COMPARISON RESULTS FOR THE VOLUME OF GEODESIC CELESTIAL SPHERES IN LORENTZIAN MANIFOLDS

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Abstract. Volume comparison results are obtained for the volume of geodesic celestial spheres in Lorentzian manifolds and the corresponding objects in Lorentzian space forms. Also, as a rigidity result it is shown that the volume of geodesic celestial spheres is independent of the instantaneous observer if and only if the spacetime has constant curvature.

1. Introduction

Comparison theorems for the volumes of subregions of Riemannian manifolds under some curvature hypothesis have played an important role in Riemannian geometry. For instance, the Bishop-Günther inequalities show lower (resp., upper) bounds for volumes of geodesic balls and tubes by imposing upper (resp., lower) bounds on the sectional curvature. These inequalities have been improved by assuming weaker conditions on the Ricci tensor or by considering the ratio between the volumes of geodesic balls in the manifold and the model spaces (see, for example [11] and the references therein).

When the attention is turned from Riemannian manifolds to spacetimes, various difficulties emerge. For example, conditions on bounds for the sectional curvature (resp., the Ricci tensor) easily produce manifolds of constant sectional curvature (resp., Einstein) [3], [14]. This demands a revision of such conditions (see, for example [2]). However, a most difficult task is related to the consideration of those regions under investigation. An important characteristic of Riemannian manifolds is that they have a Riemannian distance function which is continuous and whose induced topology is the same as the topology of the manifold itself. Thus, several geometric objects like geodesic spheres and tubes can be defined, at least locally, by means of this function. These objects are also Riemannian manifolds whose geometric properties influence and can even characterize the geometry of the ambient manifold. Also, they have nice properties, such as compactness and an acceptable behaviour with respect to other constructions. When dealing with general semi-Riemannian manifolds, several difficulties arise, one of the most important being that there is no "semi-Riemannian distance" function. In fact, a distance-like function is only defined for spacetimes, but even in this case its properties are completely different from those in the Riemannian setting (cf. [3].) For example, the "Lorentzian distance" may not be continuous or bounded and geometric objects defined from it usually have awkward properties. Moreover, level sets of the

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Lorentzian distance function with respect to a given point are not compact and, although some properties of those sets have been previously investigated (cf. [1], [9]), they do not seem to be adequate for the investigation of volume-properties. Therefore, different families of objects have been considered in Lorentzian geometry for the purpose of investigating their volume properties. Among those, truncated light cones [15], slabs bounded by spacelike hypersurfaces [2], compact geodesic wedges in the chronological future of some point [7] and more generally some neighborhoods covered by timelike geodesic emanating from a given point [8].

In this paper we consider a different family of geometric objects, namely the geodesic celestial spheres. Roughly speaking, they are the set of points reached after a fixed time travelling along radial geodesics emanating from a point m which are orthogonal to a given timelike direction. In Relativity, a unit timelike vector represents an instantaneous observer and the vector subspace which is orthogonal to it is called the infinitesimal restspace, that is, the infinitesimal Newtonian universe where the observer perceives particles as Newtonian particles relative to his rest position. Then, a geodesic celestial sphere is nothing but the image by the exponential map of the celestial sphere in the infinitesimal restspace.

The paper is organized as follows. In Section 2 we introduce the notations and conventions used throughout the paper and state the main results as theorems 2.2 and 2.3, which compare the volumes of sufficiently small geodesic celestial spheres on a Lorentzian manifold with the corresponding ones in a Lorentzian space form. Theorem 2.6 shows that local isotropy of Lorentzian manifolds can be recovered from the properties of the volume of geodesic celestial spheres. The corresponding proofs are carried out in Section 3. Finally, in Section 4 we state some results analogous to Theorem 2.6, showing that local isotropy can be detected by considering the total curvatures of geodesic celestial spheres.

2. Geodesic celestial spheres

Let $\left(M^{n+1}, g\right)$ be a Lorentzian manifold of dimension $n+1$ and signature $(-+\cdots+)$. We denote by ∇ the Levi-Civita connection of M. The curvature tensor R is defined by using the convention $R_{XY} = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y]$ and $R_{XYVW} = g(R_{XY}V, W)$, where X, Y, V and W are vector fields on M. We will also denote by $\rho(x,y) = \mathrm{trace}\{z \mapsto R(x,z)y\}$ and $\tau = \mathrm{trace}\rho$ the Ricci tensor and the scalar curvature, respectively. With respect to an orthonormal basis $\{e_i\}$ they are written as

$$\rho(X,Y) = \sum_{i=0}^{n} \epsilon_i R_{Xe_iYe_i} \quad \text{and} \quad \tau = \sum_{i=0}^{n} \epsilon_i \rho_{e_ie_i},$$

where $\epsilon_i = g(e_i, e_i)$.

A unit timelike vector $\xi \in T_mM$ is called an *instantaneous observer*, and $\xi^\perp$ is called the *restspace* of ξ. The *celestial sphere of radius* r of ξ is defined by $\mathcal{S}^\xi(r) = \{x \in \xi^\perp; g(x,x) = 1\}$ (c.f. [16]). If $\mathfrak{U}$ is a sufficiently small neighborhood of the origin in T_mM, $\widetilde{M} = \exp_m(\mathfrak{U} \cap \xi^\perp)$ is an embedded Riemannian submanifold of M, where $\exp_m : T_mM \to M$ denotes the exponential map of M at m. We will denote by $\widetilde{\nabla}$ its Levi-Civita connection, $\widetilde{R}$ is the curvature tensor, and in general, we use the symbol $\widetilde{\ }$ to denote the corresponding geometrical objects in $\widetilde{M}$. We define the *geodesic celestial sphere of radius* r *associated to* ξ as

$$S_m^\xi(r) = exp_m\left(\left\{x \in \xi^\perp; \|x\| = r\right\}\right) = exp_m(\mathcal{S}^\xi(r)). \tag{2.1}$$

For r sufficiently small, $S_m^\xi(r)$ is a compact submanifold of $\widetilde{M}$. Therefore, by studying the volumes of geodesic celestial spheres in comparison to the volumes of the corresponding celestial spheres one obtains a measure of how the exponential map distorts volumes on spacelike directions.

As an immediate observation, note that for a given radius, the volume of celestial geodesic spheres depends both on the observer field $\xi \in T_mM$ and the center point $m \in M$. However, if (M, g) is assumed to be of constant sectional curvature, then the volumes depend only on the radii, since Lorentzian space forms are locally isotropic and conversely [17]. Indeed one may compute the volume of geodesic celestial spheres as in the following.

