POLAR ACTIONS ON COMPLEX HYPERBOLIC SPACES

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Abstract. We classify polar actions on complex hyperbolic spaces up to orbit equivalence.

1. Introduction and main results

A proper isometric Lie group action on a Riemannian manifold is called *polar* if there exists an immersed submanifold that meets every orbit orthogonally. Such a submanifold is then called a *section* of the action. In the special case where the section is flat in its induced Riemannian metric, the action is called *hyperpolar*. In this article, we classify polar actions on complex hyperbolic spaces.

The motivation for our work can be traced back to the work of Dadok [12], who classified polar representations on Euclidean spaces, and the paper of Palais and Terng [30], who proved fundamental properties of polar actions on Riemannian manifolds. Several years later, the problem of classifying hyperpolar actions on symmetric spaces of compact type was posed in [18]. Hyperpolar actions on irreducible symmetric spaces of compact type have been classified by the third-named author in [20]. The classification of polar actions on compact symmetric spaces of rank one was obtained by Podestà and Thorbergsson [31]. This classification shows that there are finitely many examples of polar, non-hyperpolar actions on each compact symmetric space of rank one.

After having completed the classification of polar actions on irreducible Hermitian symmetric spaces of compact type, Biliotti [11] formulated the following conjecture: *a polar action on an irreducible symmetric space of compact type and higher rank is hyperpolar.*

The third author proved that the conjecture holds for symmetric spaces with simple isometry group [21], and for the exceptional simple Lie groups [22]. In recent work, Lytchak [26] obtained a decomposition theorem for the more general case of singular polar foliations on nonnegatively curved, not necessarily irreducible, Riemannian symmetric spaces. In particular, his result shows that polar actions on irreducible symmetric spaces of higher rank are hyperpolar if the cohomogeneity is at least three. Kollross and Lytchak [24] then completed the proof that Biliotti's conjecture holds and hence, that the classification of polar actions on irreducible symmetric spaces of compact type follows from [20] and [31]. Note that the classification of polar actions on reducible symmetric spaces cannot be obtained from the corresponding classification in irreducible ones. However, by the structural

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result [25, Theorem 5.5], it now only remains to classify nondecomposable hyperpolar actions on reducible spaces in order to obtain a complete classification of polar actions on symmetric spaces of compact type up to orbit equivalence.

But while polar actions on nonnegatively curved symmetric spaces are almost completely classified, the situation in the noncompact case remains largely open. Wu [36] classified polar actions on real hyperbolic spaces and showed that, up to orbit equivalence, they are products of a noncompact factor (which is either the isometry group of a lower dimensional real hyperbolic space or the nilpotent part of its Iwasawa decomposition) and a compact factor (which comes from the isotropy representation of a symmetric space). In particular, there are only finitely many examples of polar actions on a real hyperbolic space up to orbit equivalence. Berndt and the first-named author obtained in [5] the classification of polar actions on the complex hyperbolic plane $\mathbb{C}H^2$, showing that there are exactly nine examples up to orbit equivalence. No other complete classification of polar actions was previously known for the symmetric spaces of noncompact type. In this paper we present the classification of polar actions on complex hyperbolic spaces of arbitrary dimension. It is a remarkable consequence of Theorems A and B below that the cardinality of the set of polar actions on a complex hyperbolic space $\mathbb{C}H^n$, $n \geq 3$, is infinite, and hence the methods used in [5] cannot be applied in this more general situation.

An important fact to bear in mind here is that, in general, the duality of Riemannian symmetric spaces cannot be applied to derive classifications of polar actions on noncompact symmetric spaces from the corresponding classifications in the compact setting. Nevertheless, there are certain situations where duality can be used to obtain partial classifications. The first and the third authors derived in [16] the classification of polar actions with a fixed point on symmetric spaces using this method. They have shown that a polar action with a fixed point in a reducible symmetric space splits as a product of polar actions on each factor. The third author explored this idea further and obtained a classification of polar actions by algebraic reductive subgroups [23].

Berndt and Tamaru [8] classified cohomogeneity one actions on complex hyperbolic spaces, the quaternionic hyperbolic plane, and the Cayley hyperbolic plane. The classification remains open in quaternionic hyperbolic spaces of higher dimension, where the first and second authors have recently obtained new examples of such actions [15], and in noncompact symmetric spaces of higher rank. See [9] for more information on cohomogeneity one actions on symmetric spaces of noncompact type.

A polar action on a symmetric space of compact type always has singular orbits. Motivated by this fact, Berndt, Tamaru and the first author studied hyperpolar actions on symmetric spaces that have no singular orbits [7] and obtained a complete classification. It was also shown in this paper that there are polar actions on symmetric spaces of noncompact type and rank higher than one that are not hyperpolar, unlike in the compact setting. This classification has been improved in complex hyperbolic spaces, where Berndt and the first author classified polar homogeneous foliations [6]. The main result of this paper contains [5], [6] and [8] as particular cases.

Let $\mathbb{C}H^n = G/K$ be the complex hyperbolic n-space, where $G = SU(1,n)$ and $K = S(U(1)U(n))$ is the isotropy group of G at some point o. Consider the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ with respect to o. Choose a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$ and let $\mathfrak{g} = \mathfrak{g}_{-2\alpha} \oplus \mathfrak{g}_{-\alpha} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$ be the root space decomposition with respect to $\mathfrak{a}$. Set $\mathfrak{k}_0 = \mathfrak{k} \cap \mathfrak{g}_0 \cong \mathfrak{u}(n-1)$. Since $\mathfrak{k}_0$ acts on the root space $\mathfrak{g}_\alpha$, the center of $\mathfrak{k}_0$ induces a natural complex structure J on $\mathfrak{g}_\alpha$ which makes it isomorphic to $\mathbb{C}^{n-1}$. On the other hand, we call a subset of $\mathfrak{g}_\alpha$ a *real subspace* of $\mathfrak{g}_\alpha$ if it is a linear subspace of $\mathfrak{g}_\alpha$, where $\mathfrak{g}_\alpha$ is viewed as a real vector space. Assume $\mathfrak{g}_\alpha$ is endowed with the inner product given by the restriction of the Killing form of $\mathfrak{g}$. A real subspace $\mathfrak{w}$ of $\mathfrak{g}_\alpha$ is said to be *totally real* if $\mathfrak{w} \perp J(\mathfrak{w})$.

In this paper, we prove the following classification result:

Theorem A. *For each of the Lie algebras $\mathfrak{h}$ below, the corresponding connected subgroup of $U(1,n)$ acts polarly on $\mathbb{C}H^n$:*

(i) $\mathfrak{h} = \mathfrak{q} \oplus \mathfrak{so}(1,k) \subset \mathfrak{u}(n-k) \oplus \mathfrak{su}(1,k)$, $k \in \{0,\dots,n\}$, *where $\mathfrak{q}$ is a subalgebra of $\mathfrak{u}(n-k)$ such that the corresponding subgroup Q of $U(n-k)$ acts polarly with a totally real section on $\mathbb{C}^{n-k}$.*

(ii) $\mathfrak{h} = \mathfrak{q} \oplus \mathfrak{b} \oplus \mathfrak{w} \oplus \mathfrak{g}_{2\alpha} \subset \mathfrak{su}(1,n)$, *where $\mathfrak{b}$ is a linear subspace of $\mathfrak{a}$, $\mathfrak{w}$ is a real subspace of $\mathfrak{g}_\alpha$, and $\mathfrak{q}$ is a subalgebra of $\mathfrak{k}_0$ which normalizes $\mathfrak{w}$ and such that the connected subgroup of $SU(1,n)$ with Lie algebra $\mathfrak{q}$ acts polarly with a totally real section on the orthogonal complement of $\mathfrak{w}$ in $\mathfrak{g}_\alpha$.*

Conversely, every nontrivial polar action on $\mathbb{C}H^n$ is orbit equivalent to one of the actions above.

In case (i) of Theorem A, one orbit of the H-action is a totally geodesic $\mathbb{R}H^k$ and the other orbits are contained in the distance tubes around it. In case (ii), if $\mathfrak{b} = \mathfrak{a}$, one H-orbit of minimum orbit type contains a geodesic line, while if $\mathfrak{b} = 0$, any H-orbit of minimum orbit type is contained in a horosphere.

We would like to remark here that Theorem A actually provides many examples of polar actions on $\mathbb{C}H^n$. Indeed, for every choice of a real subspace $\mathfrak{w}$ in $\mathfrak{g}_\alpha$, there is at least one polar action as described in part (ii) of Theorem A, see Section 3.

With the notation as in Theorem A, we can determine the orbit equivalence classes of the polar actions given in the theorem above.

Theorem B. *Let H_1 and H_2 be two subgroups of $U(1,n)$ acting polarly on $\mathbb{C}H^n$ as given by Theorem A, and let $\mathfrak{h}_1$ and $\mathfrak{h}_2$ be their corresponding Lie algebras. Then the actions of H_1 and H_2 are orbit equivalent if and only if one of the following conditions holds:*

(a) $\mathfrak{h}_i = \mathfrak{q}_i \oplus \mathfrak{so}(1,k)$, $i \in \{1,2\}$, *and the actions of Q_1 and Q_2 on $\mathbb{C}^{n-k}$ are orbit equivalent.*

(b) $\mathfrak{h}_i = \mathfrak{q}_i \oplus \mathfrak{b}_i \oplus \mathfrak{w}_i \oplus \mathfrak{g}_{2\alpha}$, $i \in \{1,2\}$, $\mathfrak{b}_1 = \mathfrak{b}_2$, *there exists an element $k \in K_0$ such that $\mathfrak{w}_2 = \mathrm{Ad}(k)\mathfrak{w}_1$, and the actions of Q_i on the orthogonal complement of $\mathfrak{w}_i$ in $\mathfrak{g}_\alpha$ are orbit equivalent for $i \in \{1,2\}$.*

By means of the concept of Kähler angle, we can give an equivalent way of characterizing the congruence of subspaces of $\mathfrak{g}_\alpha$ by an element of K_0 stated in Theorem B(b). A subspace $\mathfrak{w}$ of $\mathfrak{g}_\alpha \cong \mathbb{C}^{n-1}$ is said to have *constant Kähler angle* $\varphi \in [0,\pi/2]$ if for each nonzero vector

$v \in \mathfrak{w}$ the angle between Jv and $\mathfrak{w}$ is precisely φ. In Subsection 2.3 we show that any real subspace $\mathfrak{w}$ of $\mathfrak{g}_\alpha$ admits a decomposition $\mathfrak{w} = \oplus_{\varphi\in\Phi}\mathfrak{w}_\varphi$ into subspaces of constant Kähler angle, where Φ is the set of the different Kähler angles arising in this decomposition, and $\mathfrak{w}_\varphi$ has constant Kähler angle φ. This decomposition is unique up to the ordering of the addends. Two subspaces $\mathfrak{w}_1 = \oplus_{\varphi\in\Phi_1}\mathfrak{w}_{1,\varphi}$ and $\mathfrak{w}_2 = \oplus_{\varphi\in\Phi_2}\mathfrak{w}_{2,\varphi}$ of $\mathfrak{g}_\alpha$ are then congruent by an element of $K_0 \cong U(n-1)$ if and only if $\Phi_1 = \Phi_2$ and $\dim \mathfrak{w}_{1,\varphi} = \dim \mathfrak{w}_{2,\varphi}$ for each φ.

It follows in particular from Theorems A and B that the moduli space of polar actions on $\mathbb{C}H^n$ up to orbit equivalence is finite if $n = 2$, cf. [5], and uncountable infinite in case $n \geq 3$. Indeed, in dimension $n \geq 3$ the action of the group $U(n-1)$ on the set of real subspaces of dimension k of $\mathbb{C}^{n-1}$, with $k \in \{2, \dots, 2n-4\}$, is not transitive. The orbits of this action are determined by the decomposition of a real subspace into a sum of spaces of constant Kähler angle, and the latter are parametrized by the set $[0, \pi/2]$, which is uncountable infinite. As a consequence, there are uncountably many polar, non-hyperpolar actions on $\mathbb{C}H^n$, $n \geq 3$, up to orbit equivalence.

This paper is organized as follows. In Section 2 we review the basic facts and notations on complex hyperbolic spaces (§2.1), polar actions (§2.2), and real vector subspaces of complex vector spaces (§2.3). The results of Subsection 2.3 will be crucial for the rest of the paper. Section 3 is devoted to present the new examples that appear in Theorem A. We also present here an outline of the proof of Theorem A. This proof has two main parts depending on whether the group acting leaves a totally geodesic subspace invariant (Section 4) or is contained in a maximal parabolic subgroup of $SU(1,n)$ (Section 5). We conclude in Section 6 with the proofs of Theorems A and B.

