

# Hermitian Walker 4-manifolds

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Abstract. A local description of proper Hermitian Walker structures that are locally conformally Kähler, self-dual, \*-Einstein or Einstein is given. This is used to supply examples of indefinite Einstein strictly almost Hermitian structures showing that an integrability result in [19] does not hold for metrics of signature  $(2, 2)$ .

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## 1 Introduction

An almost Hermitian structure on a smooth manifold  $M$  consists of an almost complex structure  $J$  and a metric  $g$  satisfying the compatibility condition  $g(JX, JY) = g(X, Y)$ . If the almost complex structure  $J$  is integrable (i.e. comes from a complex structure on  $M$ ), the structure  $(g, J)$  is called Hermitian. Any almost Hermitian structure  $(g, J)$  determines a non-degenerate 2-form  $\Omega(X, Y) = g(JX, Y)$ . If  $\Omega$  is closed (i.e., it is a symplectic form) the structure is said to be almost Kähler and  $(g, J)$  is called Kähler if, in addition, the almost complex structure  $J$  is integrable.

A basic problem in almost Hermitian geometry is to relate properties of the structure  $(g, J)$  to the curvature of  $(M, g)$ . For example, the well-known Goldberg conjecture [14] claims that a compact almost Kähler manifold is Kähler provided the metric  $g$  is Einstein. This conjecture was proved by K. Sekigawa [26] in the case of non-negative scalar curvature but it is still far from being solved in the negative case. We refer to the survey [2] for an update on the integrability of almost Kähler structures. Another integrability result related to the curvature properties of a manifold is the Riemannian version of the well-known Goldberg-Sacks theorem in General Relativity. It says that an oriented Einstein 4-manifold admits locally a compatible complex structure if and only if the spectrum of the positive Weyl tensor is degenerate [23, 25]. We refer to [3] for generalizations of this result in the Riemannian setting and to [1] for analogous results for arbitrary pseudo-Riemannian 4-manifolds.

One should note that some integrability results for almost Hermitian manifolds are not true in the case of indefinite metrics. For example, it is shown in [8] that, in contrast to the Riemannian case, there are local examples of flat non-Kähler almost Kähler metrics of signature  $(2, 2)$ . Moreover, an indefinite Ricci flat strictly almost Kähler metric on 8-dimensional torus has been recently reported in [22].

The purpose of this paper is to provide a large family of non-Kähler isotropic Kähler Hermitian structures having interesting curvature properties. To this end we consider Walker metrics [27] on 4-manifolds together with the so-called proper almost complex structure [21] and obtain a local description of those metrics which are Hermitian or locally conformally Kähler and self-dual, \*-Einstein or Einstein. We also construct examples of indefinite Einstein strictly almost Hermitian structures showing that an integrability result in [19] does not hold for metrics of signature  $(2, 2)$ .

## 2 Preliminaries

Throughout this paper we use the following convention for the curvature tensor  $R(X, Y) = \nabla_{[X, Y]} - [\nabla_X, \nabla_Y]$ , where  $\nabla$  denotes the Levi-Civita connection.  $\rho(X, Y) = \text{trace}\{U \rightsquigarrow R(X, U)Y\}$  and  $\tau = \text{trace}\rho$  are the Ricci tensor and the scalar curvature, respectively. As usual,  $(M, g)$  is said to be Einstein if  $\rho = \frac{\tau}{n}g$ ,  $n = \dim M$ , in which case the scalar curvature is necessarily constant.

Let  $M$  be an oriented 4-dimensional manifold with a neutral metric  $g$ , i.e. a metric of signature  $(2, 2)$ . The metric  $g$  induces an inner product on the bundle  $\Lambda^2$  of bivectors by

$$\langle X_1 \wedge X_2, X_3 \wedge X_4 \rangle = g(X_1, X_3)g(X_2, X_4) - g(X_1, X_4)g(X_2, X_3),$$

$X_1, \dots, X_4 \in TM$ . Let  $\mathbf{e}_1, \dots, \mathbf{e}_4$  be a local oriented orthonormal frame of  $TM$  with  $\|\mathbf{e}_1\|^2 = \|\mathbf{e}_2\|^2 = 1$ ,  $\|\mathbf{e}_3\|^2 = \|\mathbf{e}_4\|^2 = -1$ . As in the Riemannian case, the Hodge star operator  $*$ :  $\Lambda^2 \rightarrow \Lambda^2$  is an involution given by

$$*(\mathbf{e}_1 \wedge \mathbf{e}_2) = \mathbf{e}_3 \wedge \mathbf{e}_4, \quad *(\mathbf{e}_1 \wedge \mathbf{e}_3) = \mathbf{e}_2 \wedge \mathbf{e}_4, \quad *(\mathbf{e}_1 \wedge \mathbf{e}_4) = -\mathbf{e}_2 \wedge \mathbf{e}_3.$$

Denote by  $\Lambda_{\pm}$  the subbundles of  $\Lambda^2$  determined by the eigenvalues  $\pm 1$  of the Hodge star operator. Set

$$\begin{aligned} s_1 &= \frac{1}{\sqrt{2}}(\mathbf{e}_1 \wedge \mathbf{e}_2 - \mathbf{e}_3 \wedge \mathbf{e}_4), & \bar{s}_1 &= \frac{1}{\sqrt{2}}(\mathbf{e}_1 \wedge \mathbf{e}_2 + \mathbf{e}_3 \wedge \mathbf{e}_4), \\ s_2 &= \frac{1}{\sqrt{2}}(\mathbf{e}_1 \wedge \mathbf{e}_3 - \mathbf{e}_2 \wedge \mathbf{e}_4), & \bar{s}_2 &= \frac{1}{\sqrt{2}}(\mathbf{e}_1 \wedge \mathbf{e}_3 + \mathbf{e}_2 \wedge \mathbf{e}_4), \\ s_3 &= \frac{1}{\sqrt{2}}(\mathbf{e}_1 \wedge \mathbf{e}_4 + \mathbf{e}_2 \wedge \mathbf{e}_3), & \bar{s}_3 &= \frac{1}{\sqrt{2}}(\mathbf{e}_1 \wedge \mathbf{e}_4 - \mathbf{e}_2 \wedge \mathbf{e}_3). \end{aligned} \quad (1)$$

Then  $\{s_1, s_2, s_3\}$  and  $\{\bar{s}_1, \bar{s}_2, \bar{s}_3\}$  are local oriented orthonormal frames of  $\Lambda_-$  and  $\Lambda_+$ , respectively, with  $\|s_1\|^2 = \|\bar{s}_1\|^2 = 1$ ,  $\|s_2\|^2 = \|\bar{s}_2\|^2 = \|s_3\|^2 = \|\bar{s}_3\|^2 = -1$ .

Further we shall often identify  $\Lambda^2$  with the bundle of skew-symmetric endomorphisms of  $TM$  by the correspondence that assigns to each  $\sigma \in \Lambda^2$  the endomorphism  $K_\sigma$  on  $T_pM$ ,  $p = \pi(\sigma)$ , defined by

$$g(K_\sigma X, Y) = g(\sigma, X \wedge Y); \quad X, Y \in T_pM. \quad (2)$$

Considering the Riemann curvature tensor as an endomorphism of  $\Lambda^2$ , we have the following  $SO(2, 2)$ -decomposition

$$R \equiv \frac{\tau}{12} Id_{\Lambda^2} + \rho_0 + W^+ + W^- : \Lambda^2 \rightarrow \Lambda^2, \quad (3)$$

where  $\tau$  is the scalar curvature,  $\rho_0$  denotes the traceless Ricci tensor,  $W = W^+ + W^-$  is the Weyl conformal curvature tensor and  $W^\pm = \frac{1}{2}(W \pm *W)$ . Recall that a pseudo-Riemannian 4-manifold is called *self-dual* (resp., *anti-self-dual*) if  $W^- = 0$  (resp.,  $W^+ = 0$ ).