Theorem 2.1. *Let $M^{n+1}(\lambda)$ be a Lorentzian manifold of constant sectional curvature λ. Then, for each point $m \in M$ and any instantaneous observer $\xi \in T_mM$, the volume of the geodesic celestial sphere $S_m^\xi(r)$ satisfies*

$$vol_{n-1}\left(S_m^\xi(r)\right) = \begin{cases} c_{n-1}\left(\frac{\sin t\sqrt{\lambda}}{\sqrt{\lambda}}\right)^{n-1} & \lambda > 0 \\ c_{n-1} & \lambda = 0 \\ c_{n-1}\left(\frac{\sinh t\sqrt{-\lambda}}{\sqrt{-\lambda}}\right)^{n-1} & \lambda < 0 \end{cases}$$

where $c_{n-1} = n\pi^{\frac{n}{2}} / \left(\frac{n}{2}\right)!$ is the volume of the $(n-1)$-dimensional Euclidean sphere of radius 1. Here $\left(\frac{n}{2}\right)! = \Gamma\left(\frac{n}{2}+1\right)$, where Γ is the Gamma function defined by

$$\Gamma(\alpha) = \int_0^\infty e^{-t}t^{\alpha-1}dt = \int_{-\infty}^\infty e^{-t^2}|t|^{2\alpha-1}dt$$

Proof. Let $\xi \in T_mM$ be an instantaneous observer and let γ be a radial geodesic leaving m orthogonally to ξ. Complete ξ to an orthonormal basis $\{e_0 = \xi, e_1, \dots, e_n\}$ of T_mM in such a way that $e_1 = \gamma'(0)$. Let $(x^0, \dots, x^n)$ denote the normal coordinates on M in a neighborhood of m associated to $\{e_0, \dots, e_n\}$. Then the $(n-1)$-dimensional volume of the geodesic celestial sphere is given by

$$\mathrm{vol}_{n-1}\left(S_m^\xi(r)\right) = r^{n-1}\int_{\mathcal{S}^\xi(1)}\left(\sqrt{\det\overline{g}}\right)(exp_m(ru))\,du \tag{2.2}$$

where du is the volume element of $\mathcal{S}^\xi$ and $\overline{g}_{ij} = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)$, $i, j \in \{1, \dots, n\}$.

Now, if M is a Lorentzian space form, the components of the metric tensor can be expressed as follows. Along the geodesic γ, $t\frac{\partial}{\partial x^i}(\gamma(t))$ with $i \in \{2, \dots, n\}$ is the variational vector field of the variation $\Gamma(t, s) = exp_m(t(e_1 + s\,e_i))$, and hence, it is a Jacobi vector field with initial conditions $\left(t\frac{\partial}{\partial x^i}\right)(\gamma(0)) = 0$ and $\left(t\frac{\partial}{\partial x^i}\right)'(\gamma(0)) = e_i$.

Put $\{E_1, \dots, E_n\}$ the parallel translation of $\{e_1, \dots, e_n\}$ along γ in M. Since M has constant sectional curvature λ and γ is a spacelike geodesic, by solving the Jacobi equation with the initial conditions above we have:

$$\begin{aligned} \frac{\partial}{\partial x^1}(\gamma(t)) &= E_1(\gamma(t)) \\ \left(t\frac{\partial}{\partial x^i}\right)(\gamma(t)) &= f(t)E_i(t), \quad i \in \{2, \cdots, n\} \end{aligned}$$

where

$$f(t) = \begin{cases} \left(\frac{\sin t\sqrt{\lambda}}{\sqrt{\lambda}}\right) & \lambda > 0 \\ 1 & \lambda = 0 \\ \left(\frac{\sinh t\sqrt{-\lambda}}{\sqrt{-\lambda}}\right)^{n-1} & \lambda < 0 \end{cases}$$

Thus, one has

$$\overline{g}(\gamma(t)) = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & \frac{f(t)^2}{t^2} Id_{n-1} \end{array} \right)$$

and hence it follows from (2.2) that

$$\mathrm{vol}_{n-1}\left(S^{\xi}(r)\right) = r^{n-1} \int_{\mathcal{S}^{\xi}} \left(\frac{f(r)}{r}\right)^{n-1} du = c_{n-1} f(r)^{n-1}$$

which proves the result. □

If $N(\lambda)$ is a Lorentzian manifold of constant sectional curvature λ, by Theorem 2.1, the volume of a geodesic celestial sphere is independent of the base point $m \in N$ and the instantaneous observer $\xi \in T_m N$, so in this case we can use the unambiguous notation

$$\mathrm{vol}_{n-1}(S(r)) = \mathrm{vol}_{n-1}\left(S^{\xi}_m(r)\right)$$

For the purpose of stating the comparison results below, we will also denote by $\mathrm{vol}^M_{n-1}\left(S^{\xi}_m(r)\right)$ the $(n-1)$-dimensional volume of the geodesic celestial sphere $S^{\xi}_m(r)$ in the manifold M of radius r and center m associated to the instantaneous observer ξ.

It is well known that the existence of a lower or upper bound on the sectional curvature of a Lorentzian manifold (or more generally, a semi-Riemannian manifold) forces it to be constant [13]. Therefore, it is natural to impose such curvature bounds on the curvature tensor itself. Following [2], we will say that $R \geq \lambda$ or $R \leq \lambda$ if and only if for all X, Y,

$$R(X,Y,X,Y) \geq \lambda\left(\langle X,X\rangle\langle Y,Y\rangle - \langle X,Y\rangle^2\right), \tag{2.3}$$

or

$$R(X,Y,X,Y) \leq \lambda\left(\langle X,X\rangle\langle Y,Y\rangle - \langle X,Y\rangle^2\right), \tag{2.4}$$

respectively.

Note that condition (2.3) (resp., (2.4)) is equivalent to the sectional curvature be bounded from below (resp., from above) on planes of signature $(++)$ and from above (resp., from below) on planes of signature $(+-)$. It is interesting to point out that such boundedness conditions hold for Robertson-Walker spacetimes, but the behavior of the curvature may change from point to point. In general, let (M,g) be a conformally flat Lorentz manifold whose Ricci tensor is diagonalizable, $\rho = diag[\mu_0, \mu_1, \ldots, \mu_n]$, where the distinguished eigenvalue μ_0 corresponds to a timelike eigenspace. If $\mu_0 \geq max\{\mu_1, \ldots, \mu_n\}$ (resp., $\mu_0 \leq min\{\mu_1, \ldots, \mu_n\}$) then $R \leq \lambda$ (resp., $R \geq \lambda$) for some constant λ.

Theorem 2.2. *Let (M^{n+1}, g) be a $n+1$-dimensional Lorentzian manifold and $N^{n+1}(\lambda)$ a Lorentzian manifold of constant sectional curvature λ. The following statements hold:*

(i) *If $R \geq \lambda$ then*

$$vol^M_{n-1}\left(S^{\xi}_m(r)\right) \leq vol^{N(\lambda)}_{n-1}(S(r))$$

for all sufficiently small r and all instantaneous observer $\xi \in T_m M$.