2. Preliminaries

In this section we introduce the main known results and notation used throughout this paper. We would like to emphasize the importance of Subsection 2.3, which is pivotal in the construction and classification of new examples of polar actions on complex hyperbolic spaces.

As a matter of notation, if U_1 and U_2 are two linear subspaces of a vector space V, then $U_1 \oplus U_2$ denotes their (not necessarily orthogonal) direct sum. We will frequently use the following notation for the orthogonal complement of a subspace of a real vector space endowed with a scalar product, namely, by $V \ominus U$ we denote the orthogonal complement of the linear subspace U in the Euclidean vector space V.

2.1. **The complex hyperbolic space.** In this subsection we recall some well-known facts and notation on the structure of the complex hyperbolic space as a symmetric space. This will be fundamental for the rest of the work. As usual, Lie algebras are written in gothic letters.

We will denote by $\mathbb{C}H^n$ the complex hyperbolic space with constant holomorphic sectional curvature -1. As a symmetric space, $\mathbb{C}H^n$ is the coset space G/K, where $G = SU(1,n)$, and $K = S(U(1)U(n))$ is the isotropy group at some point $o \in \mathbb{C}H^n$. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$ with respect to o, where $\mathfrak{p}$ is the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to the Killing form B of $\mathfrak{g}$. Denote by θ the corresponding Cartan involution, which satisfies $\theta|_{\mathfrak{k}} = \mathrm{id}$ and $\theta|_{\mathfrak{p}} = -\mathrm{id}$. Note that the orthogonal projections onto $\mathfrak{k}$ and $\mathfrak{p}$ are $\frac{1}{2}(1+\theta)$ and $\frac{1}{2}(1-\theta)$, respectively. Let ad and Ad be the adjoint maps of $\mathfrak{g}$ and G, respectively. It turns out that $\langle X, Y\rangle = -B(\theta X, Y)$ defines a positive definite inner product on $\mathfrak{g}$ satisfying the relation $\langle \mathrm{ad}(X)Y, Z\rangle = -\langle Y, \mathrm{ad}(\theta X)Y\rangle$ for all X, Y, $Z \in \mathfrak{g}$. Moreover, we can identify $\mathfrak{p}$ with the tangent space $T_o\mathbb{C}H^n$ of $\mathbb{C}H^n$ at the point o.

Since $\mathbb{C}H^n$ has rank one, any maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$ is 1-dimensional. For each linear functional λ on $\mathfrak{a}$, define $\mathfrak{g}_\lambda = \{X \in \mathfrak{g} : \mathrm{ad}(H)X = \lambda(H)X \text{ for all } H \in \mathfrak{a}\}$. Then $\mathfrak{a}$ induces the restricted root space decomposition $\mathfrak{g} = \mathfrak{g}_{-2\alpha} \oplus \mathfrak{g}_{-\alpha} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$, which is an orthogonal direct sum with respect to $\langle\cdot,\cdot\rangle$ satisfying $[\mathfrak{g}_\lambda, \mathfrak{g}_\mu] = \mathfrak{g}_{\lambda+\mu}$ and $\theta\mathfrak{g}_\lambda = \mathfrak{g}_{-\lambda}$. Moreover, $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{a}$, where $\mathfrak{k}_0 = \mathfrak{g}_0 \cap \mathfrak{k} \cong \mathfrak{u}(n-1)$ is the normalizer of $\mathfrak{a}$ in $\mathfrak{k}$. The root space $\mathfrak{g}_\alpha$ has dimension $2n-2$, while $\mathfrak{g}_{2\alpha}$ is 1-dimensional, and both are normalized by $\mathfrak{k}_0$.

We define $\mathfrak{n} = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$, which is a nilpotent subalgebra of $\mathfrak{g}$ isomorphic to the $(2n-1)$-dimensional Heisenberg algebra. The corresponding Iwasawa decomposition of $\mathfrak{g}$ is $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$. The connected subgroup of G with Lie algebra $\mathfrak{a} \oplus \mathfrak{n}$ acts simply transitively on $\mathbb{C}H^n$. One may endow AN, and then $\mathfrak{a} \oplus \mathfrak{n}$, with the left-invariant metric $\langle\cdot,\cdot\rangle_{AN}$ and the complex structure J that make $\mathbb{C}H^n$ and AN isometric and isomorphic as Kähler manifolds. Then $\langle X, Y\rangle_{AN} = \langle X_{\mathfrak{a}}, Y_{\mathfrak{a}}\rangle + \frac{1}{2}\langle X_{\mathfrak{n}}, Y_{\mathfrak{n}}\rangle$ for X, $Y \in \mathfrak{a} \oplus \mathfrak{n}$; here subscripts mean the $\mathfrak{a}$ and $\mathfrak{n}$ components respectively. The complex structure J on $\mathfrak{a}\oplus\mathfrak{n}$ leaves $\mathfrak{g}_\alpha$ invariant, turning $\mathfrak{g}_\alpha$ into an $(n-1)$-dimensional complex vector space $\mathbb{C}^{n-1}$. Moreover, $J\mathfrak{a} = \mathfrak{g}_{2\alpha}$.

Let $B \in \mathfrak{a}$ be a unit vector and define $Z = JB \in \mathfrak{g}_{2\alpha}$. Then $\langle B, B\rangle = \langle B, B\rangle_{AN} = 1$ and $\langle Z, Z\rangle = 2\langle Z, Z\rangle_{AN} = 2$. The Lie bracket of $\mathfrak{a} \oplus \mathfrak{n}$ is given by

$$[aB + U + xZ, bB + V + yZ] = -\frac{b}{2}U + \frac{a}{2}V + \left(-bx + ay + \frac{1}{2}\langle JU, V\rangle\right) Z,$$

where a, b, x, $y \in \mathbb{R}$, and U, $V \in \mathfrak{g}_\alpha$. Let us also define $\mathfrak{p}_\lambda = (1-\theta)\mathfrak{g}_\lambda$, the projection onto $\mathfrak{p}$ of the restricted root spaces. Then $\mathfrak{p} = \mathfrak{a} \oplus \mathfrak{p}_\alpha \oplus \mathfrak{p}_{2\alpha}$. If the complex structure on $\mathfrak{p}$ is denoted by i, then we have that $2iB = (1-\theta)Z$, and $i(1-\theta)U = (1-\theta)JU$ for every $U \in \mathfrak{g}_\alpha$.

We state now two lemmas that will be used frequently throughout the article.

Lemma 2.1. *We have:*

(a) $[\theta X, Z] = -JX$ *for each* $X \in \mathfrak{g}_\alpha$.

(b) $\langle T, (1+\theta)[\theta X, Y]\rangle = 2\langle [T, X], Y\rangle$, *for any* X, $Y \in \mathfrak{g}_\alpha$ *and* $T \in \mathfrak{k}_0$.

Proof. See [6, Lemma 2.1]. □

Lemma 2.2. *The orthogonal projection map* $\frac{1}{2}(1-\theta)\colon \mathfrak{a} \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha} \to \mathfrak{a} \oplus \mathfrak{p}_\alpha \oplus \mathfrak{p}_{2\alpha}$ *defines an equivalence between the adjoint* K_0*-representation on* $\mathfrak{a} \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$ *and the adjoint* K_0*-representation on* $\mathfrak{p} = \mathfrak{a} \oplus \mathfrak{p}_\alpha \oplus \mathfrak{p}_{2\alpha}$. *Moreover, this equivalence is an isometry between* $(\mathfrak{a} \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}, \langle\cdot,\cdot\rangle_{AN})$ *and* $(\mathfrak{p}, \langle\cdot,\cdot\rangle)$, *and* $\frac{1}{2}(1-\theta)\colon \mathfrak{g}_\alpha \to \mathfrak{p}_\alpha$ *is a complex linear map.*

Proof. The first part follows from the fact that θ is a K-equivariant, hence K_0-equivariant, map on $\mathfrak{g}$. The other claims follow from the facts stated above in this subsection. □

2.2. **Polar actions.** Let M be a Riemannian manifold and $I(M)$ its isometry group. It is known that $I(M)$ is a Lie group. Let H be a connected closed subgroup of $I(M)$. The action of H on M is called *polar* if there exists an immersed submanifold Σ of M such that:

(1) Σ intersects all the orbits of the H-action, and
(2) for each $p \in \Sigma$, the tangent space of Σ at p, $T_p\Sigma$, and the tangent space of the orbit through p at p, $T_p(H \cdot p)$, are orthogonal.

In such a case, the submanifold Σ is called a *section* of the H-action. The action of H is called *hyperpolar* if the section Σ is flat in its induced Riemannian metric.

Two isometric Lie group actions on two Riemannian manifolds M and N are said to be *orbit equivalent* if there is an isometry $M \to N$ which maps connected components of orbits onto connected components of orbits. They are said to be *conjugate* if there exists an equivariant isometry $M \to N$.

The final aim of our research is to classify polar actions on a given Riemannian manifold up to orbit equivalence. In this paper we accomplish this task for complex hyperbolic spaces. See the survey articles [33], [34] and [13] for more information and references on polar actions.

Since $\mathbb{C}H^n$ is of rank one, a polar action on $\mathbb{C}H^n$ is hyperpolar if and only if it is of cohomogeneity one, i.e. the orbits of maximal dimension are hypersurfaces. Conversely, any action of cohomogeneity one on $\mathbb{C}H^n$ (or any other Riemannian symmetric space) is hyperpolar. Cohomogeneity one actions on complex hyperbolic spaces have been classified by Berndt and Tamaru in [8].

From now on we focus on polar actions on complex hyperbolic spaces and recall or prove some facts that will be used later in this article. We begin with a criterion that allows us to decide whether an action is polar or not. The first such criterion of polarity is credited to Gorodski [17].

Proposition 2.3. *Let $M = G/K$ be a Riemannian symmetric space of noncompact type, and let Σ be a connected totally geodesic submanifold of M with $o \in \Sigma$. Let H be a closed subgroup of $I(M)$. Then H acts polarly on M with section Σ if and only if $T_o\Sigma$ is a section of the slice representation of H_o on $\nu_o(H \cdot o)$, and $\langle \mathfrak{h}, T_o\Sigma \oplus [T_o\Sigma, T_o\Sigma] \rangle = 0$.*

In this case, the following conditions are satisfied:

(a) $T_o\Sigma \oplus [\mathfrak{h}_o, \xi] = \nu_o(H \cdot o)$ *for each regular normal vector* $\xi \in \nu_o(H \cdot o)$.
(b) $T_o\Sigma \oplus [\mathfrak{h}_o, T_o\Sigma] = \nu_o(H \cdot o)$.
(c) $\mathrm{Ad}(H_o)T_o\Sigma = \nu_o(H \cdot o)$.

Proof. Follows from [5, Corollary 3.2] and from well-known facts on polar representations of compact groups [12]. □

If N is a submanifold of $\mathbb{C}H^n$, then N is said to be *totally real* if for each $p \in N$ the tangent space T_pN is a totally real subspace of $T_p\mathbb{C}H^n$, that is, JT_pN is orthogonal to T_pN. See §2.3 for more information of totally real subspaces of complex vector spaces. The next theorem shows that sections are necessarily totally real.

Proposition 2.4. *Let H act nontrivially, nontransitively, and polarly on the complex hyperbolic space $\mathbb{C}H^n$, and let Σ be a section of this action. Then, Σ is a totally real submanifold of $\mathbb{C}H^n$.*

Proof. Since the action of H is polar, the section Σ is a totally geodesic submanifold of $\mathbb{C}H^n$, hence Σ is either totally real or complex. Assume that Σ is complex.

Since all sections are of the form $h(\Sigma)$, with $h \in H$, and the isometries of H are holomorphic, it follows that any principal orbit is almost complex. It is a well-known fact that an almost complex submanifold in a Kähler manifold is Kähler. Since every H-equivariant normal vector field on a principal orbit is parallel with respect to the normal connection [3, Corollary 3.2.5], then this principal orbit is either a point or $\mathbb{C}H^n$ (see for example [1]), contradiction. Therefore Σ is totally real. □

2.3. **The structure of a real subspace of a complex vector space.** Let us denote by J the complex structure of the complex vector space $\mathbb{C}^n$. We view $\mathbb{C}^n$ as a Euclidean vector space with the scalar product given by the real part of the standard Hermitian scalar product. We define a *real subspace* of $\mathbb{C}^n$ to be an $\mathbb{R}$-linear subspace of the real vector space obtained from $\mathbb{C}^n$ by restricting the scalars to the real numbers. Let V be a real subspace of $\mathbb{C}^n$. We will denote by π_V the orthogonal projection map onto V.