A *Walker manifold* is a triple  $(M, g, \mathcal{D})$  where  $M$  is an  $n$ -dimensional manifold,  $g$  an indefinite metric and  $\mathcal{D}$  an  $r$ -dimensional parallel null distribution. Of special interest are those manifolds admitting a field of null planes of maximal dimension ( $r = \frac{n}{2}$ ). Since the dimension of a null plane is  $r \leq \frac{n}{2}$ , the lowest possible case of a Walker metric is that of  $(++--)$ -manifolds admitting a field of parallel null two-planes.

The Walker metrics appear in several specific pseudo-Riemannian structures like 2-step nilpotent Lie groups with degenerate center, para-Kähler and hyper-symplectic structures, hypersurfaces with nilpotent shape operator and some four-dimensional Osserman manifolds. Indecomposable metrics of neutral signature (playing a distinguished role in investigating holonomy of indefinite metrics) are also equipped with a Walker structure. This clearly motivates the study of pseudo-Riemannian manifolds carrying a parallel degenerate plane field (see [8] and the references therein for more information).

For our purposes it is convenient to use special coordinate systems associated to any Walker metric. Recall that, by a result of Walker [27], for every Walker metric  $g$  on a 4-manifold  $M$ , there exist local coordinates  $(x, y, z, t)$  around any point of  $M$  such that the matrix of  $g$  in these coordinates has the following form

$$g_{(x,y,z,t)} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & a & c \\ 0 & 1 & c & b \end{pmatrix} \quad (4)$$

for some functions  $a$ ,  $b$  and  $c$  depending on the coordinates  $(x, y, z, t)$ . As a matter of notation, throughout this work we denote by  $\partial_i$  the coordinate tangent vectors,  $i = x, \dots, t$ . Also,  $h_i$  means partial derivative  $\frac{\partial h}{\partial x_i}$ ,  $i = x, \dots, t$ , for any function  $h(x, y, z, t)$ . Expressions for the Levi-Civita connection, curvature tensor and Ricci tensor of a Walker metric (4) are available at [8, 9, 21]. We omit the details for the sake of brevity.

### 3 Proper almost hyper-parahermitian structures

An almost hyper-paracomplex structure on a  $4n$ -dimensional manifold  $M$  is a triple  $(J_1, J_2, J_3)$ , where  $J_2, J_3$  are almost paracomplex structures (cf. [16]) and  $J_1$  is an almost complex structure, satisfying the paraquaternionic identities

$$J_1^2 = -J_2^2 = -J_3^2 = -1, \quad J_1 J_2 = -J_2 J_1 = J_3.$$

A hyper-parahermitian metric is a pseudo-Riemannian metric which is compatible with the (almost) hyper-paracomplex structure in the sense that the metric  $g$  is skew-symmetric with respect to each  $J_i$ ,  $i = 1, 2, 3$ , i.e.

$$g(J_1 \cdot, J_1 \cdot) = -g(J_2 \cdot, J_2 \cdot) = -g(J_3 \cdot, J_3 \cdot) = g(\cdot, \cdot).$$

Such a structure is called hyper-parahermitian if all the structures  $J_i$  are integrable. If each  $J_i$ ,  $i = 1, 2, 3$ , is parallel with respect to the Levi-Civita connection or, equivalently, the three Kähler forms  $\Omega_i(X, Y) = g(J_i X, Y)$  are closed, then the manifold is called hyper-symplectic [15] or hyper-parakählerian. In this case  $J_2$  and  $J_3$  are parakähler structures and it follows that  $g$  is a Walker metric (see [17, 18] for more information).

If  $(g, J_1, J_2, J_3)$  is an almost hyper-parahermitian structure, then the bivectors corresponding via the metric to the two-forms  $\Omega_1, \Omega_2, \Omega_3$  define an orthonormal basis of  $\Lambda_-^2$ , and conversely, any orthonormal basis of  $\Lambda_-^2$  defines an almost hyper-parahermitian structure.

Let  $g$  be a Walker metric on  $\mathbb{R}^4$  having the form (4) Then an orthonormal frame of  $T\mathbb{R}^4$  can be specialized by using the canonical coordinates as follows:

$$\mathbf{e}_1 = \frac{1-a}{2} \partial_x + \partial_z, \quad \mathbf{e}_2 = \frac{1-b}{2} \partial_y + \partial_t - c \partial_x, \quad (5)$$

$$\mathbf{e}_3 = -\frac{1+a}{2} \partial_x + \partial_z, \quad \mathbf{e}_4 = -\frac{1+b}{2} \partial_y + \partial_t - c \partial_x.$$

Let  $\{s_1, s_2, s_3, \bar{s}_1, \bar{s}_2, \bar{s}_3\}$  be the frame of  $\Lambda^2 = \Lambda_- \oplus \Lambda_+$  defined by means of  $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$  via (1). Then

$$s_1 = \frac{1}{\sqrt{2}} \left( -\frac{a+b}{2} \partial_x \wedge \partial_y + \partial_x \wedge \partial_t - \partial_y \wedge \partial_z \right),$$

$$s_2 = \frac{1}{\sqrt{2}} (\partial_x \wedge \partial_z - \partial_y \wedge \partial_t - c \partial_x \wedge \partial_y), \quad (6)$$

$$s_3 = \frac{1}{\sqrt{2}} \left( \frac{a-b}{2} \partial_x \wedge \partial_y + \partial_x \wedge \partial_t + \partial_y \wedge \partial_z \right),$$

and

$$\bar{s}_1 = \frac{1}{\sqrt{2}} \left( \frac{1+ab}{2} \partial_x \wedge \partial_y + 2c \partial_x \wedge \partial_z - a \partial_x \wedge \partial_t + b \partial_y \wedge \partial_z + 2 \partial_z \wedge \partial_t \right),$$

$$\bar{s}_2 = \frac{1}{\sqrt{2}} (c \partial_x \wedge \partial_y + \partial_x \wedge \partial_z + \partial_y \wedge \partial_t), \quad (7)$$

$$\bar{s}_3 = \frac{1}{\sqrt{2}} \left( \frac{ab-1}{2} \partial_x \wedge \partial_y + 2c \partial_x \wedge \partial_z - a \partial_x \wedge \partial_t + b \partial_y \wedge \partial_z + 2 \partial_z \wedge \partial_t \right).$$

The bivectors  $s_1, s_2, s_3$  define via (2) endomorphisms  $J_1, J_2, J_3$  of  $T\mathbb{R}^4$  such that

$$J_1^2 = -1, \quad J_2^2 = J_3^2 = 1, \quad J_1 J_2 = -J_2 J_1 = J_3.$$

We shall say that the almost hyper-paracomplex structure  $J_1, J_2, J_3$  defined by means of  $s_1, s_2, s_3$  is *proper*. Note that  $J_1$  is an isometry of the Walker metric  $g$  while  $J_2, J_3$  are anti-isometries, i.e.  $J_1, J_2, J_3$  is an almost hyper-parahermitian structure.

