A note on the structure of algebraic curvature tensors[1]

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Abstract

It is shown that any algebraic curvature tensor on an n-dimensional vector space can be represented by at most $n(n+1)/2$ symmetric bilinear forms.

2000 Mathematics Subject Classification: 53B20.

Key words and phrases: Algebraic curvature tensor, normal coordinates, Nash embedding theorem.

[1]Supported by projects BFM2001-3778-C03-01 and BFM 2003-02949 (Spain)

Let V be an n-dimensional vector space with an inner product $\langle\,,\,\rangle$. An *algebraic curvature tensor* is an $F \in \otimes^4(V^*)$ satisfying the algebraic identities of the curvature tensor of a Riemannian manifold:

$$F(x, y, z, w) = -F(y, x, z, w) = F(z, w, x, y)$$
$$F(x, y, z, w) + F(y, z, x, w) + F(z, x, y, w) = 0.$$

The space of algebraic curvature tensors $\mathcal{R}(V)$ is a $n^2(n^2-1)/12$-dimensional vector space. Let $S(V)$ and $A(V)$ denote the spaces of symmetric and anti-symmetric bilinear forms on V and define

$$F^{\phi}(x, y, z, w) = \phi(x, z)\phi(y, w) - \phi(y, z)\phi(x, w) \tag{1}$$

for any $\phi \in S(V)$, and

$$F^{\psi}(x, y, z, w) = \psi(x, z)\psi(y, w) - \psi(y, z)\psi(x, w) - 2\psi(x, y)\psi(z, w) \tag{2}$$

for any $\psi \in A(V)$. Then it follows that F_ϕ and F_ψ are algebraic curvature tensors. Further put

$$\mathcal{A}(V) = span\{F^{\phi}\}_{\phi \in A(V)} \subset \mathbb{R}(V), \quad \mathcal{S}(V) = span\{F^{\psi}\}_{\psi \in S(V)} \subset \mathbb{R}(V).$$

Then the space of algebraic curvature tensors coincides with $\mathcal{S}(V)$ and $\mathcal{A}(V)$ (see also [1]):

Theorem 1 [3],[4] *Let $(V^n, \langle\,,\,\rangle)$ be an n-dimensional vector space with an inner product $\langle\,,\,\rangle$. Then,*

$$\mathcal{R}(V) = \mathcal{S}(V) = \mathcal{A}(V)$$

Here, it is worth to emphasize that the proof of theorem above is constructive and it relies on basic linear algebra (cf. [3, Thm. 1.8.2]). By following that proof, one can obtain an estimate of the number of different symmetric tensor fields needed to express a given algebraic curvature tensor as follows. Let F be an algebraic curvature tensor and decompose it as $F = \sum_{i=1}^{\nu} \lambda_i F^{\phi_i}$ where ϕ_i belong to one of the following:

(i) For $i < j$ define $\phi_{ij} = \phi_{ji} = 1$, $\phi_{ab} = 0$ otherwise.

$$\begin{array}{cc} & \begin{array}{cccccccc} & & & & & j & & \\ & & & & & \downarrow & & \end{array} \\ \begin{array}{c} \\ \\ i \to \\ \\ \\ \\ \\ \\ \end{array} & \begin{pmatrix} 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ & \cdot & & & & \cdot & & \cdot \\ & & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ & & & \cdot & & \cdot & & \cdot \\ & & & & \cdot & \cdot & & \cdot \\ & & & & & \cdot & & \cdot \\ & & & & & & \cdot & \cdot \\ & & & & & & & 0 \end{pmatrix} \end{array} \tag{3}$$

(ii) For $j \neq i \neq k, j < k$ define $\phi_{ij} = \phi_{ji} = \phi_{ik} = \phi_{ki} = 1$, $\phi_{ab} = 0$ otherwise. Then we have:

$$
(4) \qquad
\begin{array}{cc}
 & \begin{array}{ccccccccc} & & & & j & & k & & \\ & & & & \downarrow & & \downarrow & & \end{array} \\
\begin{array}{c} \\ \\ i \rightarrow \\ \\ \\ \\ \\ \\ \\ \end{array} &
\begin{pmatrix}
0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\
 & \cdot & & & \cdot & & \cdot & & \cdot \\
 & & \cdot & \cdot & 1 & \cdot & 1 & \cdot & \cdot \\
 & & & \cdot & \cdot & & \cdot & & \cdot \\
 & & & & \cdot & & \cdot & & \cdot \\
 & & & & & \cdot & \cdot & & \cdot \\
 & & & & & & \cdot & & \cdot \\
 & & & & & & & \cdot & \cdot \\
 & & & & & & & & 0
\end{pmatrix}
\end{array}
$$