The *Kähler angle* of a nonzero vector $v \in V$ with respect to V is defined to be the angle between Jv and V or, equivalently, the value $\varphi \in [0, \pi/2]$ such that $\langle \pi_V Jv, \pi_V Jv\rangle = \cos^2(\varphi)\langle v, v\rangle$. We say that V has *constant Kähler angle* φ if the Kähler angle of every nonzero vector $v \in V$ with respect to V is φ. In particular, V is a *complex subspace* if and only if it has constant Kähler angle 0; it is a *totally real subspace* if and only if it has constant Kähler angle $\pi/2$.

Remark 2.5. If $\{e_1, \dots, e_n\}$ and $\{f_1, \dots, f_n\}$ both are $\mathbb{C}$-orthonormal bases of $\mathbb{C}^n$, then the real subspace V_φ of $\mathbb{C}^{2n} = \mathbb{C}^n \oplus \mathbb{C}^n$ generated by

$$\{\cos(\tfrac{\varphi}{2})e_1+\sin(\tfrac{\varphi}{2})Jf_1, \cos(\tfrac{\varphi}{2})Je_1+\sin(\tfrac{\varphi}{2})f_1, \dots, \cos(\tfrac{\varphi}{2})e_n+\sin(\tfrac{\varphi}{2})Jf_n, \cos(\tfrac{\varphi}{2})Je_n+\sin(\tfrac{\varphi}{2})f_n\}$$

has constant Kähler angle $\varphi \in [0, \pi/2)$. Conversely, any subspace of constant Kähler angle $\varphi \in [0, \pi/2)$ and dimension $2n$ of $\mathbb{C}^{2n}$ can be constructed in this way, see [2]. In particular, it follows that two real subspaces of $\mathbb{C}^n$ with the same dimension and the same constant Kähler angle are congruent by an element of $U(n)$.

For general real subspaces of a complex vector space, we have the following structure result.

Theorem 2.6. *Let V be any real subspace of $\mathbb{C}^n$. Then V can be decomposed in a unique way as an orthogonal sum of subspaces V_i, $i = 1, \dots, r$, such that:*

(a) *Each real subspace V_i of $\mathbb{C}^n$ has constant Kähler angle φ_i.*
(b) *$\mathbb{C}V_i \perp \mathbb{C}V_j$, for every $i \neq j$, $i, j \in \{1, \dots, r\}$.*
(c) *$\varphi_1 < \varphi_2 < \cdots < \varphi_r$.*

Proof. The endomorphism $P = \pi_V \circ J$ of V is clearly skew-symmetric, i.e. $\langle Pv, w\rangle = -\langle v, Pw\rangle$ for every v, $w \in V$. Then, there exists an orthonormal basis of V for which P

takes a block diagonal form with 2×2 skew-symmetric matrix blocks, and maybe one zero matrix block. Since P is skew-symmetric, its nonzero eigenvalues are imaginary. Assume then that the distinct eigenvalues of P are $\pm i\lambda_1, \ldots, \pm i\lambda_r$ (maybe one of them is zero). We can and will further assume that $|\lambda_1| > \cdots > |\lambda_r|$.

Now consider the quadratic form $\Psi \colon V \to \mathbb{R}$ defined by $\Psi(v) = \langle Pv, Pv \rangle = -\langle P^2 v, v \rangle$ for $v \in V$. The matrix of this quadratic form Ψ (or of the endomorphism $-P^2$) with respect to the basis fixed above is diagonal with entries $\lambda_1^2, \ldots, \lambda_r^2$. For each $i = 1, \ldots, r$, let V_i be the eigenspace of $-P^2$ corresponding to the eigenvalue λ_i^2. Let $v \in V_i$ be a unit vector. Then

$$\langle \pi_{V_i} Jv, \pi_{V_i} Jv \rangle = \langle Pv, \pi_{V_i} Jv \rangle = \langle Pv, Pv \rangle = \Psi(v) = \lambda_i^2,$$

where in the second and last equalities we have used that $Pv \in V_i$. This means that each subspace V_i has constant Kähler angle φ_i, where φ_i is the unique value in $[0, \frac{\pi}{2}]$ such that $\lambda_i^2 = \cos^2(\varphi_i)$.

By construction, it is clear that $V_i \perp V_j$ and $JV_i \perp JV_j$ for $i \neq j$. Since for every $v \in V_i$ and $w \in V_j$, $i \neq j$, we have that $\langle Jv, w \rangle = \langle Pv, w \rangle = 0$, we also get that $JV_i \perp V_j$ if $i \neq j$. Hence $\mathbb{C}V_i \perp \mathbb{C}V_j$ if $i \neq j$.

Property (c) follows from the assumption that $|\lambda_1| > \cdots > |\lambda_r|$, and this also implies the uniqueness of the decomposition. □

It is convenient to change the notation of Theorem 2.6 slightly. Let V be any real subspace of $\mathbb{C}^n$, and let $V = \bigoplus_{\varphi \in \Phi} V_\varphi$ be the decomposition stated in Theorem 2.6, where V_φ has constant Kähler angle $\varphi \in [0, \pi/2]$, and Φ is the set of all Kähler angles arising in this decomposition. Note that according to Theorem 2.6, this decomposition is unique up to the order of the factors. We agree to write $V_\varphi = 0$ if $\varphi \notin \Phi$. The subspaces V_0 and $V_{\pi/2}$ (which can be zero) play a somewhat distinguished role in the calculations that follow, so we will denote $\Phi^* = \{\varphi \in \Phi : \varphi \neq 0, \pi/2\}$. Then, the above decomposition is written as

$$V = V_0 \oplus \left(\bigoplus_{\varphi \in \Phi^*} V_\varphi \right) \oplus V_{\pi/2}.$$

For each $\varphi \in \Phi^* \cup \{0\}$, we define $J_\varphi \colon V_\varphi \to V_\varphi$ by $J_\varphi = \frac{1}{\cos(\varphi)}(\pi_{V_\varphi} \circ J)$. This is clearly a skew-symmetric and orthogonal endomorphism of V_φ (see the proof of Theorem 2.6). Therefore (V_φ, J_φ) is a complex vector space for every $\varphi \in \Phi^* \cup \{0\}$. Note that $J_0 = J|_{V_0}$. Let $U(V_\varphi)$ be the group of all unitary transformations of the complex vector space (V_φ, J_φ).

Lemma 2.7. *Let V be a real subspace of constant Kähler angle $\varphi \neq 0$ in $\mathbb{C}^n$. Then the real subspace $\mathbb{C}V \ominus V$ of $\mathbb{C}^n$ has the same dimension as V and constant Kähler angle φ.*

Proof. See for example [4, page 135]. □

Let $V^\perp = \mathbb{C}^n \ominus V$, where as usual $\ominus$ denotes the orthogonal complement. Then, Lemma 2.7 implies that the decomposition stated in Theorem 2.6 can be written as

$$V^\perp = V_0^\perp \oplus \left(\bigoplus_{\varphi \in \Phi^*} V_\varphi^\perp \right) \oplus V_{\pi/2}^\perp, \quad \text{where } \mathbb{C}V_\varphi = V_\varphi \oplus V_\varphi^\perp \text{ for each } \varphi \in \Phi^* \cup \{\pi/2\}.$$

We define $m_\varphi = \dim V_\varphi$ and $m_\varphi^\perp = \dim V_\varphi^\perp$. For every $\varphi \neq 0$ we have $m_\varphi = m_\varphi^\perp$ by Lemma 2.7, but V_0 and $V_0^\perp$ are both complex subspaces of $\mathbb{C}^n$, possibly of different dimension.

Lemma 2.8. *Let V be a real subspace of $\mathbb{C}^n$. Let $U(n)_V$ be the subgroup of $U(n)$ consisting of all the elements $A \in U(n)$ such that $AV = V$. Then, we have the canonical isomorphism*

$$U(n)_V \cong \left[\prod_{\varphi \in \Phi^* \cup \{0\}} U(V_\varphi) \right] \times O(V_{\pi/2}) \times U(V_0^\perp).$$

where we assume that V_φ, $\varphi \in \Phi^* \cup \{0\}$, is endowed with the complex structure given by $J_\varphi = \frac{1}{\cos(\varphi)}(\pi_{V_\varphi} \circ J)$, and that $V_0^\perp$ is endowed with the complex structure given by the restriction of J.

Proof. Let $A \in U(n)$ be such that $AV = V$. Then A commutes with J and π_V and hence leaves the eigenspaces of $-P^2$ invariant (see the proof of Theorem 2.6). Thus $AV_\varphi = V_\varphi$. Since we also have $AV^\perp = V^\perp$, it follows that $AV_\varphi^\perp = V_\varphi^\perp$.

Let $\varphi \in \Phi \cup \{0\}$. Since $AV_\varphi = V_\varphi$ and $AV_\varphi^\perp = V_\varphi^\perp$ we have $A\mathbb{C}V_\varphi = \mathbb{C}V_\varphi$. Clearly, $A \circ \pi_{V_\varphi}|_{V_\varphi} = \pi_{V_\varphi} \circ A|_{V_\varphi}$, and $A \circ \pi_{V_\varphi}|_{\mathbb{C}^n \ominus V_\varphi} = 0 = \pi_{V_\varphi} \circ A|_{\mathbb{C}^n \ominus V_\varphi}$. Hence, $A \circ \pi_{V_\varphi} = \pi_{V_\varphi} \circ A$. Since $AJ = JA$ as well, we have that $A \circ J_\varphi|_{V_\varphi} = J_\varphi \circ A|_{V_\varphi}$ on V_φ, and thus, $A|_{V_\varphi} \in U(V_\varphi)$. If $\varphi = \pi/2$ then we have $AV_{\pi/2} = V_{\pi/2}$, and clearly, $A|_{V_{\pi/2}}$ is an orthogonal transformation of $V_{\pi/2}$. Moreover, we have $A|_{V_0^\perp} \in U(V_0^\perp)$. We define a map

$$F \colon U(n)_V \to \left[\prod_{\varphi \in \Phi^* \cup \{0\}} U(V_\varphi) \right] \times O(V_{\pi/2}) \times U(V_0^\perp)$$

by requiring that the projection onto each factor is given by the corresponding restriction, that is, the $U(V_\varphi)$-projection of $F(A)$ is given by $A|_{V_\varphi}$, the $O(V_{\pi/2})$-projection of $F(A)$ is $A|_{V_{\pi/2}}$, and the $U(V_0^\perp)$-projection of $F(A)$ is $A|_{V_0^\perp}$.

Since every element in $U(n)_V$ leaves the subspaces V_φ, $\varphi \in \Phi$, and $V_0^\perp$ invariant, the map thus defined is a homomorphism. Let us show injectivity and surjectivity. Let $A_\varphi \in U(V_\varphi)$ for each $\varphi \in \Phi^* \cup \{0\}$, let $A_{\pi/2} \in O(V_{\pi/2})$, and let $A_0^\perp \in U(V_0^\perp)$. If $A \in U(n)_V$ and $v \in JV_\varphi$ for $\varphi \in \Phi$, then Av is determined by A_φ and v, since $Av = -AJ^2v = -JAJv = -JA_\varphi(Jv)$. Since we have the direct sum decomposition

$$\mathbb{C}^n = \left[\bigoplus_{\varphi \in \Phi} \mathbb{C}V_\varphi \right] \oplus V_0^\perp,$$

it follows that the unitary map A on $\mathbb{C}^n$ is uniquely determined by the maps A_φ, $\varphi \in \Phi$, and $A_0^\perp$. This shows injectivity.