(ii) *If* $R \leq \lambda$ *then*

$$vol_{n-1}^{M}\left(S_m^{\xi}(r)\right) \geq vol_{n-1}^{N(\lambda)}\left(S(r)\right)$$

for all sufficiently small r *and all instantaneous observer* $\xi \in T_mM$.

Moreover, the equality holds at (i) or (ii) for all $\xi \in T_mM$ *if and only if* M *has constant sectional curvature* λ *at* m.

The previous theorem shows that the quotient $\mathrm{vol}_{n-1}^{M}\left(S_m^{\xi}(r)\right)/\mathrm{vol}_{n-1}^{N(\lambda)}(S(r)) \leq 1$ (resp., ≥ 1) if (2.3) (resp., (2.4)) hold. Now, a more precise result can be stated as follows

Theorem 2.3. *Let* (M^{n+1}, g) *be a* $n+1$*-dimensional Lorentzian manifold and* $N^{n+1}(\lambda)$ *a Lorentzian manifold of constant sectional curvature* λ.

(i) *If* $R \geq \lambda$ *then*

$$\frac{vol_{n-1}^{M}\left(S_m^{\xi}(r)\right)}{vol_{n-1}^{N(\lambda)}\left(S(r)\right)}$$

is nonincreasing for sufficiently small r *and all instantaneous observer* $\xi \in T_mM$.

(ii) *If* $R \leq \lambda$ *then*

$$\frac{vol_{n-1}^{M}\left(S_m^{\xi}(r)\right)}{vol_{n-1}^{N(\lambda)}\left(S(r)\right)}$$

is nondecreasing for sufficiently small r *and all instantaneous observer* $\xi \in T_mM$.

Remark 2.4. Under the hypothesis in Theorem 2.3, if there exists $0 < r_0 < r_1$ such that $\dfrac{\mathrm{vol}_{n-1}^{M}\left(S_m^{\xi}(r_0)\right)}{\mathrm{vol}_{n-1}^{N(\lambda)}\left(S(r_0)\right)} = \dfrac{\mathrm{vol}_{n-1}^{M}\left(S_m^{\xi}(r_1)\right)}{\mathrm{vol}_{n-1}^{N(\lambda)}\left(S(r_1)\right)}$, then the sectional curvature is constant. Indeed, since the quotient above is monotone, then it must be constant and thus $R = \lambda$ (see proof of Theorem 2.3).

Remark 2.5. We point out here that the proofs of Theorem 2.2 and Theorem 2.3 will only require conditions (2.3) - (2.4) to hold for spacelike planes.

Recall here that a Lorentzian manifold is said to be locally isotropic if for any point $m \in M$ and nonzero vectors $X, Y \in T_mM$, with $\langle X, X\rangle = \langle Y, Y\rangle$ there exists a local isometry of (M, g) fixing m which sends X to Y. Thus, $\mathrm{vol}_{n-1}^{M}\left(S_m^{\xi}(r)\right)$ does not depend on the instantaneous observer $\xi \in T_mM$ for such manifolds. Moreover, since locally isotropic manifolds are locally homogeneous, it follows that $\mathrm{vol}_{n-1}^{M}\left(S_m^{\xi}(r)\right)$ does not depend on the center $m \in M$ either. The next theorem shows that local isotropy can be recovered from the properties of the volume of geodesic celestial spheres.

Theorem 2.6. *Let* $\left(M^{n+1}, g\right)$ *be a Lorentzian manifold. If the volume of the geodesic celestial spheres* $S_m^{\xi}(r)$ *is independent of the observer field* $\xi \in TM$*, then* M *has constant sectional curvature.*

3. Proofs of the theorems

The technique we will use to prove the theorems above relies on the possibility of writing down the first terms in the power series expansion of the function $r \mapsto \mathrm{vol}_{n-1}^M\left(S_m^\xi(r)\right)$, for sufficiently small r. Recall that a *scalar curvature invariant* is a polynomial in the components of the curvature tensor that does not depend on the choice of orthonormal basis used to build it. For instance, $\tau = \sum_{ij} \epsilon_i \epsilon_j R_{ijij}$ is a first order scalar curvature invariant while

$$\begin{array}{rclrcl} \|R\|^2 & = & \sum_{ijkl} \epsilon_i\epsilon_j\epsilon_k\epsilon_l R_{ijkl}^2, & \tau^2, & & \\ \|\rho\|^2 & = & \sum_{ij} \epsilon_i\epsilon_j\rho_{ij}^2, & \Delta\tau & = & \sum_i \epsilon_i \nabla_{ii}^2\tau, \end{array} \tag{3.1}$$

are second order scalar curvature invariants, where $\{e_0, \ldots, e_n\}$ is an orthonormal basis of M and Δ is the Laplacian operator in (M, g). Scalar curvature invariants are a powerful tool in Riemannian geometry, but they may become useless when the metric is allowed to have indefinite signature [4], [5].

Theorem 3.1. *Let $\left(M^{n+1}, g\right)$ be a Lorentzian manifold and $\xi \in T_mM$ an instantaneous observer. The $(n-1)$-dimensional volume of the geodesic celestial spheres associated to $\xi \in T_mM$ satisfies*

$$vol_{n-1}\left(S_m^\xi(r)\right) = c_{n-1}r^{n-1}\left(1 + \frac{A(\xi)}{n}r^2 + \frac{B(\xi)}{n(n+2)}r^4 + O(r^6)\right)(m)$$

where

$$\begin{aligned} A(\xi) &= -\frac{1}{6}(\tau + 2\rho_{\xi\xi}), \\ B(\xi) &= -\frac{1}{120}\|R\|^2 + \frac{1}{45}\|\rho\|^2 + \frac{1}{72}\tau^2 - \frac{1}{20}\Delta\tau - \frac{1}{15}\sum_{i,j,k=1}^{n} R_{\xi ijk}^2 \\ &\quad + \frac{1}{18}\sum_{i,j=1}^{n} R_{\xi i\xi j}^2 + \frac{2}{45}\sum_{i,j=1}^{n}\rho_{ij}R_{\xi i\xi j} + \frac{1}{45}\sum_{i=1}^{n}\rho_{\xi i}^2 \\ &\quad + \frac{1}{30}\rho_{\xi\xi}^2 + \frac{1}{18}\tau\rho_{\xi\xi} - \frac{1}{10}\Delta\rho_{\xi\xi} - \frac{1}{20}\nabla_{\xi\xi}^2\tau - \frac{1}{10}\nabla_{\xi\xi}^2\rho_{\xi\xi}. \end{aligned}$$