(iii) For i, j, k, l different, define $\phi_{ik} = \phi_{ki} = \phi_{jl} = \phi_{lj} = 1$, $\phi_{ab} = 0$ otherwise. Then in matrix representation:

$$
(5) \qquad
\begin{array}{cc}
 & \begin{array}{ccccccccc} & & & & j & & l & & \\ & & & & \downarrow & & \downarrow & & \end{array} \\
\begin{array}{c} \\ \\ i \rightarrow \\ \\ k \rightarrow \\ \\ \\ \\ \\ \end{array} &
\begin{pmatrix}
0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\
 & \cdot & & & \cdot & & \cdot & & \cdot \\
 & & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
 & & & \cdot & \cdot & & \cdot & & \cdot \\
 & & & & \cdot & \cdot & 1 & \cdot & \cdot \\
 & & & & & \cdot & \cdot & & \cdot \\
 & & & & & & \cdot & & \cdot \\
 & & & & & & & \cdot & \cdot \\
 & & & & & & & & 0
\end{pmatrix}
\end{array}.
$$

Therefore, the number of different symmetric tensors ϕ needed to express any given algebraic curvature tensor is at most $n(n-1)(n^2-n+2)/8$. Our purpose in this note is to provide an alternative proof of $\mathcal{R}(V) = \mathcal{S}(V)$ which gives a better (although not optimal) estimation.

Theorem 2 *Let $(V^n, \langle\,,\,\rangle)$ be an n-dimensional vector space with an inner product $\langle\,,\,\rangle$. Then, for any algebraic curvature tensor $F \in \otimes^4(V^*)$ there exists at most $\frac{1}{2}n(n+1)$ symmetric tensor ϕ on V such that F is a linear combination of the associated algebraic curvature tensors F^ϕ.*

Proof. First of all note that any algebraic curvature tensor F is geometrically realizable, i.e., there exists a smooth manifold M and a metric g on M such that the curvature tensor of (M, g) at some point $m \in M$ is exactly F. We mean that there is a linear isometry $\Phi : (V, \langle\,,\,\rangle) \rightarrow (T_m M, g_m)$ such that $F = \Phi^* R_m$, where R is the curvature tensor of g. For example, by using the theory of normal coordinates, define a metric in a neighborhood of the origin of $\mathbb{R}^n$ as follows:

$$
(6) \qquad g_{ij}(x^1, \ldots, x^n) = \delta_{ij} - \frac{1}{3} \sum_{\alpha,\beta=1}^{n} F_{i\alpha j\beta} x^\alpha x^\beta
$$

where $F_{ijkl} = F(e_i, e_j, e_k, e_l)$ and $\{e_1, \ldots, e_n\}$ is an orthonormal basis of $\mathbb{R}^n$. Then, we have at the origin $R_m = F$ with the identification $V = \mathbb{R}^n$ by means of the previous basis.

Now, it follows from the Nash embedding theorem [6] that M is isometrically embedded in $\mathbb{R}^{n+\nu}$ for some $\nu = \frac{3n(n+3)}{2}$. Next, let $\{e_1, \ldots, e_\nu\}$ be an orthonormal basis of the normal space $T_m^\perp M$ at m and let II denote the second fundamental form of the immersion. For each tangent vectors x, $y \in T_m M$ one has $II(x,y) = \sum_{i=1}^{\nu} \phi_i(x,y) e_i$, where $\phi_i(x,y) = g(II(x,y), e_i)$ is a symmetric bilinear tensor for all $i = 1, \ldots, \nu$.