Conversely, let $A \in \left[\prod_{\varphi \in \Phi^* \cup \{0\}} U(V_\varphi) \right] \times O(V_{\pi/2}) \times U(V_0^\perp)$, and denote by A_φ the $U(V_\varphi)$-projection, by $A_{\pi/2}$ the $O(V_{\pi/2})$-projection, and by $A_0^\perp$ the $U(V_0^\perp)$-projection. Then, we may construct a map $A \in U(n)_V$ be defining $A(v + Jw) = A_\varphi v + JA_\varphi w$ for all v, $w \in V_\varphi$,

$\varphi \in \Phi$, $Av = A_0^\perp v$ for $v \in V_0^\perp$, and extending linearly. For the map A thus defined we have $A|_{V_\varphi} = A_\varphi$ for $\varphi \in \Phi$, and $A|_{V_0^\perp} = A_0^\perp$. This proves surjectivity. □

Remark 2.9. Let V and W be two real subspaces of $\mathbb{C}^n$ whose Kähler angle decompositions have the same set of Kähler angles Φ and the same dimensions, that is, the decompositions given by Theorem 2.6 are $V = \oplus_{\varphi\in\Phi} V_\varphi$ and $W = \oplus_{\varphi\in\Phi} W_\varphi$, with $\dim V_\varphi = \dim W_\varphi$ for all $\varphi \in \Phi$. Then, it follows from the results of this subsection that there exists a unitary automorphism A of $\mathbb{C}^n$ such that $AV_\varphi = W_\varphi$ for all $\varphi \in \Phi$, and in particular, $AV = W$.

3. New examples of polar actions

We will now construct new examples of polar actions on complex hyperbolic spaces. We will use the notation from Subsection 2.1.

Recall that the root space $\mathfrak{g}_\alpha$ is a complex vector space, which we will identify with $\mathbb{C}^{n-1}$. Let $\mathfrak{w}$ be a real subspace of $\mathfrak{g}_\alpha$ and

$$\mathfrak{w} = \bigoplus_{\varphi\in\Phi} \mathfrak{w}_\varphi = \mathfrak{w}_0 \oplus \left(\bigoplus_{\varphi\in\Phi^*} \mathfrak{w}_\varphi \right) \oplus \mathfrak{w}_{\pi/2}$$

its decomposition as in Theorem 2.6, where Φ is the set of all possible Kähler angles of vectors in $\mathfrak{w}$, $\Phi^* = \{\varphi \in \Phi : \varphi \neq 0, \pi/2\}$, and $\mathfrak{w}_\varphi$ has constant Kähler angle $\varphi \in [0, \pi/2]$. Similarly, define $\mathfrak{w}^\perp = \mathfrak{g}_\alpha \ominus \mathfrak{w}$ and let

$$\mathfrak{w}^\perp = \mathfrak{w}_0^\perp \oplus \left(\bigoplus_{\varphi\in\Phi^*} \mathfrak{w}_\varphi^\perp \right) \oplus \mathfrak{w}_{\pi/2}^\perp$$

be the corresponding decomposition as in Theorem 2.6. We define $m_\varphi = \dim \mathfrak{w}_\varphi$ and $m_\varphi^\perp = \dim \mathfrak{w}_\varphi^\perp$, and recall that $m_\varphi = m_\varphi^\perp$ if $\varphi \in (0, \pi/2]$. Recall also that K_0, the connected subgroup of $G = SU(1,n)$ with Lie algebra $\mathfrak{k}_0$, is isomorphic to $U(n-1)$ and acts on $\mathfrak{g}_\alpha \cong \mathbb{C}^{n-1}$ in the standard way. We denote by $N_{K_0}(\mathfrak{w})$ the normalizer of $\mathfrak{w}$ in K_0, and by $\mathfrak{n}_{\mathfrak{k}_0}(\mathfrak{w})$ the normalizer of $\mathfrak{w}$ in $\mathfrak{k}_0$. We know from Lemma 2.8 that

$$N_{K_0}(\mathfrak{w}) \cong \left[\prod_{\varphi\in\Phi^*\cup\{0\}} U(\mathfrak{w}_\varphi) \right] \times O(\mathfrak{w}_{\pi/2}) \times U(\mathfrak{w}_0^\perp). \tag{1}$$

This group leaves invariant each $\mathfrak{w}_\varphi$ and each $\mathfrak{w}_\varphi^\perp$, and acts transitively on the unit sphere of these subspaces of constant Kähler angle. Moreover, it acts polarly on $\mathfrak{w}^\perp$, see Remark 3.2 below.

The following result provides a large family of new examples of polar actions on $\mathbb{C}H^n$.

Theorem 3.1. *Let $\mathfrak{w}$ be a real subspace of $\mathfrak{g}_\alpha$ and $\mathfrak{b}$ a subspace of $\mathfrak{a}$. Let $\mathfrak{h} = \mathfrak{q}\oplus\mathfrak{b}\oplus\mathfrak{w}\oplus\mathfrak{g}_{2\alpha}$, where $\mathfrak{q}$ is any Lie subalgebra of $\mathfrak{n}_{\mathfrak{k}_0}(\mathfrak{w})$ such that the corresponding connected subgroup Q of K acts polarly on $\mathfrak{w}^\perp$ with section $\mathfrak{s}$. Assume $\mathfrak{s}$ is a totally real subspace of $\mathfrak{g}_\alpha$. Then the connected subgroup H of G with Lie algebra $\mathfrak{h}$ acts polarly on $\mathbb{C}H^n$ with section $\Sigma = \exp_o((\mathfrak{a} \ominus \mathfrak{b}) \oplus (1-\theta)\mathfrak{s})$.*

Proof. We have that $T_o\Sigma = (\mathfrak{a} \ominus \mathfrak{b}) \oplus (1-\theta)\mathfrak{s}$ and $\nu_o(H \cdot o) = (\mathfrak{a} \ominus \mathfrak{b}) \oplus (1-\theta)\mathfrak{w}^\perp$. Since $\mathfrak{s} \subset \mathfrak{w}^\perp$, it follows that $T_o\Sigma \subset \nu_o(H \cdot o)$. The slice representation of H_o on $\nu_o(H \cdot o)$ leaves the subspaces $\mathfrak{a} \ominus \mathfrak{b}$ and $(1-\theta)\mathfrak{w}^\perp$ invariant. For the first one the action is trivial, while for the second one the action is equivalent to the representation of Q on $\mathfrak{w}^\perp$ (see Lemma 2.2), which is polar with section $\mathfrak{s}$. Hence, the slice representation of H_o on $\nu_o(H \cdot o)$ is polar and $T_o\Sigma$ is a section of it. Let v, $w \in \mathfrak{s} \subset \mathfrak{w}^\perp$. We have:

$$[(1-\theta)v, (1-\theta)w] = (1+\theta)[v,w] - (1+\theta)[\theta v, w] = -(1+\theta)[\theta v, w].$$

The last equality holds because v and w lie in $\mathfrak{s}$, which is a totally real subspace of $\mathfrak{g}_\alpha$, and then $[v,w] = \frac{1}{2}\langle Jv, w\rangle Z = 0$. Since v, $w \in \mathfrak{g}_\alpha$, then $\theta v \in \mathfrak{g}_{-\alpha}$ and $[\theta v, w] \in \mathfrak{g}_0$. Hence $-(1+\theta)[\theta v, w] \in \mathfrak{k}_0$. Let $X = T + aB + U + xZ \in \mathfrak{h}$, where $T \in \mathfrak{q}$, $U \in \mathfrak{w}$ and a, $x \in \mathbb{R}$. Since $\mathfrak{k}_0$ is orthogonal to $\mathfrak{a} \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$, we have:

$$\langle [(1-\theta)v, (1-\theta)w], X\rangle = -\langle (1+\theta)[\theta v, w], T\rangle = -2\langle [T,v], w\rangle = -4\langle [T,v], w\rangle_{AN} = 0,$$

where in the last equality we have used that the action of Q on $\mathfrak{w}^\perp$ is a polar representation with section $\mathfrak{s}$. If $\mathfrak{b} = \mathfrak{a}$, the result then follows using the criterion in Proposition 2.3.

If $\mathfrak{b} \neq \mathfrak{a}$ then $\mathfrak{b} = 0$. In this case, let $v \in \mathfrak{s}$ and $X = T + U + xZ \in \mathfrak{h}$, where $T \in \mathfrak{q}$, $U \in \mathfrak{w}$, $x \in \mathbb{R}$. Then:

$$\langle [B, (1-\theta)v], X\rangle = \langle (1+\theta)[B,v], X\rangle = \frac{1}{2}\langle (1+\theta)v, U\rangle = 0.$$

Since $[B,B] = 0$, by linearity and the skew-symmetry of the Lie bracket, it follows that $\langle [T_o\Sigma, T_o\Sigma], \mathfrak{h}\rangle = 0$. Again by Proposition 2.3, the result follows also in case $\mathfrak{b} \neq \mathfrak{a}$. ☐

Remark 3.2. In the special case $Q = N_{K_0}(\mathfrak{w})$, we obtain a polar action on $\mathbb{C}H^n$, since the whole normalizer $N_{K_0}(\mathfrak{w})$ acts polarly on $\mathfrak{w}^\perp$. Indeed, let $\mathfrak{s}_\varphi$ be any one-dimensional subspace of $\mathfrak{w}^\perp_\varphi$ if $\mathfrak{w}^\perp_\varphi \neq 0$, and define $\mathfrak{s} = \bigoplus_{\varphi \in \Phi \cup \{0\}} \mathfrak{s}_\varphi$. Then $\mathfrak{s}$ is a section of the action of $N_{K_0}(\mathfrak{w})$ on $\mathfrak{w}^\perp$. The cohomogeneity one examples introduced in [2] correspond to the case where $\mathfrak{w}^\perp$ has constant Kähler angle, $\mathfrak{b} = \mathfrak{a}$ and $Q = N_{K_0}(\mathfrak{w})$.

Remark 3.3. It is straightforward to describe all polar actions of closed subgroups Q in Theorem 3.1 up to orbit equivalence. In fact, the action of the group $N_{K_0}(\mathfrak{w})$ is given by the products of the natural representations of the direct factors in (1) on the spaces $\mathfrak{w}^\perp_\varphi$. By the main result of Dadok [12], a representation is polar if and only if it is orbit equivalent to the isotropy representation of some Riemannian symmetric space. Therefore, we obtain a representative for each orbit equivalence class of polar actions on $\mathfrak{w}^\perp$ given by closed subgroups of $N_{K_0}(\mathfrak{w})$ in the following manner. Given $\mathfrak{w}$, for each $\varphi \in \Phi \cup \{0\}$ choose a Riemannian symmetric space M_φ such that $\dim M_\varphi = \dim \mathfrak{w}^\perp_\varphi$. In case $\pi/2 \in \Phi$, choose the symmetric spaces such that all of them except possibly $M_{\pi/2}$ are Hermitian symmetric; in case $\pi/2 \notin \Phi$, choose all these symmetric spaces to be Hermitian without exception. Then the isotropy representation of $\prod_{\varphi \in \Phi \cup \{0\}} M_\varphi$ defines a closed subgroup of $N_{K_0}(\mathfrak{w})$, which acts polarly on $\mathfrak{w}^\perp$ with a section $\mathfrak{s}$, which is a totally real subspace of $\mathfrak{g}_\alpha$, see [31]. This construction exhausts all orbit equivalent classes of closed subgroups in K_0 leaving $\mathfrak{w}$ invariant and acting polarly on $\mathfrak{w}^\perp$ with totally real section.

Remark 3.4. There is a curious relation between some of the new examples of polar actions in Theorem 3.1 and certain isoparametric hypersurfaces constructed by the first two authors in [14]. The orbit $H \cdot o$ of any of the polar actions described in Theorem 3.1 with $\mathfrak{b} = \mathfrak{a}$ is always a minimal (even austere) submanifold of $\mathbb{C}H^n$ that satisfies the following property: the distance tubes around it are isoparametric hypersurfaces which are hence foliated by orbits of the H-action. Moreover, these hypersurfaces have constant principal curvatures if and only if they are homogeneous (i.e. they are the principal orbits of the cohomogeneity one action resulting from choosing $\mathfrak{q} = \mathfrak{n}_{\mathfrak{k}_0}(\mathfrak{w})$ in Theorem 3.1); this happens precisely when the real subspace $\mathfrak{w}^\perp$ of $\mathfrak{g}_\alpha$ has constant Kähler angle. See [14] for more details.

The rest of the paper will be devoted to the proof of the classification result stated in Theorems A and B. In order to justify the content of the following sections, we will give here a sketch of the proof of Theorem A, and leave the details for the following sections.