Recall that an indefinite almost Hermitian (resp., almost parahermitian) structure  $(g, J)$  is said to be *isotropic Kähler* (resp., *parakähler*) if  $\|\nabla J\|^2 = 0$ . Isotropic Kähler structures were first investigated in [13] in dimension four and subsequently in [4] in dimension six. It has been shown by the authors in [8] that any proper almost Hermitian structure is isotropic Kähler. Moreover, we have the following:

**Theorem 1** *Any proper almost hyper-parahermitian structure  $(g, J_1, J_2, J_3)$  on a Walker 4-manifold satisfies  $\|\nabla J_i\|^2 = 0$ ,  $\|d\Omega_i\|^2 = 0$ ,  $\|\delta\Omega_i\|^2 = 0$  and  $\|N_{J_i}\|^2 = 0$ , where  $\Omega_i$  denotes the fundamental 2-form and  $N_{J_i}$  the Nijenhuis tensor of  $J_i$ ,  $i = 1, 2, 3$ .*

**Proof.** From the expressions (4) and (6) one gets the description in local coordinates of the structures  $J_i$  ( $i = 1, 2, 3$ ) as follows:

$$J_1 = \begin{pmatrix} 0 & -1 & -c & \frac{a-b}{2} \\ 1 & 0 & \frac{a-b}{2} & c \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 1 & 0 & a & 0 \\ 0 & -1 & 0 & -b \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad J_3 = \begin{pmatrix} 0 & 1 & c & \frac{a+b}{2} \\ 1 & 0 & \frac{a+b}{2} & c \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

Now the result follows after some calculations as in [8].  $\square$

**Theorem 2** *A proper almost hyper-parahermitian structure  $J_1, J_2, J_3$  is hyper-parahermitian (i.e.  $J_1, J_2, J_3$  are integrable) if and only if the functions  $a, b$  and  $c$  in (4) have the forms*

$$\begin{aligned} a &= x^2\mathcal{B}(z, t) + xP(z, t) + \xi(z, t), \\ b &= y^2\mathcal{B}(z, t) + yT(z, t) + \eta(z, t), \\ c &= xy\mathcal{B}(z, t) + \frac{1}{2}xT(z, t) + \frac{1}{2}yP(z, t) + \gamma(z, t), \end{aligned} \quad (8)$$

where the capital, calligraphic and Greek letters stand for arbitrary smooth functions depending only on the coordinates  $(z, t)$ .

**Proof.** Let  $\Omega_i(X, Y) = g(J_i X, Y)$  be the fundamental 2-form of  $(g, J_i)$ ,  $i = 1, 2, 3$ . Then we have

$$\begin{aligned} \Omega_1 &= dx \wedge dt - dy \wedge dz + \frac{a+b}{2} dz \wedge dt, \\ \Omega_2 &= -dx \wedge dz + dy \wedge dt + cdz \wedge dt, \\ \Omega_3 &= -dx \wedge dt - dy \wedge dz - \frac{a-b}{2} dz \wedge dt. \end{aligned}$$

The differential of  $\Omega_i$  has the form  $d\Omega_i = \omega_i \wedge \Omega_i$ ,  $i = 1, 2, 3$ , where the Lee forms  $\omega_i$  are given by

$$\begin{aligned} \omega_1 &= -\frac{1}{2}(a+b)_x dz - \frac{1}{2}(a+b)_y dt, \\ \omega_2 &= -c_y dz - c_x dt, \\ \omega_3 &= -\frac{1}{2}(a-b)_x dz + \frac{1}{2}(a-b)_y dt. \end{aligned} \quad (9)$$

Now the result follows from (9) and the well-known theorem [5, 17] that an almost hyper-parahermitian structure is hyper-parahermitian if and only if the three Lee forms coincide.  $\square$

**Remark 1** Observe that all metrics (8) are self-dual (see (14)) and moreover, the Ricci operator has a unique eigenvalue  $\lambda = \frac{3}{2}\mathcal{B}(z, t)$ , which is a double root of its minimal polynomial.

Formulas (9) imply also the following

**Theorem 3** *A proper almost hyper-parahermitian structure  $J_1, J_2, J_3$  is hyper-parakählerian if and only if the functions  $a, b$  and  $c$  do not depend on  $x$  and  $y$ .*

It is well-known [18] that, as in the definite case [5], any hyper-parakählerian structure is Ricci flat. Neutral Ricci flat non-flat metrics on complex tori and primary Kodaira surfaces have been constructed in [24]. These metrics are induced by proper hyper-parakählerian Walker structures on  $\mathbb{R}^4$ . Further observe that proper hyper-parakähler structures correspond to Walker metrics admitting two orthogonal parallel null vector fields  $\{\partial_x, \partial_y\}$ .

## 4 Proper Hermitian Walker structures

The existence of a metric of signature  $(++--)$  with structure group  $SO_0(2, 2)$  is equivalent to the existence of a pair of commuting almost complex structures [20], and moreover, any such pseudo-Riemannian metric may be viewed as an indefinite almost Hermitian metric for a suitable almost complex structure. Such almost complex structures are not uniquely determined. One such structure

associated with a four-dimensional Walker metric in the form (4) was given in [21] and called the *proper* almost complex structure. Our purpose here is to investigate curvature properties of Walker metrics by considering the associated proper structure. It turns out that this structure exhibits a very rich behavior providing examples, as we mentioned above, of indefinite Ricci flat (non flat) Kähler structures on tori and primary Kodaira surfaces [24] as well as flat non Kähler almost Kähler structures [8]. This is in a sharp contrast to the Riemannian case and it is important to recognize that such an exceptional behavior comes from the fact that any proper almost Hermitian structure is isotropic Kähler but not necessarily Kähler.

The proper almost complex structure associated with the metric (4) coincides with the structure  $J_1$  defined in Section 3 and is given by

$$\begin{aligned} J\partial_x &= \partial_y, & J\partial_z &= -c\partial_x + \frac{1}{2}(a-b)\partial_y + \partial_t, \\ J\partial_y &= -\partial_x, & J\partial_t &= \frac{1}{2}(a-b)\partial_x + c\partial_y - \partial_z. \end{aligned} \quad (10)$$

According to [21], the proper almost Hermitian structure  $(g, J)$  is :

- almost Kähler if and only if

$$a_x + b_x = 0, \quad a_y + b_y = 0. \quad (11)$$

- Hermitian if and only if

$$a_x - b_x = 2c_y, \quad a_y - b_y = -2c_x. \quad (12)$$

- Kähler if and only if

$$a_x = -b_x = c_y, \quad a_y = -b_y = -c_x. \quad (13)$$

Self-dual Walker metrics have been previously investigated in [7, 9] where it has been shown that a metric (4) is self-dual if and only if the functions  $a, b, c$  have the forms

$$\begin{aligned} a(x, y, z, t) &= x^3\mathcal{A} + x^2\mathcal{B} + x^2y\mathcal{C} + xy\mathcal{D} + xP + yQ + \xi, \\ b(x, y, z, t) &= y^3\mathcal{C} + y^2\mathcal{E} + xy^2\mathcal{A} + xy\mathcal{F} + xS + yT + \eta, \\ c(x, y, z, t) &= \frac{1}{2}x^2\mathcal{F} + \frac{1}{2}y^2\mathcal{D} + x^2y\mathcal{A} + xy^2\mathcal{C} + \frac{1}{2}xy(\mathcal{B} + \mathcal{E}) + xU + yV + \gamma, \end{aligned} \quad (14)$$

where the capital, calligraphic and Greek letters stand for arbitrary smooth functions depending only on the coordinates  $(z, t)$ .

Now it follows from (14) using (12) and (13) that

**Theorem 4** *A proper Hermitian structure  $(g, J)$  on a Walker 4-manifold is self-dual if and only if*

$$\begin{aligned} a &= x^2\mathcal{B} + xy\mathcal{D} + xP + yQ + \xi, \\ b &= y^2\mathcal{B} - xy\mathcal{D} + xS + yT + \eta, \\ c &= \frac{1}{2}(y^2 - x^2)\mathcal{D} + xy\mathcal{B} - \frac{1}{2}x(Q - T) + \frac{1}{2}y(P - S) + \gamma, \end{aligned} \quad (15)$$

where all capital, calligraphic and Greek letters are arbitrary smooth functions depending only on the coordinates  $(z, t)$ .

**Remark 2** Note that the Ricci operator of any Walker metric (15) has complex eigenvalues  $\lambda = \frac{3}{2}\mathcal{B} \pm \sqrt{-1}\mathcal{D}$  of multiplicity two.

An indefinite Hermitian manifold  $(M, g, J)$  is called *locally conformally Kähler* if for any point  $p \in M$  there exists an open neighborhood  $U$  and a function  $f : U \rightarrow \mathbb{R}$  such that  $(U, e^{-f}g, J)$  is an indefinite Kähler manifold [10].