Now, it follows from the Gauss Equation that

$$\begin{aligned} F(x,y,v,w) &= R_m(x,y,v,w) \\ &= g\left(II(x,v), II(x,v)\right) - g\left(II(x,w), II(y,v)\right) \\ &= \sum_{i=1}^{\nu} \{\phi_i(x,v)\phi_i(y,w) - \phi_i(x,w)\phi_i(y,v)\} \\ &= \sum_{i=1}^{\nu} F^{\phi_i}(x,y,v,w) \end{aligned}$$

which proves the result.

Finally note that a geometric realization of an algebraic curvature tensor can be done locally in an analytic manifold, and thus the codimension in Nash's theorem can be reduced to $n(n+1)/2$ as desired [5]. $\square$

Remark 3 A tensor $K \in \otimes^5 (V^*)$ is called an *algebraic covariant derivative curvature tensor* if it satisfies

$$\begin{aligned} &K(x,y,z,v,w) = -K(x,z,y,v,w) = K(x,v,w,y,z) \\ &K(x,y,z,v,w) + K(x,z,v,y,w) + K(x,v,y,z,w) = 0 \\ &K(x,y,z,v,w) + K(y,z,x,v,w) + K(z,x,y,v,w) = 0. \end{aligned}$$

Systems of generators of the space of algebraic covariant derivative curvature tensors have been investigated by Fiedler [2], where such generators are constructed from symmetric tensors of type 2 and 3. Next, consider

$$\bar{S}(V) = \{\Phi \in \otimes^3 (V^*) : \Phi(x,y,z) = \Phi(x,z,y) \, \forall \, x,y,z \in V\}$$

and define an algebraic covariant derivative curvature tensor $K^{\phi,\Phi}$ by

$$\begin{aligned} K^{\phi,\Phi}(x,y,z,v,w) &= \Phi(x,y,v)\phi(z,w) + \phi(y,v)\Phi(x,z,w) \\ &\quad -\Phi(x,y,w)\phi(z,v) - \phi(y,v)\Phi(x,z,w) \end{aligned}$$

where ϕ is symmetric 2-tensor $\phi \in S(V)$ and $\Phi \in \bar{S}(V)$.

Now, let K be an algebraic covariant derivative curvature tensor and note that it is geometrically realizable just extending (6) to

$$g_{ij}(x^1,\dots,x^n)=\delta_{ij}-\frac{1}{3}\sum_{\alpha,\beta=1}^{n}F_{i\alpha j\beta}\,x^\alpha x^\beta-\frac{1}{6}\sum_{\alpha,\beta,\gamma=1}^{n}K_{\alpha i\beta j\gamma}\,x^\alpha x^\beta x^\gamma.$$

Now, proceeding in the same way as in Theorem 2 we get

$$K(x,y,z,v,w)=\sum_{i=1}^{\nu}K^{\phi,\Phi}(x,y,z,v,w) \tag{7}$$

where $\Phi_i(x,y,z)=(\nabla_x\phi_i)(y,z)$ in the notation of Theorem 2. This shows that the $K^{\phi,\Phi}$'s are a system of generators, and moreover an estimate for number of terms in (7) is obtained as $\nu=\frac{n(n+1)}{2}$.

1 Examples and applications

First of all, observe that for any two-dimensional manifold, the curvature tensor is expressed by the Ricci tensor and thus, any algebraic curvature tensor on a two-dimensional vector space is completely determined by exactly one F^ϕ.

The situation is more complicated in dimension three but, since the curvature tensor in that dimension is completely determined by the Ricci tensor, we still have the following

Theorem 4 *Let F be an algebraic curvature tensor in a three-dimensional vector space. Then*

(a) there exists exactly one symmetric $(0,2)$-tensor ϕ such that $F=F^\phi$ or, otherwise

(b) there exists exactly two distinct symmetric $(0,2)$-tensors ϕ_1 and ϕ_2 such that

$$F=\kappa_1F^{\phi_1}+\kappa_2F^{\phi_2}$$

the second case occurs if and only if the Ricci tensor has eigenvalues $\lambda_1\neq 0\neq\lambda_2$, $\lambda_3=\lambda_1+\lambda_2$.