Assume that H is a closed subgroup of $SU(1,n)$ that acts polarly on $\mathbb{C}H^n$. Any subgroup of $SU(1,n)$ is contained in a maximal proper subgroup L of $SU(1,n)$. We will see that each maximal subgroup of $SU(1,n)$ either leaves a totally geodesic proper subspace of $\mathbb{C}H^n$ invariant or it is a parabolic subgroup. In the first case, L leaves invariant a lower dimensional complex hyperbolic space $\mathbb{C}H^k$, $k \in \{0,\dots,n-1\}$, or a real hyperbolic space $\mathbb{R}H^n$. The first possibility is tackled in Subsection 4.1, and it follows from this part of the paper that, roughly, the action of H splits, up to orbit equivalence, as the product of a polar action on the totally geodesic $\mathbb{C}H^k$, and a polar action with a fixed point on its normal space. Hence, the problem is reduced to the classification of polar actions on lower dimensional complex hyperbolic spaces, which will allow us to use an induction argument. The second possibility is addressed in Subsection 4.2 where we show that the action of H is orbit equivalent to the action of $SO(1,n)$, which is a cohomogeneity one action whose orbits are tubes around a totally geodesic $\mathbb{R}H^n$. If the group L is parabolic, then its Lie algebra is of the form $\mathfrak{l} = \mathfrak{k}_0 \oplus \mathfrak{a} \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$, for some root space decomposition of $\mathfrak{su}(1,n)$ (see §2.1). We show in Section 5 that the Lie algebra of H (up to orbit equivalence) must be of the form $\mathfrak{q} \oplus \mathfrak{b} \oplus \mathfrak{w} \oplus \mathfrak{g}_{2\alpha}$, with $\mathfrak{q} \subset \mathfrak{k}_0$, $\mathfrak{b} \subset \mathfrak{a}$, and $\mathfrak{w} \subset \mathfrak{g}_\alpha$, or of the form $\mathfrak{q} \oplus \mathfrak{a}$, with $\mathfrak{q} \subset \mathfrak{k}_0$. A bit more work leads us to the examples described in Theorem 3.1. Combining the different cases, we will conclude in Section 6 the proofs of Theorems A and B.

4. Actions leaving a totally geodesic subspace invariant

The results in this section show that in order to classify polar actions leaving a totally geodesic complex hyperbolic subspace invariant it suffices to study polar actions on the complex hyperbolic spaces of lower dimensions. We will also show that actions leaving a totally geodesic $\mathbb{R}H^n$ invariant are orbit equivalent to the cohomogeneity one action of $SO(1,n)$. Note that if an isometric action leaves a totally geodesic $\mathbb{R}H^k$ invariant, it also leaves a totally geodesic $\mathbb{C}H^k$ invariant.

The following is well known. Let H be closed connected subgroup of $SU(1,n)$. If the natural action of H on $\mathbb{C}H^n$ leaves a totally geodesic proper submanifold of $\mathbb{C}H^n$ invariant, then there is an element $g \in SU(1,n)$ such that gHg^{-1} is contained in one of the subgroups $S(U(1,k)U(n-k))$ or $SO(1,n)$ of $SU(1,n)$.

4.1. **Actions leaving a totally geodesic complex hyperbolic space invariant.** Let $L = S(U(1,k)U(n-k)) \subset G = SU(1,n)$. Let M_1 be the totally geodesic $\mathbb{C}H^k$ given by the orbit $L \cdot o$. Let M_2 be the totally geodesic $\mathbb{C}H^{n-k}$ which is the image of the normal space $\nu_o M_1$ under the Riemannian exponential map $\exp_o$. Let H be a closed connected subgroup of L. Then the H-action on $\mathbb{C}H^n$ leaves M_1 invariant and the H-action on $\mathbb{C}H^n$ restricted to the isotropy subgroup H_o leaves M_2 invariant. Let $\pi_1 \colon L \to U(1,k)$ and $\pi_2 \colon L \to U(n-k)$ be the natural projections.

Theorem 4.1. *Assume the H-action on $\mathbb{C}H^n$ is nontrivial. Then it is polar if and only if the following hold:*

(i) *The action of H on M_1 is polar and nontrivial.*
(ii) *The action of H_o on M_2 is polar and nontrivial.*
(iii) *The action of $\pi_1(H) \times \pi_2(H_o)$ on $\mathbb{C}H^n$ is orbit equivalent to the H-action.*

Proof. Assume first that the H-action on $\mathbb{C}H^n$ is polar and Σ is a section. Let Σ_i be the connected component of $\Sigma \cap M_i$ containing o for $i = 1, 2$. Obviously, the H-orbits on M_1 intersect Σ_1 orthogonally. Let p be an arbitrary point in M_1. Then the intersection of the orbit $H \cdot p$ with Σ is non-empty. Let $q \in (H \cdot p) \cap \Sigma$. Since H leaves M_1 invariant, we have that $q \in M_1$. Both the Riemannian exponential maps of M_1 and of Σ at the point o are diffeomorphisms by the Cartan-Hadamard theorem. Hence there is a unique shortest geodesic segment β in Σ connecting o with q and there is also a unique shortest geodesic segment γ in M_1 connecting o with q. Since both Σ and M_1 are totally geodesic submanifolds of $\mathbb{C}H^n$ it follows that β and γ are both also totally geodesic segments of $\mathbb{C}H^n$ connecting the points o and q and must coincide by the Cartan-Hadamard theorem. Hence $\beta = \gamma$ both lie in Σ_1. This shows that Σ_1 meets the H-orbit through p (namely, at the point q) and completes the proof that (i) holds.

Obviously, the H_o-orbits on M_2 intersect Σ_2 orthogonally. Since $T_o M_2$ is a submodule of the slice representation of H_o on $\nu_o(H \cdot o)$, the linear H_o-action on $T_o M_2$ is polar with section $T_o\Sigma_2$. The map $\exp_o \colon T_o M_2 \to M_2$ is an H_o-equivariant diffeomorphism by the Cartan-Hadamard theorem. In particular, it follows that Σ_2 meets all H_o-orbits in M_2, since $T_o\Sigma_2$ meets all H_o-orbits in $T_o M_2$. Thus (ii) holds.

Consider the polar slice representation of H_o at $T_o\mathbb{C}H^n$ with section $T_o\Sigma$. By [12, Theorem 4], it follows that $T_o\Sigma = T_o\Sigma_1 \oplus T_o\Sigma_2$. Since $H \subset \pi_1(H) \times \pi_2(H)$, it follows that the actions of the two groups on $\mathbb{C}H^n$ are orbit equivalent.

Now let us prove the other direction of the equivalence. Assume $H \subset L$ is a closed subgroup such that (i), (ii) and (iii) hold. Because of (iii) we may replace H by $\pi_1(H) \times \pi_2(H)$. Let Σ_1 be the section of the H-action on M_1 and let Σ_2 be the section of H-action on M_2. Then by Proposition 2.4, the tangent spaces $T_o\Sigma_1$ and $T_o\Sigma_2$ are totally real subspaces of $T_o\mathbb{C}H^n$; moreover, $\mathbb{C}T_o\Sigma_1 \perp \mathbb{C}T_o\Sigma_2$. Thus the sum $T_o\Sigma_1 \oplus T_o\Sigma_2$ is totally real Lie triple system in $T_o\mathbb{C}H^n$. Let Σ be the corresponding totally geodesic submanifold.

Using Proposition 2.3, we will show that the H-action on $\mathbb{C}H^n$ is polar and Σ is a section. Consider the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ with respect to $o \in \mathbb{C}H^n$. We have

$\mathfrak{p} = T_oM_1 \oplus T_oM_2$. Furthermore, the direct sum decomposition

$$\nu_o(H \cdot o) = (\nu_o(H \cdot o) \cap T_oM_1) \oplus T_oM_2 \tag{2}$$

holds. The slice representation of the H-action on M_1 at the point o is orbit equivalent to the submodule $\nu_o(H \cdot o) \cap T_oM_1$ of the slice representation of the H-action on $\mathbb{C}H^n$ at o. The slice representation of the H_o-action on M_2 at the point o is orbit equivalent to the submodule T_oM_2 of the slice representation of the H-action on $\mathbb{C}H^n$ at o. By [12, Theorem 4], we conclude that the slice representation of H_o on $\nu_o(H \cdot o)$ is polar and a section is $T_o\Sigma = T_o\Sigma_1 \oplus T_o\Sigma_2$. We have to show $\langle [v,w], X\rangle = -B([v,w], \theta(X)) = 0$ for all v, $w \in T_o\Sigma \subset \mathfrak{p}$ and all $X \in \mathfrak{h}$. We may identify the tangent space $T_o\mathbb{C}H^n = \mathfrak{p}$ with the space of complex $(n+1)\times(n+1)$-matrices of the form

$$\left(\begin{array}{c|ccc} 0 & \bar{z}_1 & \dots & \bar{z}_n \\ \hline z_1 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ z_n & 0 & \dots & 0 \end{array}\right). \tag{3}$$

The subspace T_oM_1 is given by the matrices where $z_{k+1} = \ldots = z_n = 0$. On the other hand, T_oM_2 consists of those matrices where $z_1 = \ldots = z_k = 0$. Let $v, w \in T_o\Sigma_1$. Then $[v,w]$ is a matrix all of whose nonzero entries are located in the $(k+1)\times(k+1)$-submatrix in the upper left-hand corner, and it follows from (i) and Proposition 2.3 that all vectors in $\mathfrak{h}$ are orthogonal to $[v,w]$. Now assume $v, w \in T_o\Sigma_2$. Then $[v,w]$ is a matrix all of whose nonzero entries are located in the $(n-k)\times(n-k)$-submatrix in the bottom right-hand corner. It follows from (ii) and Proposition 2.3 that all vectors in $\mathfrak{h}$ are orthogonal to $[v,w]$. Finally assume $v \in T_o\Sigma_1$ and $w \in T_o\Sigma_2$. In this case, the bracket $[v,w]$ is contained in the orthogonal complement of the Lie algebra of L in $\mathfrak{su}(1,n)$; in particular, $[v,w]$ is orthogonal to $\mathfrak{h}$. We conclude that the H-action on $\mathbb{C}H^n$ is polar by Proposition 2.3. ☐

4.2. **Actions leaving a totally geodesic real hyperbolic space invariant.** Now we assume that the polar action leaves a totally geodesic $\mathbb{R}H^n$ invariant. We have:

Theorem 4.2. *Assume that H is a closed subgroup of $SO(1,n) \subset SU(1,n)$. If the H-action on $\mathbb{C}H^n$ is polar and nontrivial, then it is orbit equivalent to the $SO(1,n)$-action on $\mathbb{C}H^n$; in particular, it is of cohomogeneity one.*

Proof. This proof is divided in three steps.

Claim 1. The group H induces a homogeneous polar foliation on the totally geodesic submanifold $\mathbb{R}H^n$ given by the $SO(1,n)$-orbit through o.

Let M_1 be the totally geodesic $\mathbb{R}H^n$ given by the $SO(1,n)$-orbit through o. Obviously, the H-action leaves M_1 invariant. Assume the H-action on M_1 has a singular orbit $H \cdot p$, where $p = g(o) \in M_1$. Consider the action of H' on $\mathbb{C}H^n$, where H' is the conjugate subgroup $H' = gHg^{-1}$ of $SU(1,n)$. The action of H' is conjugate to the H-action on $\mathbb{C}H^n$, hence polar. We have the splitting (2) for the normal space of the H'-orbit through o as in the proof of Lemma 4.1, where in this case M_2 is the totally geodesic $\mathbb{R}H^n$ such that

$T_oM_2 = i(T_oM_1)$. Since o is a singular orbit of the H'-action on M_1, the slice representation of H'_o on $V = \nu_o(H' \cdot o) \cap T_oM_1$ is nontrivial. The space T_oM_1 consists of all matrices in (3) where the entries $z_1, \dots, z_n$ are real. Consequently, the space iV is contained in the normal space $\nu_o(H' \cdot o)$ and it follows that the slice representation of H'_o with respect to the H'-action on $\mathbb{C}H^n$ contains the submodule $V \oplus iV$ with two equivalent nontrivial H'_o-representations and is hence non-polar by [20, Lemma 2.9], a contradiction. Hence the H-action on M_1 does not have singular orbits, i.e. H induces a homogeneous foliation on M_1.

Claim 2. The homogeneous polar foliation induced on the invariant totally geodesic real hyperbolic space consists of only one leaf or all the leaves are points.