**Theorem 5** *The structure  $(g, J)$  is locally conformally Kähler if and only if the functions  $a$ ,  $b$  and  $c$  have the form*

$$\begin{aligned} a &= xP(z, t) + yQ(z, t) + \mathcal{A}(x, y, z, t) \\ b &= xP(z, t) + yQ(z, t) + \eta(z, t) - \mathcal{A}(x, y, z, t) \\ c &= \mathcal{B}(x, y, z, t), \end{aligned} \tag{16}$$

where  $P$ ,  $Q$  and  $\eta$  are smooth functions depending only on the coordinates  $(z, t)$ , the function  $\mathcal{A} + i\mathcal{B}$  is holomorphic with respect to  $w = x + iy$  and  $P_t = Q_z$ . Moreover,  $(g, J)$  is Kähler if and only if  $P = Q = 0$ .

**Proof.** Recall that a Hermitian structure  $(g, J)$  is locally conformally Kähler if and only if the Lee form  $\omega$  defined by  $d\Omega = \omega \wedge \Omega$  is a closed 1-form [11]. Now, observe that the Lee form  $\omega$  of the proper almost Hermitian structure  $(g, J)$  is given by

$$\omega = -\frac{1}{2}(a+b)_x dz - \frac{1}{2}(a+b)_y dt \tag{17}$$

(see the first equation in (9)). Therefore,  $d\omega = 0$  if and only if

$$(a+b)_{xx} = (a+b)_{xy} = (a+b)_{yy} = 0, \quad (a+b)_{xt} = (a+b)_{yz}. \tag{18}$$

In particular, the function  $a+b$  is linear with respect to  $x$  and  $y$ , i.e., it has the form

$$a+b = 2xP(z, t) + 2yQ(z, t) + \eta(z, t), \tag{19}$$

where  $P$ ,  $Q$  and  $\eta$  are smooth functions. Moreover, since the almost complex structure  $J$  is integrable, we have by (12) and (19) that

$$(a - xP(z, t) - yQ(z, t))_x = c_y, \quad (a - xP(z, t) - yQ(z, t))_y = -c_x,$$

which are the Cauchy-Riemann equations for the functions  $\mathcal{A} = a - xP(z, t) - yQ(z, t)$  and  $\mathcal{B} = c$  with respect to the variables  $x$  and  $y$ . Thus the function  $\mathcal{A} + i\mathcal{B}$  is holomorphic with respect to  $w = x + iy$  and  $a$ ,  $b$ ,  $c$  have the form (16). Note also that the equation  $(a+b)_{xt} = (a+b)_{yz}$  implies  $P_t = Q_z$ . Finally, the Kähler condition follows from (13).  $\square$

**Remark 3** The metrically equivalent vector field  $B$  of the Lee form  $\omega$  is called the *Lee vector field* and  $JB$  is usually named as the *anti-Lee vector field*. A special class of locally conformally Kähler structures, the so-called *Vaisman manifolds*, corresponds to the case of parallel Lee form. Note that although proper locally conformally Kähler structures are not necessarily Vaisman, the distribution generated by  $B$  and  $JB$  is parallel since  $B = -P(z, t)\partial_x - Q(z, t)\partial_y$ . Further observe that many of the striking differences between the positive definite locally conformally Kähler structures and the indefinite counterpart lie on the fact that the distribution  $\{B, JB\}$  may be degenerate, which indeed holds in the case under consideration (see [10]).

**Theorem 6** *Let  $(g, J)$  be a proper Hermitian structure with nowhere vanishing Lee form  $\omega$ . Then  $\omega$  is parallel if and only if the functions  $a$ ,  $b$ ,  $c$  have the forms*

$$\begin{aligned} a &= -2x \frac{PP_z + QQ_z}{P^2 + Q^2} + 2y \frac{PQ_z - QP_z}{P^2 + Q^2} + \xi \\ b &= 2x \frac{QP_t - PQ_t}{P^2 + Q^2} - 2y \frac{PP_t + QQ_t}{P^2 + Q^2} + \eta \\ c &= x \frac{Q(P_z - Q_t) - 2PQ_z}{P^2 + Q^2} - y \frac{P(P_z - Q_t) + 2QQ_z}{P^2 + Q^2} + \gamma, \end{aligned} \tag{20}$$

where  $P$ ,  $Q$ ,  $\xi$ ,  $\eta$ ,  $\gamma$  are smooth functions of  $(z, t)$  and

$$P_t - Q_z = 0, \quad P_z + Q_t = -(P^2 + Q^2). \tag{21}$$

**Proof.** The Levi-Civita connection of a Walker metric (4) is determined by (see for example [9]):

$$\begin{aligned}
\nabla_{\partial_x} \partial_z &= \frac{1}{2} a_x \partial_x + \frac{1}{2} c_x \partial_y, & \nabla_{\partial_x} \partial_t &= \frac{1}{2} c_x \partial_x + \frac{1}{2} b_x \partial_y, \\
\nabla_{\partial_y} \partial_z &= \frac{1}{2} a_y \partial_x + \frac{1}{2} c_y \partial_y, & \nabla_{\partial_y} \partial_t &= \frac{1}{2} c_y \partial_x + \frac{1}{2} b_y \partial_y, \\
\nabla_{\partial_z} \partial_z &= \frac{1}{2} (a a_x + c a_y + a_z) \partial_x + \frac{1}{2} (c a_x + b a_y - a_t + 2c_z) \partial_y - \frac{a_x}{2} \partial_z - \frac{a_y}{2} \partial_t, \\
\nabla_{\partial_z} \partial_t &= \frac{1}{2} (a_t + a c_x + c c_y) \partial_x + \frac{1}{2} (b_z + c c_x + b c_y) \partial_y - \frac{c_x}{2} \partial_z - \frac{c_y}{2} \partial_t, \\
\nabla_{\partial_t} \partial_t &= \frac{1}{2} (a b_x + c b_y - b_z + 2c_t) \partial_x + \frac{1}{2} (c b_x + b b_y + b_t) \partial_y - \frac{b_x}{2} \partial_z - \frac{b_y}{2} \partial_t.
\end{aligned} \tag{22}$$

Straightforward computations making use of (17) and (22) show that the Lee form  $\omega$  is parallel if and only if the structure  $(g, J)$  is locally conformally Kähler and the functions  $a, b, c$  satisfy the following equations

$$\begin{aligned}
a_x(a+b)_x + a_y(a+b)_y &= -2(a+b)_{xz}, & b_x(a+b)_x + b_y(a+b)_y &= -2(a+b)_{yt}, \\
c_x(a+b)_x + c_y(a+b)_y &= -2(a+b)_{yz} = -2(a+b)_{xt}.
\end{aligned}$$

In view of Theorem 5, the latter conditions are equivalent to the requirement that the functions  $\mathcal{A}$  and  $\mathcal{B}$  have the form

$$\mathcal{A} = \alpha x + \beta y + \xi, \quad \mathcal{B} = -\beta x + \alpha y + \gamma,$$

where  $\alpha, \beta, \xi, \gamma$  are functions of  $(z, t)$ ,

$$\alpha = -\frac{P(P_z - Q_t) + 2QQ_z}{P^2 + Q^2}, \quad \beta = -\frac{Q(P_z - Q_t) - 2PQ_z}{P^2 + Q^2}$$

and  $P, Q$  satisfy (21). Now the result follows from (16).