Since radial geodesics starting from m orthogonally to ξ are the same for M and $\widetilde{M}$, it is clear that the geodesic celestial sphere $S_m^\xi(r)$ of (M, g) associated to the instantaneous observer $\xi \in T_mM$ coincides with the geodesic sphere $G_m^{\widetilde{M}}(r)$ of radius r centered at m on the Riemannian manifold $\widetilde{M}$ for sufficiently small radius. Now, the first terms in the power series expansion of the volume of sufficiently small geodesic spheres are well known [11]:

$$\begin{aligned} \mathrm{vol}\left(G_m^{\widetilde{M}}(r)\right) &= c_{n-1}r^{n-1}\Big\{1 - \frac{\widetilde{\tau}}{6n}r^2 \\ &\qquad - \frac{r^4}{n(n+2)}\left(\frac{\|\widetilde{R}\|^2}{120} - \frac{\|\widetilde{\rho}\|^2}{45} - \frac{\widetilde{\tau}^2}{72} + \frac{\widetilde{\Delta}\widetilde{\tau}}{20}\right) \\ &\qquad + O(r^6)\Big\}(m) \end{aligned} \tag{3.2}$$

For our purpose it is necessary to relate the scalar curvature invariants of the Lorentzian manifold (M, g) with those of the Riemannian $(\widetilde{M}, \widetilde{g})$ at the base point m as follows

Lemma 3.2. *The first and second order curvature invariants of* (M, g) *and* $(\widetilde{M}, \widetilde{g})$ *at the base point* m *satisfy*

$$\begin{aligned}
\|\widetilde{R}\|^2 &= \|R\|^2 + 4 \sum_{i,j,k=1}^{n} R^2_{\xi ijk} - 4 \sum_{i,j=1}^{n} R^2_{\xi i \xi j} \\
\|\widetilde{\rho}\|^2 &= \|\rho\|^2 + 2 \sum_{i=1}^{n} \rho^2_{\xi i} - \rho^2_{\xi\xi} + \sum_{i,j=1}^{n} R^2_{\xi i \xi j} + 2 \sum_{i,j=1}^{n} \rho_{ij} R_{\xi i \xi j} \\
\widetilde{\tau} &= \tau + 2\rho_{\xi\xi} \\
\widetilde{\Delta}\widetilde{\tau} &= \Delta\tau + 2\Delta\rho_{\xi\xi} + \nabla^2_{\xi\xi}\tau + 2\nabla^2_{\xi\xi}\rho_{\xi\xi} + \frac{4}{9} \sum_{i=1}^{n} \rho^2_{\xi i} + \frac{2}{3} \sum_{i,j,k=1}^{n} R^2_{\xi ijk}.
\end{aligned}$$

Proof. It will follow from equations (3.12), (3.14) and (3.15) after straightforward calculations. To prove the desired (3.12), (3.14) and (3.15) we proceed as follows. Denote by ξ a local extension of $\xi \in T_m M$ to the normal bundle of $\widetilde{M}$. We recall the following conventions for the shape tensor II, defined by $\widetilde{\nabla}_X Y = \nabla_X Y + II(X, Y)$, the second fundamental form σ given by $II(X, Y) = -\sigma(X, Y)\xi$ and the shape operator T defined as $\widetilde{g}(TX, Y) = \sigma(X, Y) = g(II(X, Y), \xi)$. Then the Weingarten, Gauss and Codazzi equations read as follows:

$$TX = \nabla_X \xi \tag{3.3}$$

$$\widetilde{R}_{XYVW} = R_{XYVW} - \sigma_{XV}\sigma_{YW} + \sigma_{XW}\sigma_{YV} \tag{3.4}$$

$$R_{XYZ\xi} = \widetilde{\nabla}_X \sigma_{YZ} - \widetilde{\nabla}_Y \sigma_{XZ} \tag{3.5}$$

If γ is a radial geodesic in $\widetilde{M}$ starting from m, then

$$\widetilde{\nabla}_{\gamma'}\gamma' = \nabla_{\gamma'}\gamma' = II(\gamma', \gamma') = 0$$

since $\widetilde{M} = \exp_m(\xi^\perp)$. Thus taking covariant derivatives and evaluating at m, we get

$$\widetilde{\nabla}^k_{u\cdots u}\sigma_{uu} = 0, \quad k \geq 0, \tag{3.6}$$

for all $u \in T_m\widetilde{M}$. For $k = 0$ we immediately get by polarization that

$$\sigma_{uv} = 0 \qquad \text{for all } u, v \in T_m\widetilde{M}. \tag{3.7}$$

Now put $k = 1$ and take arbitrary $a, b, c \in \mathbb{R}$, $u, v, w \in T_m\widetilde{M}$. Then

$$\begin{aligned}
0 &= \widetilde{\nabla}_{au+bv+cw}\sigma_{au+bv+cw,au+bv+cw} \\
&= \cdots + 2abc\left(\widetilde{\nabla}_u\sigma_{vw} + \widetilde{\nabla}_v\sigma_{uw} + \widetilde{\nabla}_w\sigma_{uv}\right) + \cdots
\end{aligned}$$

and hence

$$\widetilde{\nabla}_u\sigma_{vw} + \widetilde{\nabla}_v\sigma_{uw} + \widetilde{\nabla}_w\sigma_{uv} = 0. \tag{3.8}$$

Then it follows from the Riccati equation that

$$R_{uvw\xi} = \widetilde{\nabla}_u\sigma_{vw} - \widetilde{\nabla}_v\sigma_{uw} \tag{3.9}$$

$$R_{uwv\xi} = \widetilde{\nabla}_u\sigma_{vw} - \widetilde{\nabla}_w\sigma_{uv} \tag{3.10}$$

and from the last three equations we get

$$\widetilde{\nabla}_u \sigma_{vw} = \frac{1}{3}\left(R_{uvw\xi} + R_{uwv\xi}\right). \tag{3.11}$$

Now, an immediate application of the Gauss equation and (3.7) shows that

$$\widetilde{R}_{xyvw} = R_{xyvw} \tag{3.12}$$

for all $x, y, v, w \in T_m\widetilde{M}$. Also, taking covariant derivatives in (3.4) we get

$$\begin{aligned} \widetilde{\nabla}_Z \widetilde{R}_{XYVW} \;=\;& \nabla_Z R_{XYVW} + \sigma_{ZX} R_{\xi YVW} + \sigma_{ZY} R_{X\xi VW} \\ & + \sigma_{ZV} R_{XY\xi W} + \sigma_{ZW} R_{XYV\xi} - \sigma_{YW} \widetilde{\nabla}_Z \sigma_{XV} \\ & - \sigma_{XV} \widetilde{\nabla}_Z \sigma_{YW} + \sigma_{YV} \widetilde{\nabla}_Z \sigma_{XW} + \sigma_{XW} \widetilde{\nabla}_Z \sigma_{YV} \end{aligned} \tag{3.13}$$

for all X, Y, Z, V, W vector fields on $\widetilde{M}$. Using (3.7) we get

$$\widetilde{\nabla}_z \widetilde{R}_{xyvw} = \nabla_z R_{xyvw} \tag{3.14}$$

for all $z, x, y, v, w \in T_m\widetilde{M}$.