Proof. Let F be an algebraic curvature tensor in a three-dimensional vector space V with inner product $\langle\,,\,\rangle=g(\cdot,\cdot)$. Let ρ^F denote the Ricci tensor and τ^F the scalar curvature of F. Then, it is known that:

$$F_{xyvw}=\frac{\tau^F}{2}\left(g_{xv}g_{yw}-g_{xw}g_{yv}\right)-\left(\rho^F_{xv}g_{yw}+\rho^F_{yw}g_{xv}-\rho^F_{xw}g_{yv}-\rho^F_{yv}g_{xw}\right). \tag{8}$$

Next, let $\{e_1,e_2,e_3\}$ be an orthonormal basis diagonalizing ρ^F and put

$$\rho^F=\begin{pmatrix}\lambda_1&0&0\\0&\lambda_2&0\\0&0&\lambda_3\end{pmatrix}.$$

Then, using (8) we have

$$F_{ijkl} = \left(\frac{\lambda_1 + \lambda_2 + \lambda_3}{2} - \lambda_i - \lambda_j \right) F^{Id}_{ijkl}$$

where $F^{Id}_{ijkl} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}$. Define:

$$\begin{array}{rcrcl} \alpha_1 & = & \lambda_1 - \lambda_2 - \lambda_3 & = & 2F_{2323} \\ \alpha_2 & = & -\lambda_1 + \lambda_2 - \lambda_3 & = & 2F_{1313} \\ \alpha_3 & = & -\lambda_1 - \lambda_2 + \lambda_3 & = & 2F_{1212} \end{array}$$

and consider the following cases:

(a.1) α_1, α_2 and α_3 are different from zero.

Let $\epsilon_i = \pm 1$ denote the sign of α_i and put

$$\epsilon = \epsilon_1 \epsilon_2 \epsilon_3 \qquad \text{and} \qquad \beta = \sqrt{\frac{\epsilon \alpha_1 \alpha_2 \alpha_3}{2}}.$$

Now, define a symmetric tensor ϕ with respect to the basis above by

$$\phi = \beta \begin{pmatrix} \frac{1}{\alpha_1} & 0 & 0 \\ 0 & \frac{1}{\alpha_2} & 0 \\ 0 & 0 & \frac{1}{\alpha_3} \end{pmatrix}$$

or, $\phi_{ij} = \frac{\beta}{\alpha_i}\delta_{ij}$. Then $F = \epsilon F^{\phi}$.

(a.2) $\alpha_1 \neq 0$ and $\alpha_2 = \alpha_3 = 0$.

Define

$$\phi = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{\alpha_1}{2} \end{pmatrix}.$$

It is straightforward to check that $F = F^{\phi}$.

(a.3) If $\alpha_1 = \alpha_2 = \alpha_3 = 0$, then $F = 0$.

(b) $\alpha_1, \alpha_2 \neq 0$ and $\alpha_3 = 0$.

Next we will show that it is not possible to express the given algebraic curvature tensor as $F = \gamma F^{\phi}$. On the contrary, suppose this can be achieved for certain γ and ϕ. Since $\alpha_1, \alpha_2 \neq 0$ we have $F \neq 0$ and hence $\gamma \neq 0$. Then $F = \gamma F^{\phi}$ is equivalent to solving the following system:

$$\begin{array}{rcl} \phi_{11}\phi_{22} - \phi_{12}^2 & = & 0 \\ \phi_{11}\phi_{23} & = & \phi_{13}\phi_{12} \\ \phi_{12}\phi_{23} & = & \phi_{13}\phi_{22} \\ \phi_{11}\phi_{33} - \phi_{13}^2 & = & \dfrac{\alpha_2}{2\gamma} \\ \phi_{12}\phi_{33} & = & \phi_{13}\phi_{23} \\ \phi_{22}\phi_{33} & = & \dfrac{\alpha_1}{2\gamma} \end{array}$$

which is not hard to see that has no solution.

Nevertheless, it is possible to write $F = F^{\phi_1} + F^{\phi_2}$. For example take

$$\phi_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{\alpha_1}{2} \end{pmatrix} \quad \phi_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{\alpha_2}{2} \end{pmatrix}$$

and the equality follows after a simple computation.

□

Remark 5 Note that, as an immediate application of Theorem 2, we obtain a criteria for non existence of immersions of a given manifold into a space of constant curvature. For instance, no Riemannian 3-manifold whose curvature tensor is as in Theorem 4-(b) at some point can be isometrically immersed as a hypersurface in a space of constant curvature.

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