**Example.** A particular solution of (21) is given by

$$P = \frac{p}{pz + qt + r} \quad \text{and} \quad Q = \frac{q}{pz + qt + r},$$

where  $p, q, r$  are constants and  $p^2 + q^2 \neq 0$ . In this case

$$a = \frac{2px}{pz + qt + r} + \xi, \quad b = \frac{2qy}{pz + qt + r} + \eta, \quad c = \frac{qx + py}{pz + qt + r} + \gamma.$$

For the particular case of self-dual Walker metrics one has the following:

**Theorem 7** *A proper Hermitian structure  $(g, J)$  on a Walker 4-manifold is locally conformally Kähler self-dual if and only if*

$$\begin{aligned}
a &= xy\mathcal{D} + xP + yQ + \xi, \\
b &= -xy\mathcal{D} + xS + yT + \eta, \\
c &= \frac{1}{2}(y^2 - x^2)\mathcal{D} - \frac{1}{2}x(Q - T) + \frac{1}{2}y(P - S) + \gamma,
\end{aligned} \tag{23}$$

where all calligraphic and Greek letters are arbitrary smooth functions depending only on the coordinates  $(z, t)$  while capital letters depend only on the coordinates  $(z, t)$  and satisfy

$$(Q + T)_z = (P + S)_t. \tag{24}$$

**Proof.** Considering the characterization (18), the result is easily obtained imposing  $(a+b)_{xx} = 0$  and  $(a+b)_{xt} = (a+b)_{yz}$  on (15).  $\square$

**Corollary 8** *A proper Hermitian structure  $(g, J)$  is Kähler self-dual if and only if*

$$\begin{aligned}
a &= xy\mathcal{D} + xP + yQ + \xi, \\
b &= -xy\mathcal{D} - xP - yQ + \eta, \\
c &= \frac{1}{2}(y^2 - x^2)\mathcal{D} - xQ + yP + \gamma.
\end{aligned} \tag{25}$$

## 5 \*-Einstein proper Hermitian structures

Associated to an almost Hermitian structure  $(g, J)$  we consider the \*-Ricci tensor defined by  $\rho^*(X, Y) = \text{trace}\{U \rightsquigarrow -\frac{1}{2}JR(X, JY)U\}$  and the \*-scalar curvature  $\tau^* = \text{trace}\rho^*$ . Note that both  $\rho$  and  $\rho^*$  coincide in the Kähler setting but  $\rho^*$  is not symmetric in general. An  $n$ -dimensional almost Hermitian manifold  $(M, g, J)$  is called *weakly \*-Einstein* if  $\rho^* = \frac{\tau^*}{n}g$  and is said to be *\*-Einstein* if, in addition,  $\tau^*$  is constant.

**Theorem 9** *The structure  $(g, J)$  is Hermitian and \*-Einstein if and only if the functions  $a, b, c$  have one of the following three forms:*

$$\begin{aligned} a &= \kappa(x^2 - y^2) + xP + yQ + \xi, \\ b &= \kappa(y^2 - x^2) - xP - yQ - \xi + \frac{1}{\kappa}(P_z - Q_t), \\ c &= 2\kappa xy - xQ + yP + \gamma, \end{aligned} \tag{26}$$

or

$$\begin{aligned} a &= \kappa x^2 + xP + yQ + \xi, \\ b &= \kappa y^2 + xS + yT - \xi - \frac{1}{4\kappa}\{4(3S_z - P_z + 3Q_t - T_t) - (P + S)^2 - (Q + T)^2\}, \\ c &= \kappa xy - \frac{1}{2}x(Q - T) + \frac{1}{2}y(P - S) + \gamma, \end{aligned} \tag{27}$$

or

$$\begin{aligned} a &= xP + yQ + \xi, \\ b &= xS + yT + \eta, \\ c &= -\frac{1}{2}x(Q - T) + \frac{1}{2}y(P - S) + \gamma, \end{aligned} \tag{28}$$

where in the last case

$$4(3S_z - P_z + 3Q_t - T_t) = (P + S)^2 + (Q + T)^2. \tag{29}$$

Here  $\kappa$  is a non-zero constant and capital and Greek letters are smooth functions depending only on  $(z, t)$ . In the first case  $\tau^* = 8\kappa$ , in the second one  $\tau^* = 2\kappa$  and in the third case  $\tau^* = 0$ .

**Proof.** The \*-Einstein equation  $(\rho_0^* = \rho^* - \frac{\tau^*}{4}g = 0)$  for a proper almost Hermitian structure  $(g, J)$  defined by (4) and (10) can be written as a system of PDEs as follows (we refer to [9], [21] for the curvature formulas to determine the \*-Ricci tensor):

$$\begin{aligned} (\rho_0^*)_{xz} &= -(\rho_0^*)_{yt} = -(\rho_0^*)_{zx} = (\rho_0^*)_{ty} = \frac{1}{4}(a_{yy} - b_{xx}) = 0, \\ (\rho_0^*)_{xt} &= -(\rho_0^*)_{zy} = -\frac{1}{2}(a_{xy} - c_{xx}) = 0, \\ (\rho_0^*)_{yz} &= -(\rho_0^*)_{tx} = -\frac{1}{2}(b_{xy} - c_{yy}) = 0, \\ (\rho_0^*)_{zz} &= \frac{1}{4}\{a_x b_x + a_y(b_y - c_x) + b_y c_x + c_y(a_x - b_x) - c_x^2 - c_y^2 + 2c(a_{xy} - c_{xx}) \\ &\quad + ba_{yy} - 2a_{yt} + ab_{xx} - 2b_{xz} - (a + b)c_{xy} + 2c_{xt} + 2c_{yz}\} = 0, \\ (\rho_0^*)_{zt} &= -\frac{1}{4}\{(a - b)(a_{xy} - c_{xx}) + c(a_{yy} - b_{xx})\} = 0, \\ (\rho_0^*)_{tz} &= \frac{1}{4}\{(a - b)(b_{xy} - c_{yy}) + c(a_{yy} - b_{xx})\} = 0, \\ (\rho_0^*)_{tt} &= \frac{1}{4}\{a_x b_x + a_y(b_y - c_x) + b_y c_x + c_y(a_x - b_x) - c_x^2 - c_y^2 + 2c(b_{xy} - c_{yy}) \\ &\quad + ba_{yy} - 2a_{yt} + ab_{xx} - 2b_{xz} - (a + b)c_{xy} + 2c_{xt} + 2c_{yz}\} = 0. \end{aligned} \tag{30}$$



Note that the \*-scalar curvature is given by

$$\tau^* = -a_{yy} - b_{xx} + 2c_{xy}. \quad (31)$$

It follows from (12) and (30) that  $(g, J)$  is Hermitian and weakly \*-Einstein if and only if

$$a_x - b_x = 2c_y, \quad a_y - b_y = -2c_x, \quad a_{xy} = c_{xx}, \quad b_{xy} = c_{yy}, \quad a_{yy} = b_{xx}, \quad (32)$$

and

$$\begin{aligned} & a_x b_x + a_y b_y - a_y c_x + b_y c_x + a_x c_y - b_x c_y - c_x^2 - c_y^2 \\ & + b a_{yy} - 2a_{yt} + a b_{xx} - 2b_{xz} - (a + b)c_{xy} + 2c_{xt} + 2c_{yz} = 0. \end{aligned} \quad (33)$$

As an immediate consequence from (32) we get  $a_{xy} = b_{xy} = c_{xx} = c_{yy} = 0$  and therefore

$$\begin{aligned} a &= A(x, z, t) + \bar{A}(y, z, t), \\ b &= B(y, z, t) + \bar{B}(x, z, t), \\ c &= xyC(z, t) + xU(z, t) + yV(z, t) + \gamma(z, t). \end{aligned} \quad (34)$$

Now,  $a_{yy} = b_{xx}$  in (32) means  $\bar{A}_{yy}(y, z, t) = \bar{B}_{xx}(x, z, t)$ . Hence

$$\bar{A}(y, z, t) = \frac{1}{2}y^2 D(z, t) + yQ(z, t) + \bar{\xi}(z, t), \quad \bar{B}(x, z, t) = \frac{1}{2}x^2 D(z, t) + xS(z, t) + \bar{\eta}(z, t), \quad (35)$$

and  $a_x - b_x = 2c_y$ ,  $a_y - b_y = -2c_x$  in (32) lead to

$$\begin{aligned} A(x, z, t) &= \frac{1}{2}x^2(2C(z, t) + D(z, t)) + x(S(z, t) + 2V(z, t)) + \tilde{\xi}(z, t), \\ B(y, z, t) &= \frac{1}{2}y^2(2C(z, t) + D(z, t)) + y(Q(z, t) + 2U(z, t)) + \tilde{\eta}(z, t). \end{aligned} \quad (36)$$

