}}{\tau\sqrt{2}}E^+_1 - \sqrt{2}E^+_2 + \frac{8W^+_{12}}{\tau\sqrt{2}}E^+_3,$$

that verifies $\langle\Omega, \Omega\rangle = -2$ with respect to the induced metric on Λ^+M. Hence it gives rise to an almost para-Hermitian structure on the Walker manifold M.

Indeed, let us define $J \in \mathrm{End}(TM)$ by $g(JX, Y) = \Omega(X, Y)$ for all vector fields X and Y. Expressing the E^+_i's in terms of the e^i's it is easy to get that J has the matrix representation with respect to $\{e_1, e_2, e_3, e_4\}$:

$$J = \begin{pmatrix} 0 & 4W^+_{12}/\tau & -1 & 4W^+_{12}/\tau \\ -4W^+_{12}/\tau & 0 & -4W^+_{12}/\tau & -1 \\ -1 & -4W^+_{12}/\tau & 0 & -4W^+_{12}/\tau \\ 4W^+_{12}/\tau & -1 & 4W^+_{12}/\tau & 0 \end{pmatrix}.$$

From here, it is easy to check that $J^2 = \mathrm{Id}$ and $g(JX, JY) = -g(X, Y)$ for all vectors X and Y. In what follows we will refer to (g, J) as *the almost para-Hermitian Walker structure*, which is globally defined on any Walker manifold with nowhere zero scalar curvature.

Remark 3 A coordinate description of the almost para-Hermitian structure J is obtained by a direct calculation from (2), namely

$$J = \begin{pmatrix} 1 & 0 & a & 2c + \frac{4}{\tau}W^+_{12} \\ 0 & 1 & -\frac{4}{\tau}W^+_{12} & b \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Analogously, Ω may be written as

$$\Omega = dx_1 \wedge dx_3 + dx_2 \wedge dx_4 - \Big(c + \frac{4W^+_{12}}{\tau}\Big)dx_3 \wedge dx_4.$$

3.1 Isotropic almost para-Hermitian Walker structures

Following the terminology in [19], an almost para-Hermitian structure (g, J) is called *isotropic para-Kähler* if $\|\nabla J\|^2 = 0$ but $\nabla J \neq 0$. The existence of isotropic Kähler structures has been already reported in [5], [13] in connection with the Goldberg conjecture in the pseudo-Riemannian situation. Indeed, such property is the underlying fact that supports the existence of Einstein and *-Einstein almost Kähler structures that are not Kähler. We first consider the following.

Theorem 4 *The almost para-Hermitian Walker structure* (g, J) *is isotropic para-Kähler, isotropic almost para-Kähler and isotropic para-Hermitian, that is,* $\|\nabla\Omega\|^2 = \|d\Omega\|^2 = \|N_J\|^2 = 0$.

Proof. For the sake of simplicity in what follows we denote $f = 4W_{12}^+/\tau$, and hence,

$$\Omega = dx_1 \wedge dx_3 + dx_2 \wedge dx_4 - (c+f)dx_3 \wedge dx_4, \tag{4}$$

with respect to the coordinate frame fields. Put $(\nabla\Omega)_{ijk} = (\nabla_{\partial_i}\Omega)(\partial_j, \partial_k)$. After some calculations we get that the non-zero components of $\nabla\Omega$ are

$$\begin{aligned}
(\nabla\Omega)_{134} &= -(\nabla\Omega)_{143} = -c_1 - f_1, \\
(\nabla\Omega)_{234} &= -(\nabla\Omega)_{243} = -c_2 - f_2, \\
(\nabla\Omega)_{334} &= -(\nabla\Omega)_{343} \\
&= \frac{1}{2}\Big(2a_4 - 4c_3 - 2f_3 - ba_2 - fc_2 - (2c+f)a_1 + ac_1\Big), \\
(\nabla\Omega)_{434} &= -(\nabla\Omega)_{443} \\
&= \frac{1}{2}\Big(-2f_4 - 2b_3 - fb_2 - bc_2 + ab_1 - (2c+f)c_1\Big).
\end{aligned} \tag{5}$$

Now, $\|\nabla\Omega\|^2 = \sum g^{ip}g^{jq}g^{kr}(\nabla\Omega)_{ijk}(\nabla\Omega)_{pqr}$ yields $\|\nabla\Omega\|^2 = 0$, which shows that (g, J) is isotropic Kähler.