Finally, taking covariant derivatives in (3.13) we obtain

$$\begin{aligned} \widetilde{\nabla}^2_{XX} \widetilde{R}_{YZYZ} \;=\;& \nabla^2_{XX} R_{YZYZ} + \sigma_{XX} \nabla_\xi R_{YZYZ} + 2\,\sigma^2_{XY} R_{\xi X \xi Z} \\ & + 2\,\sigma^2_{XZ} R_{\xi Y \xi Y} - 4\,\sigma_{XY}\sigma_{XZ} R_{\xi Y \xi Z} + 2\,\sigma_{XY} R_{TXZYZ} \\ & + 2\,\sigma_{XZ} R_{YTXYZ} + 2\,\widetilde{\nabla}_X \sigma_{XY} R_{\xi ZYZ} + 2\,\widetilde{\nabla}_X \sigma_{XZ} R_{Y\xi YZ} \\ & - \sigma_{YY} \widetilde{\nabla}_{XX} \sigma_{ZZ} - \sigma_{ZZ} \widetilde{\nabla}_{XX} \sigma_{YY} + \sigma_{YZ} \widetilde{\nabla}_{XX} \sigma_{YZ} \\ & - 2 \widetilde{\nabla}_X \sigma_{YY} \widetilde{\nabla}_X \sigma_{ZZ} + 2\left(\widetilde{\nabla}_X \sigma_{YZ}\right)^2 \\ & + 4\sigma_{XY} \nabla_X R_{\xi ZYZ} + 4\sigma_{XZ} \nabla_X R_{Y\xi YZ}, \end{aligned}$$

and using (3.7) and (3.11) we get

$$\begin{aligned} \widetilde{\nabla}^2_{xx} \widetilde{R}_{yzyz} \;=\;& \nabla^2_{xx} R_{yzyz} + \tfrac{2}{3} R_{xyx\xi} R_{yz\xi z} + \tfrac{2}{3} R_{xzx\xi} R_{yzy\xi} \\ & - \tfrac{8}{9} R_{xy\xi y} R_{xz\xi z} + \tfrac{2}{9} R^2_{xyz\xi} + \tfrac{2}{9} R^2_{xzy\xi} + \tfrac{4}{9} R_{xyz\xi} R_{xzy\xi} \end{aligned} \tag{3.15}$$

for all $x, y, z \in T_m\widetilde{M}$.

Now Lemma 3.2 follows from the definitions of τ, $\|R\|^2$, $\|\rho\|^2$ and $\Delta\tau$. □

Proof of Theorem 3.1. It follows immediately from (3.2) using the relations in Lemma 3.2.

Proof of Theorem 2.2. Let $\{e_0 = \xi, e_1, \ldots, e_n\}$ be an orthonormal basis of T_mM. It follows after some calculations that $\tau + 2\rho_{\xi\xi} = \sum_{i,j=1}^n R_{ijij}$. Hence by assuming (i) (resp. (ii)) to hold for spacelike planes, we have $\tau + 2\rho_{\xi\xi} \geq n(n-1)\lambda$ (resp., $\leq n(n-1)\lambda$). Thus, by Theorem 2.1 and Theorem 3.1, we have for sufficiently small r

$$\begin{aligned} \mathrm{vol}^M_{n-1}\left(S^\xi_m(r)\right) &= c_{n-1} r^{n-1}\left(1 - \frac{\tau + 2\rho_{\xi\xi}}{6n} r^2 + O(r^4)\right) \\ &\leq c_{n-1} r^{n-1}\left(1 - \frac{n-1}{6}\lambda\, r^2 + O(r^4)\right) \\ &= \mathrm{vol}^{N(\lambda)}_{n-1}\left(S(r)\right), \end{aligned}$$

which proves (i). (ii) is obtained in an analogous way.

Now, suppose the equality holds for sufficiently small r and all $\xi \in T_mM$. Then $\tau+2\rho_{\xi\xi} = n(n-1)\lambda$ for all $\xi \in T_mM$ and thus the sectional curvature K is constant λ on planes of signature $(++)$, since $R \geq \lambda$. Indeed, given π a plane of signature $(++)$, take an orthonormal basis $\{x, y\}$ of π and complete it to an orthonormal basis $\{e_0, e_1 = x, e_2 = y, \dots, e_n\}$ of T_mM with e_0 timelike. Then

$$\sum_{i,j=1}^{n} R_{ijij} = \tau + 2\rho_{e_0e_0} = n(n-1)\lambda$$

and since $R_{ijij} \geq \lambda$ by assumption, it follows that the sectional curvature $K(\pi) = \lambda$. Now the constancy of the sectional curvature at m follows from [14]. □

Proof of Theorem 2.3. By using the results in Theorem 2.1 and Theorem 3.1 one gets the first terms in the power series expansion of the quotient

$$\frac{\mathrm{vol}_{n-1}^{M}\left(S_m^{\xi}(r)\right)}{\mathrm{vol}_{n-1}^{N(\lambda)}\left(S(r)\right)} = 1 + \left(\frac{n(n-1)\lambda - (\tau + 2\rho_{\xi\xi})}{6n}\right) r^2 + O(r^4).$$

Therefore, if $R > \lambda$, then $\tau+2\rho_{\xi\xi} > n(n-1)\lambda$. Hence the derivative of the quotient is negative for small r, and thus the quotient is decreasing, which shows (i), since in case $R = \lambda$ the quotient above is constant for sufficiently small r. The proof of (ii) is completely analogous. □

Recall at this point that, by Theorem 2.1, the volume of a geodesic celestial sphere in a constant curvature Lorentzian manifold is independent of center $m \in M$ and the instantaneous observer $\xi \in T_mM$. In what follows we will show the converse result in proving Theorem 2.6. For that, we need some algebraic preliminaries:

Lemma 3.3. *Let $(V, \langle\, ,\, \rangle)$ a Lorentzian vector space and let W denote a covariant tensor of type $(0, 2k)$. If $W_{\zeta\cdots\zeta} = 0$ for all ζ with $\langle \zeta, \zeta\rangle = -1$, then $W_{x\cdots x} = 0$ for all $x \in V$.*