Collecting together (34), (35), (36) and setting  $2L = 2C + D$ ,  $M = \frac{1}{2}D$ ,  $P = S + 2V$ ,  $T = Q + 2U$ ,  $\xi = \bar{\xi} + \tilde{\xi}$ ,  $\eta = \bar{\eta} + \tilde{\eta}$  we obtain

$$\begin{aligned} a &= x^2 L + y^2 M + xP + yQ + \xi, \\ b &= x^2 M + y^2 L + xS + yT + \eta, \\ c &= xy(L - M) - \frac{1}{2}x(Q - T) + \frac{1}{2}y(P - S) + \gamma. \end{aligned} \quad (37)$$

Now, using (31), the (constant) \*-scalar curvature is given by  $\tau^* = 2(L - 3M)$ , from where

$$M = \frac{1}{6}(2L - \tau^*).$$

Plugging  $a$ ,  $b$ ,  $c$  in (33) and differentiating twice by  $x$  we have

$$(\tau^*)^2 - 10\tau^*L(z, t) + 16L(z, t)^2 = 0.$$

Thus,  $L(z, t) = \kappa$  must be constant, and

$$\begin{aligned} a &= x^2 \kappa + \frac{1}{6}y^2(2\kappa - \tau^*) + xP + yQ + \xi, \\ b &= \frac{1}{6}x^2(2\kappa - \tau^*) + y^2 \kappa + xS + yT + \eta, \\ c &= \frac{1}{6}xy(4\kappa + \tau^*) - \frac{1}{2}x(Q - T) + \frac{1}{2}y(P - S) + \gamma, \end{aligned} \quad (38)$$

where  $\kappa = \frac{\tau^*}{8} \neq 0$ ,  $\kappa = \frac{\tau^*}{2} \neq 0$  or  $\kappa = \tau^* = 0$ . The proof finishes analyzing each of these cases by separate.

Starting with the case  $\kappa = \frac{\tau^*}{8} \neq 0$  and differentiating (33) by  $x$  and by  $y$  we have

$$\tau^*(P + S) = 0, \quad \tau^*(Q + T) = 0,$$

which implies  $S = -P$  and  $T = -Q$ . Thus, (33) reduces to  $\tau^*(\xi + \eta) - 8(P_z - Q_t) = 0$ , from where

$$\eta = \frac{1}{\kappa}(P_z - Q_t) - \xi$$

and (26) is obtained from (38). Now, if  $\kappa = \frac{\tau^*}{2}$ , (33) reduces to

$$2\tau^*(\xi + \eta) + 4(3S_z - P_z + 3Q_t - T_t) = (Q + T)^2 + (P + S)^2. \quad (39)$$

Hence, for  $\tau^* \neq 0$ , this last equation lets determine  $\eta$  and (27) is obtained from (38). Otherwise, for  $\tau^* = 0$ , (38) reduces to (28) and (39) to (29), finishing the proof.  $\square$

**Remark 4** Metrics (28) can be viewed as *Riemann extensions*, i.e. they are locally isometric to the cotangent bundle  $(T^*\Sigma, g_D + \pi^*\phi)$  of a torsion-free affine surface  $(\Sigma, D)$  equipped with the metric  $g_D(X^C, Y^C) = -\iota(D_X Y + D_Y X)$ , where  $X^C$  and  $Y^C$  are complete lifts to the cotangent bundle,  $\iota Z^k \partial_k = x_{k'} Z^k$  for any vector field  $Z = Z^k \partial_k$ ,  $\pi^*\phi$  is the pull-back on  $T^*\Sigma$  of a symmetric  $(0, 2)$ -tensor field on  $\Sigma$ , and  $(x_k, x_{k'})$  are natural coordinates on  $T^*\Sigma$  induced by coordinates  $(x_k)$  on  $\Sigma$ . (See [6, 28] and the references therein for more information on the geometry of Riemann extensions).

**Theorem 10** *The structure  $(g, J)$  is strictly locally conformally Kähler and \*-Einstein if and only if the functions  $a, b$  and  $c$  have the form*

$$\begin{aligned} a &= xP(z, t) + yQ(z, t) + \xi(z, t), \\ b &= xS(z, t) + yT(z, t) + \eta(z, t), \\ c &= -\frac{1}{2}x(Q(z, t) - T(z, t)) + \frac{1}{2}y(P(z, t) - S(z, t)) + \gamma(z, t), \end{aligned} \quad (40)$$

where at least one of  $P + S$  and  $Q + T$  does not vanish everywhere and

$$4(3S_z - P_z + 3Q_t - T_t) = (P + S)^2 + (Q + T)^2, \quad (Q + T)_z = (P + S)_t. \quad (41)$$

**Proof.** We analyze the three families of metrics obtained in Theorem 9 using the characterization (18). First note that metrics (26) are Kähler. For metrics (27),  $(a + b)_{xx} = 2\kappa \neq 0$ , and therefore the locally conformally Kähler condition does not hold. Finally, for metrics (28), one easily checks that the locally conformally Kähler condition for this family of Hermitian \*-Einstein Walker metrics is equivalent to  $(Q + T)_z = (P + S)_t$ . Moreover, these metrics are Kähler if and only if  $P + S = Q + T = 0$  (see (13)), from where the result follows.  $\square$

**Remark 5** A proper almost Hermitian structure  $(g, J)$  is Hermitian, self-dual and \*-Einstein if and only if the functions  $a, b$  and  $c$  are given by (27) or (28) in Theorem 9.

**Remark 6** If the function  $c$  depends only on  $(z, t)$ , then the structure  $(g, J)$  is Hermitian and \*-Einstein if and only if the functions  $a$  and  $b$  have the forms

$$\begin{aligned} a &= xP(z, t) + yQ(z, t) + \xi(z, t), \\ b &= xP(z, t) + yQ(z, t) + \eta(z, t), \end{aligned} \quad (42)$$

where

$$2(P_z + Q_t) = P^2 + Q^2. \quad (43)$$

A particular solution of (43) is given by

$$P = -\frac{2}{z + f(t)}, \quad Q = -\frac{2}{t + g(z)},$$

where  $f(t)$  and  $g(z)$  are smooth functions. According to Corollary 6 the structure  $(g, J)$  is Hermitian and \*-Einstein, but it is neither Einstein nor locally conformally Kähler in general.

## 6 Einstein proper Hermitian structures

The Einstein equation ( $\rho_0 = \rho - \frac{\tau}{4}g = 0$ ) for a Walker metric (4) is a system of PDEs as follows (cf. [21]):

$$\begin{aligned}
(\rho_0)_{xz} &= -(\rho_0)_{yt} = (\rho_0)_{zx} = -(\rho_0)_{ty} = \frac{1}{4}(a_{xx} - b_{yy}) = 0, \\
(\rho_0)_{xt} &= (\rho_0)_{tx} = \frac{1}{2}(b_{xy} + c_{xx}) = 0, \\
(\rho_0)_{yz} &= (\rho_0)_{zy} = \frac{1}{2}(a_{xy} + c_{yy}) = 0, \\
(\rho_0)_{zt} &= (\rho_0)_{tz} = \frac{1}{4}\{-2a_y b_x + 2c_x c_y - c a_{xx} + 2a_{xt} - c b_{yy} \\
&\quad + 2b_{yz} + 2a c_{xx} + 2c c_{xy} - 2c_{xz} + 2b c_{yy} - 2c_{yt}\} = 0, \\
(\rho_0)_{zz} &= \frac{1}{4}\{2a_x c_y + 2a_y b_y - 2a_y c_x - 2c_y^2 + a a_{xx} + 4c a_{xy} \\
&\quad + 2b a_{yy} - 4a_{yt} - a b_{yy} - 2a c_{xy} + 4c_{yz}\} = 0, \\
(\rho_0)_{tt} &= \frac{1}{4}\{2a_x b_x - 2b_x c_y + 2b_y c_x - 2c_x^2 - b a_{xx} + 2a b_{xx} \\
&\quad + 4c b_{xy} - 4b_{xz} + b b_{yy} - 2b c_{xy} + 4c_{xt}\} = 0.
\end{aligned} \tag{44}$$