Moreover, it follows from (4) that

$$d\Omega = -(c_1 + f_1)dx_1 \wedge dx_3 \wedge dx_4 - (c_2 + f_2)dx_2 \wedge dx_3 \wedge dx_4, \tag{6}$$

and thus $\|d\Omega\|^2 = 0$.

Finally recall that J is an integrable almost para-complex structure if the associated Nijenhuis tensor, $N_J(X, Y) = [JX, JY] - J[JX, Y] - J[X, JY] + [X, Y]$, vanishes. The non-zero components of N_J are

$$\begin{aligned}
&N_{13}^4 = -N_{14}^3 \\
&\quad = 4((2c_1 + f_1)a - ba_2 - 2ca_1 - (a_1 + 2c_2 + f_2)f + a_4 - 4c_3 - 2f_3), \\
&N_{23}^4 = -N_{24}^3 \\
&\quad = 4(ab_1 + bf_2 + 2cf_1 + (f_1 - b_2)f - 2b_3 - 2f_4),
\end{aligned} \tag{7}$$

where $N_J(\partial_i, \partial_j) = \sum_k N_{ij}^k \partial_k$. The result follows from the formula $\|N_J\|^2 = \sum g^{ip}g^{jq}g_{kr}N_{ij}^k N_{pq}^r$ after some calculations. □

Remark 5 We emphasize that although $\nabla\Omega$, $d\Omega$ and N_J are isotropic, the corresponding tensor fields are not zero. Indeed the following are inmediate consequences of (5)–(7)

(i) Ω is symplectic (equivalently (g, J) is called *almost para-Kähler*) if and only if $c + f$ does not depend on x_1 and x_2.

(ii) J is integrable (equivalently (g, J) is called *para-Hermitian*) if and only if

$$(ii.1) \quad (2c_1 + f_1)a - ba_2 - 2ca_1 - (a_1 + 2c_2 + f_2)f + a_4 - 4c_3 - 2f_3 = 0,$$
$$(ii.2) \quad ab_1 + bf_2 + 2cf_1 + (f_1 - b_2)f - 2b_3 - 2f_4 = 0.$$

Moreover (g, J) is para-Kähler if and only if (i) and (ii) are satisfied.

3.2 Global structure of 4-dimensional Walker manifolds

Due to the existence of coordinates (1) the local properties of Walker metrics have been widely investigated but very few is know as concerns their global structure (see for example [14], [20]). However, if the scalar curvature is nowhere zero, the metric becomes almost para-Hermitian, from where some conclusions can be obtained.

The Hitchin-Thorpe inequalities for Einstein 4-manifolds have been generalized to the $(--++)$-setting in [24], showing that the Hirzebruch index $\tau[M]$ and the Euler characteristic $\chi[M]$ satisfy $\frac{3}{2}|\tau[M]| \leq \chi[M]$. Walker manifolds of nowhere zero scalar curvature provide examples where the equality is attained.

Theorem 6 *Let (M, g) be a compact Walker 4-manifold whose scalar curvature is nowhere zero. Then $\frac{3}{2}\tau[M] = -\chi[M]$.*

Proof. First of all, note that the existence of an almost para-Hermitian structure (g, J) is an equivalent condition to the existence of an almost anti-Hermitian structure $(g, \mathcal{J})$ (i.e., $\mathcal{J}^2 = -Id$, $g(\mathcal{J}X, \mathcal{J}Y) = -g(X, Y)$ for all vector fields X, Y on M). Indeed, let (g, J) be an almost para-Hermitian structure and let h be an arbitrary Riemannian metric on M such that $h(JX, JY) = h(X, Y)$ for all X, Y. Define an almost product structure Q by $h(QX, Y) = g(X, Y)$ and put $\mathcal{J} = -JQ$. An straightforward calculation shows that $\mathcal{J}$ is an almost complex structure on M and moreover

$$g(\mathcal{J}X, \mathcal{J}Y) = g(JQX, JQY) = -g(QX, QY) = -g(X, Y),$$

which shows that $(g, \mathcal{J})$ is almost anti-Hermitian.

Moreover, a straightforward calculation shows that $\mathcal{J}$ is h-orthogonal and the fundamental forms Ω and $\Omega^h_{\mathcal{J}}(X, Y) = h(\mathcal{J}X, Y)$ coincide. Hence, both J and $\mathcal{J}$ induce the same orientation on M (see [6], [7] for more information on anti-Hermitian and anti-Kähler structures). Chern classes of almost complex manifolds with anti-Hermitian metrics where studied in [6], [7], where it is shown that all odd Chern classes vanish (i.e., $c_{2k+1}[M] = 0$, for all k).