Consider the point $o \in M_1$ as in the proof of Claim 1. The tangent space of M_1 at o splits as

$$T_oM_1 = T_o(H \cdot o) \oplus (\nu_o(H \cdot o) \cap T_oM_1).$$

The action of the isotropy group H_o on T_oM_1 respects this splitting. Moreover, the action is trivial on $V = \nu_o(H \cdot o) \cap T_oM_1$, as this is a submodule of the slice representation at o, which lies in a principal orbit of the H-action on M_1. It follows that the action of H_o on iV is trivial as well and the only possibly nontrivial submodule of the slice representation at o is iW, where we define $W = T_o(H \cdot o)$. It follows that the action of the isotropy group H_o on iW is polar by Proposition 2.3. Let Σ' be a section of this action. Let Σ be a section of the H-action on $\mathbb{C}H^n$. Then we have

$$T_o\Sigma = V \oplus iV \oplus \Sigma'.$$

By Proposition 2.4, Σ is either totally real or $\Sigma = \mathbb{C}H^n$. In the first case, V must be 0, so the action of H on M_1 is transitive. In the second case, the action of H on $\mathbb{C}H^n$ is trivial.

Claim 3. The H-action on $\mathbb{C}H^n$ is orbit equivalent to the $SO(1, n)$-action.

Assume the H-action is nontrivial and polar with section Σ. We will use the notation of Subsection 2.1. By Claim 2, H acts transitively on $M_1 = \mathbb{R}H^n$. By Lemma 2.2, the tangent space $T_o(H \cdot o) = T_oM_1$ coincides with $\mathfrak{a} \oplus (1 - \theta)\mathfrak{g}_\alpha^{\mathbb{R}}$, where $\mathfrak{g}_\alpha^{\mathbb{R}}$ is a totally real subspace of the root space $\mathfrak{g}_\alpha$ satisfying $\mathbb{C}\mathfrak{g}_\alpha^{\mathbb{R}} = \mathfrak{g}_\alpha$. Moreover $\nu_oM_1 = i(T_oM_1)$. The action of the isotropy subgroup $H_o = H \cap K$ on ν_oM_1 by the slice representation is polar with section $T_o\Sigma$. Since $iB \in \nu_oM_1$, by conjugating the section with a suitable element in H_o we can then assume that $iB \in T_o\Sigma$.

According to [7, Proposition 2.2], the group H contains a solvable subgroup S which acts transitively on $M_1 = \mathbb{R}H^n$. Since S is solvable, it is contained in a Borel subgroup of $SO(1, n)$. As shown in the proof of [6, Proposition 4.2], we may assume that the Lie algebra of such a Borel subgroup is maximally noncompact, i.e. its Lie algebra is $\mathfrak{t} \oplus \mathfrak{a} \oplus \mathfrak{g}_\alpha^{\mathbb{R}}$, where $\mathfrak{t}$ is an abelian subalgebra of $\mathfrak{k} \cap \mathfrak{so}(n)$ such that $\mathfrak{t} \oplus \mathfrak{a}$ is a Cartan subalgebra of $\mathfrak{so}(1, n)$, see [28]. Note that the Cartan decomposition of $\mathfrak{so}(1, n)$ with respect to the point $o \in M_1 = \mathbb{R}H^n$ is $\mathfrak{so}(1, n) = (\mathfrak{k} \cap \mathfrak{so}(1, n)) \oplus \mathfrak{p}^{\mathbb{R}}$, where $\mathfrak{p}^{\mathbb{R}} = \mathfrak{a} \oplus (1 - \theta)\mathfrak{g}_\alpha^{\mathbb{R}} \cong T_oM_1$, and $\mathfrak{g}_\alpha^{\mathbb{R}}$ is the only positive root space of $\mathfrak{so}(1, n)$ with respect to the maximal abelian subalgebra $\mathfrak{a}$ of $\mathfrak{p}^{\mathbb{R}}$, for a fixed order in the roots.

Now assume the H-action on $\mathbb{C}H^n$ is not of cohomogeneity one. Then $T_o\Sigma \subset \nu_o M_1$ is a Lie triple system containing iB and a nonzero vector iw such that iB, $iw \in \mathfrak{p}$ are orthogonal. By Lemma 2.2, there is a vector $W \in \mathfrak{g}_\alpha^{\mathbb{R}}$ such that $w = (1-\theta)W$. Then, using Lemma 2.1(a), we have

$$[iB, iw] = \frac{1}{2}[(1-\theta)Z, (1-\theta)JW] = \frac{1}{2}(1+\theta)[\theta JW, Z] = \frac{1}{2}(1+\theta)W.$$

Since $T_o(S\cdot o) = T_oM_1$, it follows that the orthogonal projection of the Lie algebra of S onto $\mathfrak{p}$ is $\mathfrak{p}^{\mathbb{R}} = \mathfrak{a}\oplus(1-\theta)\mathfrak{g}_\alpha^{\mathbb{R}}$. This implies that $\mathfrak{a}\oplus\mathfrak{g}_\alpha^{\mathbb{R}}$ is contained in the Lie algebra of S, and hence, also in $\mathfrak{h}$. But then $W\in\mathfrak{h}$ and

$$\langle [iB, iw], W\rangle = \frac{1}{2}\langle (1+\theta)W, W\rangle = \frac{1}{2}\langle W, W\rangle \neq 0,$$

so we have arrived at a contradiction with the criterion for polarity in Proposition 2.3. □

5. The parabolic case

As above, let $G = SU(1,n)$ be the identity connected component of the isometry group of $\mathbb{C}H^n$, and $K = S(U(1)U(n))$ the isotropy group at some point o. Let $\mathfrak{g} = \mathfrak{k}\oplus\mathfrak{p}$ be the Cartan decomposition of the Lie algebra of G with respect to o, and choose a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$. As usual we consider $\mathfrak{n} = \mathfrak{g}_\alpha\oplus\mathfrak{g}_{2\alpha}$, where α is a simple positive restricted root. The normalizer of $\mathfrak{n}$ in $\mathfrak{k}$ is denoted by $\mathfrak{k}_0$. Then $\mathfrak{k}_0\oplus\mathfrak{a}\oplus\mathfrak{n}$ is a maximal parabolic subalgebra, and a maximal parabolic subgroup can be written as the semi-direct product K_0AN.

The aim of this section is to prove the following decomposition theorem.

Theorem 5.1. *Let H be a connected closed subgroup of K_0AN acting polarly and non-trivially on $\mathbb{C}H^n$. Then the action of H is orbit equivalent to the action of a subgroup of K_0AN whose Lie algebra can be written as one of the following:*

(a) $\mathfrak{q}\oplus\mathfrak{a}$, *where $\mathfrak{q}$ is a subalgebra of $\mathfrak{k}_0$.*
(b) $\mathfrak{q}\oplus\mathfrak{a}\oplus\mathfrak{w}\oplus\mathfrak{g}_{2\alpha}$, *where $\mathfrak{w}$ is a subspace of $\mathfrak{g}_\alpha$, and $\mathfrak{q}$ is a subalgebra of $\mathfrak{k}_0$.*
(c) $\mathfrak{q}\oplus\mathfrak{w}\oplus\mathfrak{g}_{2\alpha}$, *where $\mathfrak{w}$ is a subspace of $\mathfrak{g}_\alpha$, and $\mathfrak{q}$ is a subalgebra of $\mathfrak{k}_0$.*

Recall that an isometric action on a complete connected Riemannian manifold by a closed subgroup of its isometry group is a proper action. In particular, this implies that isotropy groups are compact, orbits are closed, and the orbit space is Hausdorff. One can then talk about types of orbits: two orbits have the same type if the isotropy groups at any given points of these orbits are conjugate. The set of conjugacy classes of isotropy groups of orbits is a partially ordered set by inclusion. For a proper action there is always a maximum orbit type, that is, a type of orbit whose isotropy groups are contained, up to conjugacy, in the isotropy groups of any of the other orbits. Orbits belonging to this type are called principal orbits. They have the maximum possible dimension, and their union constitutes an open dense subset of the ambient manifold. Orbits that are not principal but have maximum dimension are called exceptional. The rest of the orbits are said to be singular. For proper isometric actions on Hadamard manifolds there is also a minimum orbit type.

Proposition 5.2. *Let M be a Hadamard manifold and H a closed subgroup of its isometry group acting on M. Then, there is a minimum orbit type, that is, there is an orbit type whose isotropy groups contain, up to conjugation in H, the isotropy groups of any other orbits.*

Proof. Let Q be a maximal compact subgroup of H. Any two maximal compact subgroups of a connected Lie group H are connected and conjugate by an element of H [29, p. 148–149]. By Cartan's fixed point theorem, Q fixes a point $p \in M$, and hence $Q = H_p$, the isotropy group of H at p. If $q \in M$, then H_q is compact, and since all maximal compact subgroups of H are conjugate it follows that there exists $h \in H$ such that $H_q \subset hQh^{-1}$. Thus, the orbit through p is of the minimum orbit type. □

Coming back to the problem in $\mathbb{C}H^n$, consider from now on a connected closed subgroup H of K_0AN acting polarly and nontrivially on $\mathbb{C}H^n$. Proposition 5.2 and its proof assert that there is a maximal compact subgroup Q of H with a fixed point $p \in \mathbb{C}H^n$. The orbit through p is of minimum type, and $Q = H_p$. Since AN acts simply transitively on $\mathbb{C}H^n$, we can take the unique element g in AN such that $g(o) = p$, and consider the group $H' = I_{g^{-1}}(H) = g^{-1}Hg$, whose action on $\mathbb{C}H^n$ is conjugate to the one of H. Moreover, $Q' = I_{g^{-1}}(Q) = g^{-1}Qg$ fixes the point o. Since $\mathfrak{a} \oplus \mathfrak{n}$ normalizes $\mathfrak{k}_0 \oplus \mathfrak{a} \oplus \mathfrak{n}$, we get that AN normalizes $\mathfrak{k}_0 \oplus \mathfrak{a} \oplus \mathfrak{n}$. In particular, $\mathrm{Ad}(g^{-1})\mathfrak{h} \subset \mathfrak{k}_0 \oplus \mathfrak{a} \oplus \mathfrak{n}$, and therefore $H' \subset K_0AN$. Since we are interested in the study of polar actions up to orbit equivalence, it is not restrictive to assume that the group $H \subset K_0AN$ acting polarly on $\mathbb{C}H^n$ admits a maximal connected compact subgroup Q that fixes the point o, and hence $Q \subset K_0$. We will assume this throughout this section.

As a matter of notation, given two subspaces $\mathfrak{m}$, $\mathfrak{l}$, and a vector v of $\mathfrak{g}$, by $\mathfrak{m}_{\mathfrak{l}}$ (resp. by $v_{\mathfrak{l}}$) we will denote the orthogonal projection of $\mathfrak{m}$ (resp. of v) onto $\mathfrak{l}$.

The crucial part of the proof of Theorem 5.1 is contained in the following assertion:

Proposition 5.3. *Let H be a connected closed subgroup of K_0AN acting polarly on $\mathbb{C}H^n$. Let Q be a maximal subgroup of H that fixes the point $o \in \mathbb{C}H^n$. Let $\mathfrak{b}$ be a subspace of $\mathfrak{a}$, $\mathfrak{w}$ a subspace of $\mathfrak{g}_\alpha$, and $\mathfrak{r}$ a subspace of $\mathfrak{g}_{2\alpha}$. Assume that $\hat{\mathfrak{h}} = \mathfrak{q} \oplus \mathfrak{b} \oplus \mathfrak{w} \oplus \mathfrak{r}$ is a subalgebra of $\mathfrak{k}_0 \oplus \mathfrak{a} \oplus \mathfrak{n}$, and let $\hat{H}$ be the connected subgroup of K_0AN whose Lie algebra is $\hat{\mathfrak{h}}$. If $\mathfrak{h}_{\mathfrak{a}\oplus\mathfrak{n}} = \mathfrak{b} \oplus \mathfrak{w} \oplus \mathfrak{r}$, then the actions of H and $\hat{H}$ are orbit equivalent.*

The proof of Proposition 5.3 is carried out in several steps. We start with a few basic remarks.

Since $\mathfrak{a}$ and $\mathfrak{g}_{2\alpha}$ are one dimensional, $\mathfrak{b}$ is either 0 or $\mathfrak{a}$, and $\mathfrak{r}$ is either 0 or $\mathfrak{g}_{2\alpha}$. Moreover, if $\mathfrak{r} = 0$ then $\mathfrak{w}$ has to be a totally real subspace of the complex vector space $\mathfrak{g}_\alpha \cong \mathbb{C}^{n-1}$, so that $\hat{\mathfrak{h}}$ is a Lie subalgebra. Using the properties of the root space decomposition, it is then easy to check that $\hat{\mathfrak{h}} = \mathfrak{q} \oplus \mathfrak{b} \oplus \mathfrak{w} \oplus \mathfrak{r}$ is a subalgebra of $\mathfrak{k}_0 \oplus \mathfrak{a} \oplus \mathfrak{n}$ if and only if $[\mathfrak{q}, \mathfrak{w}] \subset \mathfrak{w}$.