Note that the scalar curvature is given by

$$\tau = a_{xx} + b_{yy} + 2c_{xy}. \tag{45}$$

**Theorem 11** *The structure  $(g, J)$  is Hermitian Einstein if and only if the functions  $a, b, c$  have one of the following three forms:*

$$\begin{aligned}
a &= \kappa(x^2 - y^2) + xP + yQ + \xi, \\
b &= \kappa(y^2 - x^2) - xP - yQ - \xi + \frac{1}{\kappa}(P_z - Q_t), \\
c &= 2\kappa xy - xQ + yP + \gamma,
\end{aligned} \tag{46}$$

or

$$\begin{aligned}
a &= \kappa x^2 + xP + yQ + \frac{1}{4\kappa}\{4(P - S)_z - 8Q_t + 2Q(Q + T) + (P^2 - S^2)\}, \\
b &= \kappa y^2 + xS + yT + \frac{1}{4\kappa}\{-4(Q - T)_t - 8S_z + 2S(P + S) - (Q^2 - T^2)\}, \\
c &= \kappa xy - \frac{1}{2}x(Q - T) + \frac{1}{2}y(P - S) \\
&\quad + \frac{1}{4\kappa}\{2(P + S)_t + 2(Q + T)_z + T(P - S) - Q(P + 3S)\},
\end{aligned} \tag{47}$$

or

$$\begin{aligned}
a &= xP + yQ + \xi, \\
b &= xS + yT + \eta, \\
c &= -\frac{1}{2}x(Q - T) + \frac{1}{2}y(P - S) + \gamma,
\end{aligned} \tag{48}$$

where in the last case

$$\begin{aligned}
8Q_t - 4(P - S)_z &= 2Q(Q + T) + (P^2 - S^2), \\
8S_z + 4(Q - T)_t &= 2S(P + S) - (Q^2 - T^2), \\
2(P + S)_t + 2(Q + T)_z &= Q(P + 3S) - T(P - S).
\end{aligned} \tag{49}$$

Here  $\kappa$  is a non-zero constant and capital and Greek letters are smooth functions depending only on  $(z, t)$ . In the first case  $\tau = 8\kappa$ , in the second one  $\tau = 6\kappa$  and in the third case  $\tau = 0$ .

**Remark 7** Metrics (48)-(49) have received attention in the literature of Osserman manifolds. Since any four-dimensional Riemann extension is self-dual [6], metrics (48)-(49) are Osserman with nilpotent Jacobi operators and moreover they are realized as Riemann extensions of torsion free connections with skew-symmetric Ricci tensor [9, 12].

**Proof.** It follows from (12) and the first three equations in (44) that  $(g, J)$  - Hermitian Einstein implies

$$a_x - b_x = 2c_y, \quad a_y - b_y = -2c_x, \quad a_{xy} = -c_{yy}, \quad b_{xy} = -c_{xx}, \quad a_{xx} = b_{yy}. \quad (50)$$

Hence, we get  $a_{xy} = b_{xy} = c_{xx} = c_{yy} = 0$  and therefore

$$\begin{aligned} a &= A(y, z, t) + \bar{A}(x, z, t), \\ b &= B(x, z, t) + \bar{B}(y, z, t), \\ c &= xyC(z, t) + xU(z, t) + yV(z, t) + \gamma(z, t). \end{aligned} \quad (51)$$

Now,  $a_{xx} = b_{yy}$  in (50) means  $\bar{A}_{xx}(x, z, t) = \bar{B}_{yy}(y, z, t)$ . Hence

$$\bar{A}(x, z, t) = \frac{1}{2}x^2D(z, t) + xP(z, t) + \bar{\xi}(z, t), \quad \bar{B}(y, z, t) = \frac{1}{2}y^2D(z, t) + yT(z, t) + \bar{\eta}(z, t). \quad (52)$$

Moreover,  $a_x - b_x = 2c_y$ ,  $a_y - b_y = -2c_x$  in (50) lead to

$$\begin{aligned} A(y, z, t) &= \frac{1}{2}y^2(D(z, t) - 2C(z, t)) + y(T(z, t) - 2U(z, t)) + \tilde{\xi}(z, t), \\ B(x, z, t) &= \frac{1}{2}x^2(D(z, t) - 2C(z, t)) + x(P(z, t) - 2V(z, t)) + \tilde{\eta}(z, t). \end{aligned} \quad (53)$$

Collecting together (51), (52), (53) and setting  $2M = D - 2C$ ,  $L = \frac{1}{2}D$ ,  $Q = T - 2U$ ,  $S = P - 2V$ ,  $\xi = \tilde{\xi} + \bar{\xi}$ ,  $\eta = \tilde{\eta} + \bar{\eta}$  we obtain

$$\begin{aligned} a &= x^2L + y^2M + xP + yQ + \xi, \\ b &= x^2M + y^2L + xS + yT + \eta, \\ c &= xy(L - M) - \frac{1}{2}x(Q - T) + \frac{1}{2}y(P - S) + \gamma. \end{aligned} \quad (54)$$

Now, the (constant) scalar curvature is given by  $\tau = 2(3L - M)$ , from where

$$M = \frac{1}{2}(6L - \tau).$$

Then, plugging  $a, b, c$  in the fourth equation of (44) and differentiating twice by  $x, y$  we get

$$\tau^2 - 14\tau L(z, t) + 48L(z, t)^2 = 0,$$

which implies that  $L(z, t) = \kappa$  must be constant and

$$\begin{aligned} a &= x^2\kappa + \frac{1}{2}y^2(6\kappa - \tau) + xP + yQ + \xi, \\ b &= \frac{1}{2}x^2(6\kappa - \tau) + y^2\kappa + xS + yT + \eta, \\ c &= \frac{1}{2}xy(\tau - 4\kappa) - \frac{1}{2}x(Q - T) + \frac{1}{2}y(P - S) + \gamma \end{aligned} \quad (55)$$

where  $\kappa = \frac{\tau}{8} \neq 0$ ,  $\kappa = \frac{\tau}{6} \neq 0$  or  $\kappa = \tau = 0$ . Next we analyze each of these cases by separate.

For  $\kappa = \frac{\tau}{8} \neq 0$ , we differentiate the last equation in (44) by  $x$  and by  $y$ , and get

$$\tau(P + S) = 0, \quad \tau(Q + T) = 0,$$

and therefore  $S = -P$  and  $T = -Q$ . With these conditions, a further analysis shows that (44) holds if and only if

$$\eta = \frac{1}{\kappa}(P_z - Q_t) - \xi,$$

from where (55) reduces to (46).

Now, if  $\kappa = \frac{\tau}{6}$ , the last three equations in (44) transform into

$$\begin{aligned}\frac{2\tau}{3}\xi &= 4(P - S)_z - 8Q_t + 2Q(Q + T) + (P^2 - S^2), \\ \frac{2\tau}{3}\eta &= -4(Q - T)_t - 8S_z + 2S(P + S) - (Q^2 - T^2), \\ \frac{2\tau}{3}\gamma &= 2(P + S)_t + 2(Q + T)_z + T(P - S) - Q(P + 3S).\end{aligned}\tag{56}$$

Hence, for  $\tau \neq 0$ , these three equations determine  $\xi$ ,  $\eta$  and  $\gamma$  and (47) is obtained from (55). Otherwise, for  $\tau = 0$ , (55) reduces to (48) and (56) to (49), showing the result.  $\square$

**Corollary 12** *Any proper Hermitian Einstein structure is \*-Einstein.*

**Remark 8** A proper almost Hermitian structure  $(g, J)$  is Hermitian, self-dual and Einstein if and only if the functions  $a$ ,  $b$  and  $c$  are given by (47) or (48) in Theorem 11.