Proof. If ζ is a timelike vector, we have

$$0 = W\left(\frac{\zeta}{\sqrt{-\langle\zeta,\zeta\rangle}}, \dots, \frac{\zeta}{\sqrt{-\langle\zeta,\zeta\rangle}}\right) = \Big(-\langle\zeta,\zeta\rangle\Big)^{-k} W_{\zeta\cdots\zeta}$$

and thus $W_{\zeta\cdots\zeta} = 0$. Now, if x is an arbitrary vector, for sufficiently small ϵ, $\zeta + \epsilon x$ is timelike if ζ is timelike. Then,

$$0 = W(\zeta + \epsilon x, \dots, \zeta + \epsilon x) = W_{\zeta\cdots\zeta} + \cdots + \epsilon^{2k} W_{x\cdots x}$$

Taking into account that ϵ is arbitrary, this immediately implies that

$$W_{x\cdots x} = 0$$

which proves the result. □

Lemma 3.4. *Let $\big(M^{n+1}, g\big)$ be a Lorentzian manifold and let a, b, c be real numbers with $b \neq 0$. If $a\,\tau + b\,\rho_{\zeta\zeta} = c$ at some point $m \in M$ for all vector $\zeta \in T_mM$ with $\langle\zeta,\zeta\rangle = -1$, then the manifold is Einstein at m.*

Proof. The hypothesis can equivalently be written as

$$-a\langle\zeta,\zeta\rangle\tau + b\rho_{\zeta\zeta} + c\langle\zeta,\zeta\rangle = 0 \tag{3.16}$$

which, using Lemma 3.3, implies that

$$-a\langle x,x\rangle\tau + b\rho_{xx} + c\langle x,x\rangle = 0 \tag{3.17}$$

for all $x \in T_mM$. The result follows by linearity and symmetry of the Ricci tensor. $\square$

For the purpose of analyzing the coefficient $B(\xi)$ in Theorem 3.1, we define the following two tensors:

$$\eta(x,y) = \sum_{i,j,k=0}^{n} \epsilon_i\epsilon_j\epsilon_k R(x,e_i,e_j,e_k)R(y,e_i,e_j,e_k) \tag{3.18}$$

$$\omega(x,y,v,w) = \sum_{i,j=0}^{n} \epsilon_i\epsilon_j R(x,e_i,y,e_j)R(v,e_i,w,e_j) \tag{3.19}$$

where, $\epsilon_i = \langle e_i, e_i\rangle$ and x, y, v, $w \in T_mM$. Note that the definitions above are independent of the orthonormal basis chosen, and thus ω and η are well defined tensors at a given point $m \in M$. We have the following result:

Lemma 3.5. *Let $\left(M^{n+1}, g\right)$ be an Einstein Lorentzian manifold. If*

$$a\,\|R\|^2 + b\,\eta_{\zeta\zeta} + c\,\omega_{\zeta\zeta\zeta\zeta} = d$$

for all vectors $\zeta \in T_mM$ with $\langle \zeta,\zeta\rangle = -1$ and some a, b, c, $d \in \mathbb{R}$ with $c \neq 0$, $3c \neq (n+5)b$, then M has constant sectional curvature at m.

Proof. Using Lemma 3.3, the hypothesis can be rewritten as

$$a\|R\|^2\langle x,x\rangle^2 - b\langle x,x\rangle\eta(x,x) + c\,\omega(x,x,x,x) = d\langle x,x\rangle^2 \tag{3.20}$$

for all $x \in T_mM$. For arbitrary $\alpha,\beta \in \mathbb{R}$ and tangent vectors x and y we get

$$\begin{gathered} a\|R\|^2\langle \alpha x+\beta y, \alpha x+\beta y\rangle^2 - b\langle \alpha x+\beta y, \alpha x+\beta y\rangle\eta(\alpha x+\beta y, \alpha x+\beta y) \\ +c\,\omega(\alpha x+\beta y, \alpha x+\beta y, \alpha x+\beta y, \alpha x+\beta y) = d\langle \alpha x+\beta y, \alpha x+\beta y\rangle^2 \end{gathered}$$

Hence, expanding the above equality and comparing the coefficients of $\alpha^2\beta^2$ we get, since α and β are arbitrary

$$\begin{gathered} 2a\|R\|^2(\langle x,x\rangle\langle y,y\rangle + 2\langle x,y\rangle^2) - b(\langle x,x\rangle\eta_{yy} + 4\langle x,y\rangle\eta_{xy} + \langle y,y\rangle\eta_{xx}) \\ +2c(\omega_{xxyy} + \omega_{xyxy} + \omega_{xyyx}) = 2d(\langle x,x\rangle\langle y,y\rangle + 2\langle x,y\rangle^2) \end{gathered} \tag{3.21}$$

Setting $y = e_i$ in the above equality and contracting we have

$$\begin{gathered} 2a(n+3)\|R\|^2\langle x,x\rangle - b(\|R\|^2\langle x,x\rangle + (n+5)\eta_{xx}) \\ +c\left(3\eta_{xx} + 2\sum_{i,j=0}^{n}\langle e_i,e_i\rangle\langle e_j,e_j\rangle\rho_{ij}R_{xixj}\right) = 2(n+3)d\langle x,x\rangle \end{gathered} \tag{3.22}$$

Since M is Einstein, $\sum_{i,j=0}^{n}\langle e_i,e_i\rangle\langle e_j,e_j\rangle\rho_{ij}R_{xixj} = \frac{\tau^2}{(n+1)^2}\langle x,x\rangle$, and (3.22) becomes

$$(-b(n+5)+3c)\eta_{xx} = \left(-2a(n+3)\|R\|^2 + b\|R\|^2 - 2c\frac{\tau^2}{(n+1)^2} + 2d(n+3)\right)\langle x,x\rangle \tag{3.23}$$

Contracting again,

$$(-b(n+5)+3c)\|R\|^2 = \left(-2a(n+3)\|R\|^2 + b\|R\|^2 - 2c\frac{\tau^2}{(n+1)^2} + 2d(n+3)\right)(n+1) \tag{3.24}$$

Substituting in (3.23) we get

$$\eta_{xx} = \frac{\|R\|^2}{n+1}\langle x, x\rangle \tag{3.25}$$

which, by symmetry of η and the metric tensor, is equivalent to $\eta = \frac{\|R\|^2}{n+1} g$. As a consequence, using (3.20) we have

$$\omega_{xxxx} = \frac{1}{c}\left(-a\|R\|^2 + b\frac{\|R\|^2}{n+1} + d\right)\langle x, x\rangle^2. \tag{3.26}$$