Let Σ be a section of the action of H on $\mathbb{C}H^n$ through $o \in \mathbb{C}H^n$, and let $T_o\Sigma$ be its tangent space at o. The normal space of the orbit through the origin is $\nu_o(H \cdot o) = (\mathfrak{a} \ominus \mathfrak{b}) \oplus (\mathfrak{p}_\alpha \ominus (1-\theta)\mathfrak{w}) \oplus (\mathfrak{p}_{2\alpha} \ominus (1-\theta)\mathfrak{r})$. Since $[\mathfrak{k}_0, \mathfrak{a}] = [\mathfrak{k}_0, \mathfrak{g}_{2\alpha}] = 0$, $[\mathfrak{k}_0, \mathfrak{g}_\alpha] = \mathfrak{g}_\alpha$, and $\nu_o(H \cdot o) = T_o\Sigma \oplus [\mathfrak{q}, T_o\Sigma]$ (orthogonal direct sum of vector subspaces) by Proposition 2.3, it follows that $\mathfrak{a} \ominus \mathfrak{b} \subset T_o\Sigma$ and $\mathfrak{p}_{2\alpha} \ominus (1-\theta)\mathfrak{r} \subset T_o\Sigma$. Moreover, since sections are totally real by Proposition 2.4, we can write the tangent space at o of any section as $T_o\Sigma = (\mathfrak{a} \ominus \mathfrak{b}) \oplus (1-\theta)\mathfrak{s} \oplus (\mathfrak{p}_{2\alpha} \ominus (1-\theta)\mathfrak{r})$, where $\mathfrak{s}$ is a totally real subspace of $\mathfrak{g}_\alpha$, with $\mathfrak{s} \subset \mathfrak{g}_\alpha \ominus \mathfrak{w}$. Furthermore, the fact that $T_o\Sigma$ is totally real, and $i\mathfrak{a} = \mathfrak{p}_{2\alpha}$ (where i is the complex structure on $\mathfrak{p}$), implies that $\mathfrak{a} \ominus \mathfrak{b} = 0$ or $\mathfrak{p}_{2\alpha} \ominus (1-\theta)\mathfrak{r} = 0$, or equivalently, $\mathfrak{b} = \mathfrak{a}$ or $\mathfrak{r} = \mathfrak{g}_{2\alpha}$ (that is, $\mathfrak{a} \subset \mathfrak{h}_{\mathfrak{a}\oplus\mathfrak{n}}$ or $\mathfrak{g}_{2\alpha} \subset \mathfrak{h}_{\mathfrak{a}\oplus\mathfrak{n}}$).

Let $T + aB + U + xZ$ be an arbitrary element of $\mathfrak{h}$, with $T \in \mathfrak{h}_{\mathfrak{k}_0}$, $U \in \mathfrak{w}$, and $a, x \in \mathbb{R}$. Let ξ, η be arbitrary vectors of $\mathfrak{s}$. By Proposition 2.3, and since $\mathfrak{s}$ is totally real, we have, using Lemma 2.1(b):

$$0 = \langle T + aB + U + xZ, [(1-\theta)\xi, (1-\theta)\eta]\rangle = -\langle T, (1+\theta)[\theta\xi, \eta]\rangle = -2\langle [T,\xi], \eta\rangle,$$

from where it follows that $[\mathfrak{h}_{\mathfrak{k}_0}, \mathfrak{s}] \subset \mathfrak{g}_\alpha \ominus \mathfrak{s}$.

Moreover, if $T \in \mathfrak{q}$ and $S_U \in \mathfrak{h}_{\mathfrak{k}_0}$, $U \in \mathfrak{w}$ are such that $S_U + U \in \mathfrak{h}$, then $[T, S_U] + [T, U] = [T, S_U + U] \in \mathfrak{h}$, so $[T, U] \in \mathfrak{w}$. In particular, if $\xi \in \mathfrak{s}$, then $0 = \langle [T,U], \xi\rangle = -\langle [T,\xi], U\rangle$, which proves $[\mathfrak{q}, \mathfrak{s}] \subset \mathfrak{g}_\alpha \ominus (\mathfrak{w} \oplus \mathfrak{s})$.

Summarizing what we have obtained about sections we can state:

Lemma 5.4. *If Σ is a section of the action of H on $\mathbb{C}H^n$ through o, then*

$$T_o\Sigma = (\mathfrak{a} \ominus \mathfrak{b}) \oplus (1-\theta)\mathfrak{s} \oplus (\mathfrak{p}_{2\alpha} \ominus (1-\theta)\mathfrak{r}),$$

where $\mathfrak{s} \subset \mathfrak{g}_\alpha \ominus \mathfrak{w}$ is a totally real subspace of $\mathfrak{g}_\alpha$, and $\mathfrak{b} = \mathfrak{a}$ or $\mathfrak{r} = \mathfrak{g}_{2\alpha}$. Moreover, $[\mathfrak{h}_{\mathfrak{k}_0}, \mathfrak{s}] \subset \mathfrak{g}_\alpha \ominus \mathfrak{s}$, and $[\mathfrak{q}, \mathfrak{s}] \subset \mathfrak{g}_\alpha \ominus (\mathfrak{w} \oplus \mathfrak{s})$.

We will need to calculate the isotropy group at certain points.

Lemma 5.5. *Let $\xi \in \mathfrak{g}_\alpha$ and write $g = \operatorname{Exp}(\lambda\xi)$, with $\lambda \in \mathbb{R}$. Then, the Lie algebra of the isotropy group H_p of H at $p = g(o)$ is $\mathfrak{h}_p = \mathfrak{h} \cap \operatorname{Ad}(g)\mathfrak{k} = \mathfrak{q} \cap \ker \operatorname{ad}(\xi)$.*

Proof. First notice that $\mathfrak{h} \cap \operatorname{Ad}(g)\mathfrak{k}$ is the Lie algebra of $H_p = H \cap I_g(K)$. Let v be the unique element in $\mathfrak{p} = T_o\mathbb{C}H^n$ such that $\exp_o(v) = p$. We show that the isotropy group H_p coincides with the isotropy group of the slice representation of Q at v, Q_v. By [35, §2] we know that the normal exponential map $\exp\colon \nu(H\cdot o) \to \mathbb{C}H^n$ is an H-equivariant diffeomorphism. Let $h \in H_p$. Since $\exp_o(v) = p = h(p) = h\exp_o(v) = \exp_{h(o)}(h_{*o}v)$, we get that $h(o) = o$ and $h_{*o}v = v$, and hence, $h \in Q_v$. The H-equivariance of exp also shows the converse inclusion. Therefore $H_p = Q_v$.

We can write $v = aB + b(1-\theta)\xi$ for certain a, $b \in \mathbb{R}$. In fact, $\operatorname{Exp}(\lambda\xi)(o)$ belongs to the totally geodesic $\mathbb{R}H^2$ given by $\exp_o(\mathfrak{a} \oplus \mathbb{R}(1-\theta)\xi)$, and $b \neq 0$ if $\lambda\xi \neq 0$. Then, the Lie algebra of $H_p = Q_v$ is $\{T \in \mathfrak{q} : [T, aB + b(1-\theta)\xi] = 0\} = \{T \in \mathfrak{q} : [T,\xi] = 0\}$, which is $\mathfrak{q} \cap \ker\operatorname{ad}(\xi)$. ∎

By definition, we say that a vector $\xi \in \mathfrak{s}$ is *regular* if $[\mathfrak{q}, \xi] = \mathfrak{g}_\alpha \ominus (\mathfrak{w} \oplus \mathfrak{s})$. We have

Lemma 5.6. *The set $\{\xi \in \mathfrak{s} : \xi \text{ is regular}\}$ is an open dense subset of $\mathfrak{s}$.*

Proof. An element of $T_o\Sigma$ can be written, according to Lemma 5.4, as $v = aB + (1-\theta)\xi + x(1-\theta)Z$ where a, $x \in \mathbb{R}$, and $\xi \in \mathfrak{s}$. We have $[\mathfrak{q}, v] = (1-\theta)[\mathfrak{q}, \xi]$ and $\nu_o(H\cdot o) \ominus T_o\Sigma = (1-\theta)(\mathfrak{g}_\alpha \ominus (\mathfrak{w} \oplus \mathfrak{s}))$. An element of $T_o\Sigma$ is regular (that is, belongs to a principal orbit of the slice representation $Q \times \nu_o(H\cdot o) \to \nu_o(H\cdot o)$) if and only if $[\mathfrak{q}, v] = \nu_o(H\cdot o) \ominus T_o\Sigma$.

The previous equalities, and the fact that $(1-\theta)\colon \mathfrak{g}_\alpha \to \mathfrak{p}_\alpha$ is an isomorphism implies that v is regular if and only if $[\mathfrak{q},\xi] = \mathfrak{g}_\alpha \ominus (\mathfrak{w}\oplus\mathfrak{s})$. Since the set of regular points of a section is open and dense, the result follows. □

Lemma 5.7. *For each regular vector $\xi \in \mathfrak{s}$ we have $[\mathfrak{h}_{\mathfrak{k}_0},\xi] = \mathfrak{g}_\alpha \ominus (\mathfrak{w}\oplus\mathfrak{s})$.*

Proof. Let $\xi\in\mathfrak{s}$ be a regular vector, that is, $[\mathfrak{q},\xi] = \mathfrak{g}_\alpha \ominus (\mathfrak{w}\oplus\mathfrak{s})$. In order to prove the lemma, it is enough to show that $[\mathfrak{h}_{\mathfrak{k}_0},\xi]\subset \mathfrak{g}_\alpha\ominus\mathfrak{w}$, since $\mathfrak{q}\subset\mathfrak{h}_{\mathfrak{k}_0}$ and, by Lemma 5.4, $[\mathfrak{h}_{\mathfrak{k}_0},\xi]\subset\mathfrak{g}_\alpha\ominus\mathfrak{s}$.

First, consider the case $\mathfrak{r}=0$. By Lemma 5.4, $T_o\Sigma = (1-\theta)\mathfrak{s}\oplus\mathbb{R}(1-\theta)Z$ for each section Σ through o, where $\mathfrak{s}$ is some totally real subspace of $\mathfrak{g}_\alpha$. By Proposition 2.3 we have $\nu_o(H\cdot o) = \mathrm{Ad}(Q)(T_o\Sigma)$ and, thus, for any $\eta\in\mathfrak{g}_\alpha\ominus\mathfrak{w}$ we can find a section Σ through o such that $\eta\in\mathfrak{s}$ by conjugating by a suitable element in Q. Then using Lemma 2.1, we have that $(1+\theta)J\eta = [(1-\theta)\eta,(1-\theta)Z]\in[T_o\Sigma,T_o\Sigma]$. Let $W\in\mathfrak{w}$ and $T_W\in\mathfrak{h}_{\mathfrak{k}_0}$ be such that $T_W+W\in\mathfrak{h}$. Since by Proposition 2.3 we have $\langle\mathfrak{h},[T_o\Sigma,T_o\Sigma]\rangle = 0$, then $0=\langle T_W+W,(1+\theta)J\eta\rangle = \langle W,J\eta\rangle$. We have then shown that $J(\mathfrak{g}_\alpha\ominus\mathfrak{w})$ is orthogonal to $\mathfrak{w}$, that is, $\mathfrak{g}_\alpha\ominus\mathfrak{w}$ is a complex subspace of $\mathfrak{g}_\alpha$. Since $\mathfrak{w}$ is totally real, we deduce $\mathfrak{w}=0$. But then $[\mathfrak{h}_{\mathfrak{k}_0},\xi]\subset\mathfrak{g}_\alpha\ominus\mathfrak{w}$ holds trivially.

For the rest of the proof, we assume that $\mathfrak{r}=\mathfrak{g}_{2\alpha}$.