**Remark 9** In the case when the function  $c$  depends only on  $(z, t)$ , the structure  $(g, J)$  is Hermitian and Einstein if and only if the functions  $a$  and  $b$  have the forms

$$\begin{aligned}a &= xP(z, t) + yQ(z, t) + \xi(z, t), \\ b &= xP(z, t) + yQ(z, t) + \eta(z, t),\end{aligned}\tag{57}$$

where

$$2P_z = P^2, \quad 2Q_t = Q^2, \quad P_t + Q_z = PQ.\tag{58}$$

Moreover, in a neighbourhood of a point where  $P^2 + Q^2 \neq 0$  the solution of the system (58) is given by

$$P = -\frac{2p}{pz + qt + r}, \quad Q = -\frac{2q}{pz + qt + r},\tag{59}$$

where  $p, q, r$  are constants and  $p^2 + q^2 \neq 0$ . Indeed, suppose that  $P^2 + Q^2 \neq 0$  in a neighborhood of a point  $(z_0, t_0)$ . Then the first two equations in (58) imply

$$P(z, t) = -\frac{2}{z + f(t)}, \quad Q(z, t) = -\frac{2}{t + g(z)},\tag{60}$$

where  $f(t)$  and  $g(z)$  are smooth functions, and the third equation in (58) takes the form

$$\frac{f_t}{(z + f(t))^2} + \frac{g_z}{(t + g(z))^2} = \frac{2}{(z + f(t))(t + g(z))}.\tag{61}$$

Setting  $F(t) = \frac{1}{z_0 + f(t)}$  and  $\alpha = g(z_0)$ ,  $\beta = g_z(z_0)$ , the latter equation implies that

$$F_t + \frac{2F}{t + \alpha} = \frac{\beta}{(t + \alpha)^2}.$$

The solution of this linear ODE is given by

$$F(t) = \frac{\beta}{t + \alpha} + \frac{\gamma}{(t + \alpha)^2}, \quad \gamma = \text{const.}$$

Thus

$$f(t) = \frac{(t + \alpha)^2}{\gamma + \beta(t + \alpha)}.$$

In the same way we see that the function  $g(z)$  has the form

$$g(z) = \frac{(z + \lambda)^2}{\mu + \nu(z + \lambda)}.$$

Substituting these expressions of  $f$  and  $g$  into (61), clearing the denominators and comparing the coefficients of the obtained polynomials of  $z$  and  $t$ , we see from (60) that  $P$  and  $Q$  have the form (59).

Note that if  $p \cdot q \neq 0$ , then the structure  $(g, J)$  is strictly locally conformally Kähler, self-dual, Ricci flat and \*-Ricci flat, but the metric  $g$  is not flat.

**Theorem 13** *The structure  $(g, J)$  is strictly locally conformally Kähler Einstein if and only if the functions  $a$ ,  $b$  and  $c$  have the forms*

$$\begin{aligned} a &= xP(z, t) + yQ(z, t) + \xi(z, t), \\ b &= xS(z, t) + yT(z, t) + \eta(z, t), \\ c &= -\frac{1}{2}x(Q(z, t) - T(z, t)) + \frac{1}{2}y(P(z, t) - S(z, t)) + \gamma(z, t), \end{aligned} \tag{62}$$

where at least one of the functions  $P + S$  and  $Q + T$  does not vanish everywhere and

$$\begin{aligned} 8Q_t - 4(P - S)_z &= 2Q(Q + T) + (P^2 - S^2), \\ 8S_z + 4(Q - T)_t &= 2S(P + S) - (Q^2 - T^2), \\ 4(P + S)_t &= 4(Q + T)_z = Q(P + 3S) - T(P - S). \end{aligned} \tag{63}$$

**Proof.** We analyze the three families of metrics obtained in Theorem 11 using the characterization (18). First note that metrics of the type (46) are Kähler (see (13)). For metrics (47),  $(a+b)_{xx} = \frac{\tau}{3} \neq 0$ , and therefore the locally conformally Kähler condition fails. Finally, for metrics (48), the locally conformally Kähler condition is equivalent to  $(Q + T)_z = (P + S)_t$ . Moreover, these metrics are Kähler if and only if  $P + S = Q + T = 0$ , from where the result follows.  $\square$

Using (13), we easily obtain from Theorem 11 the following

**Corollary 14** *The structure  $(g, J)$  is Kähler Einstein if and only if the functions  $a$ ,  $b$ ,  $c$  have one of the following forms*

$$\begin{aligned} a &= \kappa(x^2 - y^2) + xP + yQ + \xi, \\ b &= \kappa(y^2 - x^2) - xP - yQ - \xi + \frac{1}{\kappa}(P_z - Q_t), \\ c &= 2\kappa xy - xQ + yP + \gamma, \end{aligned} \tag{64}$$

or

$$\begin{aligned} a &= xP(z, t) + yQ(z, t) + \xi(z, t), \\ b &= -xP(z, t) - yQ(z, t) + \eta(z, t), \\ c &= -xQ(z, t) + yP(z, t) + \gamma(z, t), \end{aligned} \tag{65}$$

where in the last case

$$P_z = Q_t \tag{66}$$

Here  $\kappa$  is a non-zero constant and capital and Greek letters are smooth functions depending only on  $(z, t)$ . In the first case  $\tau = 8\kappa$ , in the second one  $\tau = 0$ .

## 7 An example

It is proved in [19, Theorem 3.2] that an almost Hermitian 4-manifold  $(M, g, J)$  with positive definite metric is Hermitian if the metric  $g$  is Einstein and its positive Weyl curvature tensor satisfies the condition  $\|W^+\|^2 = \frac{1}{24}(\tau - 3\tau^*)^2 \neq 0$  at every point of  $M$ . We shall give an example showing that the analog of this result is not true in the setting of signature  $(2, 2)$ . For metrics of this signature the role of  $W^+$  is played by  $W^-$  since the almost complex structures compatible with the metric and the orientation are parametrized by sections of  $\Lambda_-$ . Note also that the metric on  $\Lambda^2$  in [19] is one half of the metric used here. Thus, in our situation, the analog of the above condition for  $W^+$  is  $\|W^-\|^2 = \frac{1}{96}(\tau - 3\tau^*)^2 \neq 0$ .

Let  $g$  be the Walker metric on  $\mathbb{R}^4$  for which

$$a = x^2 + \frac{2}{3}y^2 + \frac{2}{\sqrt{3}}xy, \quad b = y^2, \quad c = -\frac{1}{\sqrt{3}}y^2.$$

Then it follows from (44), (45) and (31) that the metric  $g$  is Einstein with  $\tau = 4$  and  $\tau^* = -\frac{4}{3}$ . Moreover, using the formula for the components of  $W^-$  with respect to the basis (6) given in [9] we get

$$W^- = \begin{bmatrix} -\frac{2}{3} & \frac{1}{\sqrt{3}} & \frac{1}{3} \\ -\frac{1}{\sqrt{3}} & \frac{2}{3} & \frac{1}{\sqrt{3}} \\ -\frac{1}{3} & \frac{1}{\sqrt{3}} & 0 \end{bmatrix}.$$

Therefore  $\|W^-\|^2 = \frac{2}{3} = \frac{1}{96}(\tau - 3\tau^*)^2$ . On the other hand, it follows from (12) that the proper almost complex structure  $J$  is not integrable. Note that  $W^-$  has degenerate spectrum  $\{\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3}\}$ . The eigenspace corresponding to the simple eigenvalue is time-like and determines an integrable almost product structure.

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