Finally, the result is obtained from the fact that an orientable four-dimensional manifold admits a $(--++)$-metric if and only if it satisfies a pair of Wu's conditions

$$c_1^2[M] = 3\tau[M] + 2\chi[M], \qquad c_1^2[-M] = -3\tau[M] + 2\chi[M],$$

where $-M$ stands for M with the opposite orientation [25]. □

4 Self-dual Walker metrics

The existence of an almost para-Hermitian structure allows one to define the $*$-*Ricci tensor*, ρ^*, by $\rho^*(X,Y) = \frac{1}{2}\,\mathrm{trace}\{Z \mapsto R(X,JY)JZ\}$ for any vector fields X and Y. The contraction of this tensor is called the $*$-*scalar curvature* and is denoted by τ^*. An almost para-Hermitian structure is said to be *weakly* $*$-*Einstein* if $\rho^* = (\tau^*/4)g$. Furthermore, if the $*$-scalar curvature is constant, (g,J) is called $*$-*Einstein.*

Lemma 7 *Assume the Walker metric g is self-dual. Then the almost para-Hermitian Walker structure (g,J) is weakly $*$-Einstein if and only if*

$$\begin{aligned} a &= x_1^2\mathcal{B} + x_1\mathcal{P} + x_2\mathcal{Q} + \xi, \\ b &= x_2^2\mathcal{B} + x_1\mathcal{S} + x_2\mathcal{T} + \eta, \\ c &= x_1x_2\mathcal{B} + x_1\mathcal{U} + x_2\mathcal{V} + \gamma, \end{aligned}$$

where all calligraphic and Greek letters depend just on x_3 and x_4. Moreover, (g,J) is $*$-Einstein if and only if $\mathcal{B}$ is constant.

Proof. The non-zero components of the $*$-Ricci tensor can be calculated from the local expression of the curvature tensor (see for example [16]):

$$\begin{aligned} &\rho^*_{13} = \rho^*_{31} = \frac{1}{2}\left(a_{11} + c_{12}\right), \qquad \rho^*_{14} = \rho^*_{41} = \frac{1}{2}\left(b_{12} + c_{11}\right), \\ &\rho^*_{23} = \rho^*_{32} = \frac{1}{2}\left(a_{12} + c_{22}\right), \qquad \rho^*_{24} = \rho^*_{42} = \frac{1}{2}\left(b_{22} + c_{12}\right), \\ &\rho^*_{33} = \frac{1}{2}\left((a_{11} + c_{12})a - (a_{12} + c_{22})f\right), \\ &\rho^*_{34} = \frac{1}{2}\left(b_{23} + c_{12} - a_{14} - c_{24} + (c_{22} + a_{12})b + (c_{12} + a_{11})(2c + f)\right), \\ &\rho^*_{43} = \frac{1}{2}\left(a_{14} + c_{24} - b_{23} - c_{13} + (b_{12} + c_{11})a - (b_{22} + c_{12})f\right), \\ &\rho^*_{44} = \frac{1}{2}\left((b_{22} + c_{12})b + (b_{12} + c_{11})(2c + f)\right). \end{aligned}$$

Now considering the $*$-Einstein equations $\mathcal{F} = \rho^* - (\tau^*/4)g \equiv 0$, one gets that (g,J) is weakly $*$-Einstein if and only if

$$a_{11} = b_{22}, \qquad c_{11} = -b_{12}, \qquad a_{12} = -c_{22}. \tag{8}$$

Recall that a Walker metric is self-dual if and only if (1) is given by (cf. [16])

$$\begin{aligned} a &= x_1^3\mathcal{A} + x_1^2\mathcal{B} + x_1^2x_2\mathcal{C} + x_1x_2\mathcal{D} + x_1\mathcal{P} + x_2\mathcal{Q} + \xi, \\ b &= x_2^3\mathcal{C} + x_2^2\mathcal{E} + x_1x_2^2\mathcal{A} + x_1x_2\mathcal{F} + x_1\mathcal{S} + x_2\mathcal{T} + \eta, \\ c &= \tfrac{1}{2}x_1^2\mathcal{F} + \tfrac{1}{2}x_2^2\mathcal{D} + x_1^2x_2\mathcal{A} + x_1x_2^2\mathcal{C} + \tfrac{1}{2}x_1x_2(\mathcal{B} + \mathcal{E}) + x_1\mathcal{U} + x_2\mathcal{V} + \gamma, \end{aligned}$$

where all calligraphic and Greek letters depend just on x_3 and x_4.