Next we show that (3.26) is an equivalent condition to constant sectional curvature for Lorentzian manifolds. We proceed as follows: Let $\pi \subset T_mM$ be a plane of signature $(-+)$ and let $\{\zeta, \vartheta\}$ be an orthonormal basis of π with $\langle \zeta, \zeta\rangle = -1 = -\langle \vartheta, \vartheta\rangle$. The Jacobi operator $R_\zeta(x) = R(\zeta, x)\zeta$ restricts to $\zeta^\perp$ by curvature identities and thus, since it is self-adjoint, it is diagonalizable with respect to an orthonormal basis $\{e_1, \ldots, e_n\}$ of $\zeta^\perp$ with eigenvalues $\lambda_1(\zeta), \ldots, \lambda_n(\zeta)$. Now, with respect to the orthonormal basis of T_mM, $\{e_0 = \zeta, e_1, \ldots, e_n\}$, equation (3.26) gives

$$\sum_{i,j=1}^{n} R^2_{\zeta e_i \zeta e_j} = \frac{1}{c}\left(\frac{b-(n+1)a}{n+1}\|R\|^2 + d\right).$$

Hence, the eigenvalues $\lambda_\alpha(\zeta)$ are bounded independently of the timelike unit ζ as

$$\lambda_\alpha(\zeta)^2 = R^2_{\zeta e_\alpha \zeta e_\alpha} \le \sum_{i,j=1}^{n} R^2_{\zeta e_i \zeta e_j} = \frac{1}{c}\left(\frac{b-(n+1)a}{n+1}\|R\|^2 + d\right)$$

for all $\alpha = 1, \ldots, n$. Next, writing $\vartheta = \sum_{i=1}^n \vartheta^i e_i$ on the basis above, one has for the sectional curvature of π:

$$K(\pi) = -R_{\zeta\vartheta\zeta\vartheta} = -\sum_{i,j=1}^{n} \vartheta^i\vartheta^j R_{\zeta e_i \zeta e_j} = -\sum_{i=1}^{n}(\vartheta^i)^2\lambda_i(\zeta).$$

Since $\langle \vartheta, \vartheta\rangle = 1 = \displaystyle\sum_{i=1}^{n}(\vartheta^i)^2$, one has

$$|K(\pi)| \le \sum_{i=1}^{n}(\vartheta^i)^2|\lambda_i(\zeta)| \le K,$$

for some constant K. This shows that the sectional curvature is bounded on planes of signature $(+,-)$ and, therefore, M has constant curvature at m (cf. [3], [13], [14]). □

Remark 3.6. Einstein Lorentzian manifolds satisfying (3.26) are called 2-*stein*. See [10] for a different proof that 2-stein Lorentz manifolds have constant curvature.

Proof of Theorem 2.6. If the volume of each geodesic celestial spheres $S^\xi_m(r)$ is independent of the instantaneous observer $\xi \in T_mM$, then the coefficients $A(\xi)$ and $B(\xi)$ in the power series expansion of $\mathrm{vol}_{n-1}(S^\xi_m(r))$ in Theorem 3.1 are independent of ξ. Now, if $-\frac{\tau+2\rho_{\xi\xi}}{6} = A(\xi) = \textit{constant}$, by Lemma 3.4, one has that that M is Einstein, and thus $\rho = \frac{\tau}{n+1}g$. Hence, it follows from the second coefficient $B(\xi)$ in

Theorem 3.1 that

$$\begin{aligned} constant = B(\xi) &= -\frac{1}{120}\|R\|^2 + \frac{5n^2+38n+61}{360(n+1)^2}\tau^2 \\ &+\frac{1}{18}\sum_{i,j=1}^{n} R^2_{\xi i\xi j} - \frac{1}{15}\sum_{i,j,k=1}^{n} R^2_{\xi ijk}. \end{aligned}$$

Now, as a consequence of Lemma 3.5, the sectional curvature of M is necessarily constant. □

4. Total curvatures of geodesic celestial spheres

Since geodesic celestial spheres of sufficiently small radius are compact manifolds, it makes sense to consider the total scalar curvatures obtained by integrating on $S^\xi_m(r)$ the corresponding scalar curvature invariants. Clearly such total curvatures are independent of the center of $S^\xi_m(r)$ and the instantaneous observer provided that (M, g) is isotropic. The next theorems show a converse of this result in the spirit of Theorem 2.6. In what follows, geometric objects defined on geodesic celestial spheres will be denoted by $\hat\tau$, $\|\hat\rho\|^2$, and so on.

Theorem 4.1. *Let (M^{n+1}, g) be a Lorentzian manifold with $n > 3$ such that $\int_{S^\xi_m(r)} \hat\tau$ only depends on the radius. Then, M has constant sectional curvature.*

Proof. It follows as in Theorem 2.6 just considering the following expansion of the total scalar curvature of $S^\xi_m(r)$ as:

$$\begin{aligned} \int_{S^\xi_m(r)} \hat\tau = & c_{n-1} r^{n-1}\Big\{ \tfrac{(n-2)(n-1)}{r^2} - \tfrac{(n-3)(n-2)}{6n}(\tau + 2\rho_{\xi\xi}) \\ & + \tfrac{1}{n(n+2)}\Big(-\tfrac{(n+2)(n+3)}{120}\|R\|^2 + \tfrac{n^2+5n+21}{45}\|\rho\|^2 + \tfrac{n^2-7n-6}{72}\tau^2 \\ & - \tfrac{(n-3)(n-2)}{20}\Delta\tau - \tfrac{n^2+6}{15}\sum_{i,j,k=1}^{n} R^2_{\xi ijk} + \tfrac{n^2+5n+12}{18}\sum_{i,j=1}^{n} R^2_{\xi i\xi j} \\ & + \tfrac{2(n^2+5n+21)}{45}\sum_{i,j=1}^{n}\rho_{ij}R_{\xi i\xi j} + \tfrac{(n+3)(n+12)}{45}\sum_{i=1}^{n}\rho^2_{\xi i} \\ & + \tfrac{n^2-15n-24}{30}\rho^2_{\xi\xi} + \tfrac{n^2-7n-6}{18}\tau\rho_{\xi\xi} - \tfrac{(n-3)(n-2)}{20}\nabla^2_{\xi\xi}\tau \\ & - \tfrac{(n-3)(n-2)}{10}\nabla^2_{\xi\xi}\rho_{\xi\xi} - \tfrac{(n-3)(n-2)}{10}\Delta\rho_{\xi\xi}\Big) + O\left(r^4\right)\Big\} \end{aligned}$$

□

Remark 4.2. If $n = 2$ then the geodesic celestial spheres are flat and thus their total scalar curvatures vanish identically. Furthermore geodesic celestial spheres are diffeomorphic with the Euclidean spheres if $n = 3$, and hence the total scalar curvature is a topological invariant equal to 8π by the Gauss–Bonnet Theorem. Therefore, it is useless for the purpose of geometrical characterizations.

Next, consider the second order scalar curvature invariants as defined in (3.1) and note that $\int_{S^\xi_m(r)} \hat\Delta\hat\tau = 0$. Hence, we will only consider the L^2-norms of the curvature tensor, the Ricci tensor and the scalar curvature of geodesic celestial spheres.

Theorem 4.3. *Let (M^{n+1}, g) be a Lorentzian manifold with $2 < n \neq 5$. Then the following statements are equivalent:*

(i) *The L^2-norm of the curvature tensor of sufficiently small geodesic celestial spheres only depends on the radius.*
(ii) *The L^2-norm of the Ricci tensor of sufficiently small geodesic celestial spheres only depends on the radius.*
(iii) *The L^2-norm of the scalar curvature of sufficiently small geodesic celestial spheres only depends on the radius.*
(iv) *(M, g) has constant sectional curvature.*

Proof. Once again, the result follows from the following expansions

$$
\begin{aligned}
\textstyle\int_{S_m^\xi(r)} \|\hat{R}\|^2 = c_{n-1} r^{n-1} \Bigg\{ & \tfrac{2(n-2)(n-1)}{r^4} - \tfrac{(n-5)(n-2)}{3n}\left(\tau + 2\rho_{\xi\xi}\right) \\
& + \tfrac{1}{n(n+2)} \Bigg(\tfrac{59n^2-93n-10}{60}\|R\|^2 + \tfrac{2(n^2-37n+60)}{45}\|\rho\|^2 + \tfrac{n^2-11n+2}{36}\tau^2 \\
& - \tfrac{(n-5)(n-2)}{10}\Delta\tau + \tfrac{2(29n^2-43n-10)}{15}\sum_{i,j,k=1}^{n} R^2_{\xi ijk} + \tfrac{(n+10)(3n-11)}{45}\rho^2_{\xi\xi} \\
& + \tfrac{4(n^2-37n+60)}{45}\sum_{i,j=1}^{n}\rho_{ij}R_{\xi i\xi j} + \tfrac{2(n^2-67n+110)}{45}\sum_{i=1}^{n}\rho^2_{\xi i} \\
& - \tfrac{35n^2-41n-30}{9}\sum_{i,j=1}^{n} R^2_{\xi i\xi j} + \tfrac{n^2-11n+2}{9}\tau\rho_{\xi\xi} - \tfrac{(n-5)(n-2)}{10}\nabla^2_{\xi\xi}\tau \\
& - \tfrac{(n-5)(n-2)}{5}\nabla^2_{\xi\xi}\rho_{\xi\xi} - \tfrac{(n-5)(n-2)}{5}\Delta\rho_{\xi\xi} \Bigg) + O\left(r^4\right) \Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
\textstyle\int_{S_m^\xi(r)} \|\hat{\rho}\|^2 = c_{n-1} r^{n-1} \Bigg\{ & \tfrac{(n-2)^2(n-1)}{r^4} - \tfrac{(n-5)(n-2)^2}{6n}\left(\tau + 2\rho_{\xi\xi}\right) \\
& + \tfrac{1}{n(n+2)} \Bigg(-\tfrac{n^3-9n^2+16n-20}{120}\|R\|^2 + \tfrac{n^3+31n^2-16n-120}{45}\|\rho\|^2 \\
& + \tfrac{n^3-13n^2-16n+44}{72}\tau^2 - \tfrac{(n-5)(n-2)^2}{20}\Delta\tau \\
& - \tfrac{n^3-9n^2+4n-20}{15}\sum_{i,j,k=1}^{n} R^2_{\xi ijk} + \tfrac{n^3+7n^2-16n-60}{18}\sum_{i,j=1}^{n} R^2_{\xi i\xi j} \\
& + \tfrac{2(n^3+31n^2-16n-120)}{45}\sum_{i,j=1}^{n}\rho_{ij}R_{\xi i\xi j} + \tfrac{n^3+71n^2-56n-220}{45}\sum_{i=1}^{n}\rho^2_{\xi i} \\
& + \tfrac{3n^3-127n^2-48n+460}{90}\rho^2_{\xi\xi} + \tfrac{n^3-13n^2-16n+44}{18}\tau\rho_{\xi\xi} \\
& - \tfrac{(n-5)(n-2)^2}{20}\nabla^2_{\xi\xi}\tau - \tfrac{(n-5)(n-2)^2}{10}\nabla^2_{\xi\xi}\rho_{\xi\xi} \\
& - \tfrac{(n-5)(n-2)^2}{10}\Delta\rho_{\xi\xi} \Bigg) + O\left(r^4\right) \Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
\int_{S^\xi_m(r)} \hat{\tau}^2 =c_{n-1}r^{n-1}\Big\{ & \tfrac{(n-2)^2(n-1)^2}{r^4} - \tfrac{(n-5)(n-2)^2(n-1)}{6n}\left(\tau + 2\rho_{\xi\xi}\right)\\
& + \tfrac{1}{n(n+2)}\Big(-\tfrac{(n-2)(n-1)(n^2+13n+10)}{120}\|R\|^2 + \tfrac{n^4-14n^3+29n^2-60n-188}{72}\tau^2\\
& + \tfrac{n^4+10n^3+43n^2-14n+120}{45}\|\rho\|^2 - \tfrac{(n-5)(n-2)^2(n-1)}{20}\Delta\tau\\
& - \tfrac{(n-2)(n-1)(n^2+3n+10)}{15}\sum_{i,j,k=1}^n R^2_{\xi ijk} + \tfrac{n^4+10n^3+n^2-8n+60}{18}\sum_{i,j=1}^n R^2_{\xi i\xi j}\\
& + \tfrac{2(n^4+10n^3+43n^2-14n+120)}{45}\sum_{i,j=1}^n \rho_{ij}R_{\xi i\xi j} - \tfrac{(n-5)(n-2)^2(n-1)}{10}\nabla^2_{\xi\xi}\rho_{\xi\xi}\\
& + \tfrac{3n^4-90n^3+59n^2-272n-1180}{90}\rho^2_{\xi\xi} + \tfrac{n^4+30n^3+53n^2+16n+220}{45}\sum_{i=1}^n \rho^2_{\xi i}\\
& - \tfrac{(n-5)(n-2)^2(n-1)}{20}\nabla^2_{\xi\xi}\tau + \tfrac{n^4-14n^3+29n^2-60n-188}{18}\tau\rho_{\xi\xi}\\
& - \tfrac{(n-5)(n-2)^2(n-1)}{10}\Delta\rho_{\xi\xi}\Big) + O\left(r^4\right)\Big\}
\end{aligned}
$$

□

The expansions in theorems 4.1 and 4.3 above are obtained after some tedious but straightforward calculations as in [6], so we omit them.

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