Considering the equations (8) we obtain

$$4x_1\mathcal{A} - 4x_2\mathcal{C} + 2\mathcal{B} - 2\mathcal{E} = 0, \quad 4x_2\mathcal{A} + 2\mathcal{F} = 0, \quad 4x_1\mathcal{C} + 2\mathcal{D} = 0,$$

which implies $\mathcal{A} = \mathcal{C} = \mathcal{D} = \mathcal{F} = 0$ and $\mathcal{E} = \mathcal{B}$, thus characterizing the weakly $*$-Einstein structures. The $*$-Einstein case now follows from $\tau^* = 6\mathcal{B}$. □

Theorem 8 *Let (M, g) be a self-dual Walker metric with nowhere zero scalar curvature. If the almost para-Hermitian Walker structure is $*$-Einstein, then it becomes almost para-Kähler.*

Proof. By Remark 5-(i), Ω is symplectic if and only if $c + f$ does not depend on (x_1, x_2), where $f = 4W_{12}^+/\tau$. A long calculation from Lemma 7 shows that

$$c + f = \frac{1}{3\mathcal{B}}(3x_1\mathcal{B}_4 - 3x_3\mathcal{B}_3 - \mathcal{T}_3 - \mathcal{U}_3 + \mathcal{P}_4 + \mathcal{V}_4).$$

This expressions does not depend on (x_1, x_2) if and only if $\mathcal{B}_3 = \mathcal{B}_4 = 0$. That is, if and only if $\tau^* = 6\mathcal{B}$ is constant. □

Remark 9 The *para-holomorphic sectional curvature* associated with a non-null tangent vector x is, by definition, the sectional curvature of the 2-plane spanned by $\{x, Jx\}$, that is, $H(x) = -g(R(x, Jx)x, Jx)/g(x, x)^2$.

A straightforward calculation from Lemma 7 shows that the para-holomorphic sectional curvature of any weakly $*$-Einstein self-dual Walker metric satisfies $H(x) = \mathcal{B}$, and thus it is pointwise constant and globally constant if (g, J) is further assumed to be $*$-Einstein.

Remark 10 Einstein self-dual Walker metrics have been studied in [16] in connection with the Osserman problem. It follows immediately from the results in [16] that any non Ricci flat self-dual Einstein Walker metric is $*$-Einstein, and thus symplectic. Moreover, the almost para-Hermitian Walker structure is never integrable unless it corresponds to a paracomplex space form (since the Bochner tensor vanishes due to $W_- = 0$) [10].

Further note that para-Kähler manifolds of non-zero constant para-holomorphic sectional curvature are locally described by the para-Kähler Walker structure (g, J) corresponding to the Walker metric [16]

$$a(x_1, x_2, x_3, x_4) = \alpha x_1^2, \quad b(x_1, x_2, x_3, x_4) = \alpha x_2^2, \quad c(x_1, x_2, x_3, x_4) = \alpha x_1 x_2.$$

Finally observe that in such case the self-dual Weyl curvature operator is diagonalizable.

Acknowledgement Some references and comments from the Referee in a previous version are greatly appreciated.

References

[1] V. Apostolov; Generalized Goldberg-Sachs theorems for pseudo-Riemannian four-manifolds, *J. Geom. Phys.* **27** (1998), 185–198.

[2] V. Apostolov, P. Gauduchon; The Riemannian Goldberg-Sachs theorem, *Int. J. Math.* **8** (1997), 421–439.

[3] V. Apostorov, T. Draghici; The curvature and the integrability of almost Kähler manifold: a survey. *Symplectic and contact topology: interactions and perspectives.* Toronto, ON/Montreal, QC, (2001), 25–53, Fields Inst. Commun. **35** Amer. Math. Soc., Providence, RI, 2003.

[4] L. Bérard Bergery, A. Ikemakhen; Sur L'holonomie des variétés pseudo-Riemanniennes de signature (n, n), *Bull. Soc. Math. France* **125** (1997), 93–114.

[5] D. Blair, J. Davidov, O. Muškarov; Isotropic Kähler hyperbolic twistor spaces, *J. Geom. Phys.* **52** (2004), 74–88.

[6] A. Bonome, R. Castro, E. García-Río, L. Hervella, Y. Matsushita; Pseudo-Chern classes and opposite Chern classes of indefinite almost Hermitian manifolds, *Acta Math. Hungar.* **75** (1997), 299–316.

[7] A. Borowiec, M. Francaviglia, I. Volovich; Anti-Kählerian manifolds, *Differential Geom. Appl.* **12** (2000), 281–289.

[8] C. P. Boyer, J. D. Finley III, J. F. Plebański; Complex general relativity, $\mathfrak{H}$ and $\mathfrak{H}\mathfrak{H}$ spaces – a survey of one approach. *General relativity and gravitation*, Vol. 2, pp. 241–281, Plenum, New York-London, 1980.

[9] M. Brozos-Vázquez, E. García–Río, R. Vázquez-Lorenzo; Conformal Osserman four-dimensional manifolds whose conformal Jacobi operators have complex eigenvalues, *Proc. Royal Society A* **462** (2006), 1425–1441.

[10] R. L. Bryant; Bochner-Kähler metrics, *J. Amer. Math. Soc.* **14** (2001), 623–715.

[11] Q. S. Chi; Curvature characterization and classification of rank-one symmetric spaces, *Pacific J. Math.* **150** (1991), 31–42.

[12] V. Cruceanu, P. Fortuny, P. M. Gadea; A survey on paracomplex geometry, *Rocky Mount. J. Math.* **26** (1996), 83–115.

[13] J. Davidov, J. C. Díaz-Ramos, E. García-Río, Y. Matsushita, O. Muškarov, R. Vázquez-Lorenzo; Almost Kähler Walker 4-manifolds, *J. Geom. Phys.*, to appear.

[14] A. Derdziński; W. Roter, Walker's theorem without coordinates, *J. Math. Phys.* **47** (2006), 062504.

[15] A. Derdziński; Self-dual Kähler manifolds and Einstein manifolds in dimension 4, *Comp. Math.* **49** (1983), 405–433.

[16] J. C. Díaz-Ramos, E. García-Río, R. Vázquez-Lorenzo; Four-dimensional Osserman metrics with nondiagonalizable Jacobi operators, *J. Geom. Anal.* **16** (2006), 39–52.

[17] J. D. Finley III, J. F. Plebański; The intrinsic spinorial structure of hyperheavens, *J. Math. Phys.* **17** (1976), 2207–2214.

[18] E. J. Flaherty; *Hermitian and Kählerian geometry in relativity*, Lect. Notes in Phys. **46**, Springer-Verlag, Berlin-New York, 1976.

[19] E. García-Río, Y. Matsushita; Isotropic Kähler structures on Engel 4-manifolds, *J. Geom. Phys.* **33** (2000), 288–294.

[20] P. Gilkey, S. Nikčević; Complete k-curvature homogeneous pseudo-Riemannian manifolds, *Ann. Global Anal. Geom.* **27** (2005), 87–100.

[21] L. Hernández-Lamoneda; Curvature vs. almost Hermitian structures, *Geom. Dedicata* **79** (2000), 205–218.

[22] S. Ivanov, S. Zamkovoy; ParaHermitian and paraquaternionc manifolds, *Differential Geom. Appl.* **23** (2005), 205–234.

[23] M. A. Magid; Indefinite Einstein hypersurfaces with nilpotent shape operators, *Hokkaido Math. J.* **13** (1984), 241–250.

[24] P. R. Law, Y. Matsushita; Hitchin-Thorpe-type inequalities for pseudo-Riemannian 4-manifolds of metric signature $(++--)$, *Geom. Dedicata* **87** (2001), 65–89.

[25] Y. Matsushita; Fields of 2-planes and two kinds of almost complex structures on compact 4-dimensional manifolds, *Math. Z.* **207** (1991), 281–291.

[26] J. F. Plebański, S. Hacyan; Null geodesic surfaces and Goldberg-Sachs theorem in complex Riemannian spaces, *J. Math. Phys.* **16** (1975), 2403–2407.

[27] M. Przanowski, B. Broda; Locally Kähler gravitational instantons, *Acta Phys. Polon. B* **14** (1983), 637–661.

[28] A. G. Walker; Canonical form for a Riemannian space with a parallel field of null planes, *Quart. J. Math. Oxford (2)* **1** (1950), 69–79.

Authors' addresses:

Alexandre Cortés-Ayaso, Eduardo García-Río; Faculty of Mathematics, University of Santiago de Compostela, 15782 Santiago de Compostela, Spain, (alancoia@usc.es, xtedugr@usc.es)

J. Carlos Díaz-Ramos; Department of Mathematics, University College Cork, Cork, Ireland, (JC.DiazRamos@ucc.ie)