Let $T_B\in\mathfrak{h}_{\mathfrak{k}_0}$ and $a\in\mathbb{R}$ such that $T_B+aB\in\mathfrak{h}$. Note that, if $\mathfrak{b}=0$, then $a=0$, $T_B\in\mathfrak{q}$ and there is nothing to prove. For each $U\in\mathfrak{w}$ take an $S_U\in\mathfrak{h}_{\mathfrak{k}_0}$ with $S_U+U\in\mathfrak{h}$. Then $[T_B,S_U]+[T_B,U]+\frac{a}{2}U = [T_B+aB,S_U+U]\in\mathfrak{h}$, so $[T_B,U]+\frac{a}{2}U\in\mathfrak{w}$, from where $[T_B,U]\in\mathfrak{w}$. Hence, $\langle[T_B,\xi],U\rangle = -\langle\xi,[T_B,U]\rangle = 0$, so we get $[T_B,\xi]\subset\mathfrak{g}_\alpha\ominus\mathfrak{w}$.

Now let $T_Z\in\mathfrak{h}_{\mathfrak{k}_0}$ and $x\in Z$ with $T_Z+xZ\in\mathfrak{h}$. For each $U\in\mathfrak{w}$ take an $S_U\in\mathfrak{h}_{\mathfrak{k}_0}$ with $S_U+U\in\mathfrak{h}$. Then $[T_Z,S_U]+[T_Z,U] = [T_Z+Z,S_U+U]\in\mathfrak{h}$, so $[T_Z,U]\in\mathfrak{w}$. As above, we conclude $[T_Z,\xi]\subset\mathfrak{g}_\alpha\ominus\mathfrak{w}$.

Finally, we have to prove that for each $U\in\mathfrak{w}$, if $T_U\in\mathfrak{h}_{\mathfrak{k}_0}$ is such that $T_U+U\in\mathfrak{h}$, then $[T_U,\xi]\in\mathfrak{g}_\alpha\ominus\mathfrak{w}$. This will require some effort.

Let $U\in\mathfrak{w}$ and $T_U\in\mathfrak{h}_{\mathfrak{k}_0}$ with $T_U+U\in\mathfrak{h}$. By Lemma 5.4, $[T_U,\xi]\in\mathfrak{g}_\alpha\ominus\mathfrak{s} = \mathfrak{w}\oplus(\mathfrak{g}_\alpha\ominus(\mathfrak{w}\oplus\mathfrak{s}))$. Since $[\mathfrak{q},\xi]=\mathfrak{g}_\alpha\ominus(\mathfrak{w}\oplus\mathfrak{s})$, we can find an $S\in\mathfrak{q}$ so that $[T_U+S,\xi]\in\mathfrak{w}$. Therefore we can define the map

$$F_\xi\colon \mathfrak{w}\to\mathfrak{w},\quad U\mapsto[T_U,\xi],\quad \text{where } T_U\in\mathfrak{h}_{\mathfrak{k}_0},\ T_U+U\in\mathfrak{h}, \text{ and } [T_U,\xi]\in\mathfrak{w}.$$

The map F_ξ is well-defined. Indeed, if T_U, $S_U\in\mathfrak{h}_{\mathfrak{k}_0}$, $U\in\mathfrak{w}$, T_U+U, $S_U+U\in\mathfrak{h}$, and $[T_U,\xi]$, $[S_U,\xi]\in\mathfrak{w}$, then $T_U-S_U\in\mathfrak{q}$, so $[T_U,\xi]-[S_U,\xi] = [T_U-S_U,\xi]\in\mathfrak{g}_\alpha\ominus(\mathfrak{w}\oplus\mathfrak{s})$, and $[T_U,\xi]-[S_U,\xi]\in\mathfrak{w}$. Hence $[T_U,\xi]=[S_U,\xi]$. It is also easy to check that F_ξ is linear.

Furthermore, F_ξ is self-adjoint. To see this, let T_U, $S_V\in\mathfrak{h}_{\mathfrak{k}_0}$, U, $V\in\mathfrak{w}$, with T_U+U, $S_V+V\in\mathfrak{h}$, and $[T_U,\xi]$, $[S_V,\xi]\in\mathfrak{w}$. Then we have

$$\begin{aligned}0 &= \langle[T_U+U,S_V+V],\xi\rangle = \langle[T_U,V],\xi\rangle - \langle[S_V,U],\xi\rangle = -\langle V,[T_U,\xi]\rangle + \langle U,[S_V,\xi]\rangle\\ &= -\langle F_\xi(U),V\rangle + \langle F_\xi(V),U\rangle.\end{aligned}$$

Assume now that $F_\xi\neq 0$. Then F_ξ admits an eigenvector $U\in\mathfrak{w}$ with nonzero eigenvalue $\lambda\in\mathbb{R}$: $F_\xi(U)=\lambda U\neq 0$. We will get a contradiction with this.

Let $g = \operatorname{Exp}(-\frac{1}{\lambda}\xi)$, and consider $T_U \in \mathfrak{h}_{\mathfrak{k}_0}$ such that $T_U + U \in \mathfrak{h}$ and $F_\xi(U) = [T_U, \xi] = \lambda U$. We also consider an element $S \in \mathfrak{h}_{\mathfrak{k}_0}$ such that $S + Z \in \mathfrak{h}$ and $[S, \xi] = 0$; this is possible because $[S, \xi] \in \mathfrak{g}_\alpha \ominus (\mathfrak{w} \oplus \mathfrak{s}) = [\mathfrak{q}, \xi]$ and $\mathfrak{q} \subset \mathfrak{h}$. If we define $R = T_U - \frac{1}{4\lambda}\langle J\xi, U\rangle S \in \mathfrak{h}_{\mathfrak{k}_0}$, then we have

$$\begin{aligned}\operatorname{Ad}(g)R = e^{-\frac{1}{\lambda}\operatorname{ad}(\xi)}R &= T_U - \frac{1}{\lambda}[\xi, T_U] + \frac{1}{2\lambda^2}[\xi, [\xi, T_U]] - \frac{1}{4\lambda}\langle J\xi, U\rangle S \\ &= (T_U + U) - \frac{1}{4\lambda}\langle J\xi, U\rangle (S + Z) \in \mathfrak{h} \cap \operatorname{Ad}(g)(\mathfrak{k}).\end{aligned}$$

However, $\operatorname{Ad}(g)R \notin \mathfrak{q} \cap \ker \operatorname{ad}(\xi)$. By virtue of Lemma 5.5, this gives a contradiction. Thus we must have $F_\xi = 0$, from where the result follows. □

Lemma 5.8. *The subspace* $\mathfrak{h}_{\mathfrak{k}_0}$ *is a subalgebra of* $\mathfrak{k}_0$ *and* $[\mathfrak{h}_{\mathfrak{k}_0}, \mathfrak{w}] \subset \mathfrak{w}$.

Proof. If $T + aB + U + xZ$, $S + bB + V + yZ \in \mathfrak{h}$, with T, $S \in \mathfrak{h}_{\mathfrak{k}_0}$, U, $V \in \mathfrak{w}$, and a, b, x, $y \in \mathbb{R}$, then the bracket $[T + aB + U + xZ, S + bB + V + yZ] = [T, S] + [T, V] - [S, U] + \frac{a}{2}V - \frac{b}{2}U + \left(\frac{1}{2}\langle JU, V\rangle + ay - bx\right) Z$ belongs to $\mathfrak{h}$. In particular $[T, S] \in \mathfrak{h}_{\mathfrak{k}_0}$, so $\mathfrak{h}_{\mathfrak{k}_0}$ is a Lie subalgebra of $\mathfrak{k}_0$. Taking $U = 0$, $a = b = x = y = 0$ we obtain that $[\mathfrak{q}, \mathfrak{w}] \subset \mathfrak{w}$ and hence $[\mathfrak{q}, \mathfrak{g}_\alpha \ominus \mathfrak{w}] \subset \mathfrak{g}_\alpha \ominus \mathfrak{w}$.

Now let $X \in \mathfrak{g}_\alpha \ominus \mathfrak{w}$. For any section through o we have $\operatorname{Ad}(Q)(T_o\Sigma) = \nu_0(H \cdot o) = (\mathfrak{a} \ominus \mathfrak{b}) \oplus (1 - \theta)(\mathfrak{g}_\alpha \ominus \mathfrak{w}) \oplus (1 - \theta)(\mathfrak{g}_{2\alpha} \ominus \mathfrak{r})$, and $(\mathfrak{a} \ominus \mathfrak{b}) \oplus (1 - \theta)(\mathfrak{g}_\alpha \ominus \mathfrak{r}) \subset T_o\Sigma$ by Lemma 5.4. Hence, for $(1 - \theta)X \in (1 - \theta)(\mathfrak{g}_\alpha \ominus \mathfrak{w})$ we can find a section Σ such that $(1 - \theta)X \in T_o\Sigma$ (after conjugation by an element of Q if necessary). Then, if X is regular, Lemma 5.7 implies $[\mathfrak{h}_{\mathfrak{k}_0}, X] \subset \mathfrak{g}_\alpha \ominus \mathfrak{w}$. Since the set of regular vectors is dense, X can always be approximated by a sequence of regular vectors, and hence, by continuity we also obtain $[\mathfrak{h}_{\mathfrak{k}_0}, X] \subset \mathfrak{g}_\alpha \ominus \mathfrak{w}$ for non-regular vectors. Therefore, $[\mathfrak{h}_{\mathfrak{k}_0}, \mathfrak{g}_\alpha \ominus \mathfrak{w}] \subset \mathfrak{g}_\alpha \ominus \mathfrak{w}$. Finally, the skew-symmetry of the elements of $\operatorname{ad}(\mathfrak{k}_0)$ implies $[\mathfrak{h}_{\mathfrak{k}_0}, \mathfrak{w}] \subset \mathfrak{w}$. □

We can now finish the proof of Proposition 5.3.

Proof of Proposition 5.3. The fact that $\hat{\mathfrak{h}} = \mathfrak{q} \oplus \mathfrak{b} \oplus \mathfrak{w} \oplus \mathfrak{r}$ is a subalgebra of $\mathfrak{k}_0 \oplus \mathfrak{a} \oplus \mathfrak{n}$, and Lemma 5.8, imply that $\tilde{\mathfrak{h}} = \mathfrak{h}_{\mathfrak{k}_0} \oplus \mathfrak{b} \oplus \mathfrak{w} \oplus \mathfrak{r}$ is a Lie subalgebra of $\mathfrak{g}$ that contains $\mathfrak{h}$ and $\hat{\mathfrak{h}}$. Let $\tilde{H}$ be the connected subgroup of G whose Lie algebra is $\tilde{\mathfrak{h}}$. Since $T_o(H \cdot o) = T_o(\tilde{H} \cdot o) = T_o(\hat{H} \cdot o) = \mathfrak{b} \oplus (1 - \theta)\mathfrak{w} \oplus (1 - \theta)\mathfrak{r}$ and $H \subset \tilde{H}$, $\hat{H} \subset \tilde{H}$, the orbits through o of the groups H, $\tilde{H}$, and $\hat{H}$ coincide. The slice representations at o of H and $\tilde{H}$ have the same principal orbits. Indeed, for a section Σ through o and $v = aB + (1-\theta)\xi + x(1-\theta)Z \in T_o\Sigma$ with $\xi \in \mathfrak{s}$ regular, Lemma 5.7 implies $[\mathfrak{h}_{\mathfrak{k}_0}, \xi] = \mathfrak{g}_\alpha \ominus (\mathfrak{w} \oplus \mathfrak{s}) = [\mathfrak{q}, \xi]$. Thus, the tangent spaces at v of the orbits of the slice representations of H and $\tilde{H}$ through v coincide, and since $H \subset \tilde{H}$, both orbits coincide. Then, the slice representations at o of H and $\tilde{H}$ are orbit equivalent. Since the codimension of an orbit of H (resp. of $\tilde{H}$) through $\exp_o(v)$ coincides with the codimension of the orbit of the slice representation of H (resp. of $\tilde{H}$) through $v \in \nu_o(H \cdot o) = \nu_o(\tilde{H} \cdot o)$, and since the orbits of H are contained in the orbits of $\tilde{H}$, we conclude that the actions of H and $\tilde{H}$ on $\mathbb{C}H^n$ have the same orbits. Similarly, an analogous argument with $\hat{H}$ instead of H allows to show that the actions of $\hat{H}$ and $\tilde{H}$ on $\mathbb{C}H^n$ are orbit equivalent, and this completes the proof. □

We now proceed with the proof of Theorem 5.1.

Let H be a closed subgroup of the isometry group of $\mathbb{C}H^n$ acting polarly on $\mathbb{C}H^n$, and assume that the Lie algebra of H is contained in a maximal parabolic subalgebra $\mathfrak{k}_0\oplus\mathfrak{a}\oplus\mathfrak{n}$. As we argued at the beginning of this section, there is a maximal compact subgroup Q of H, and we can assume that $o\in\mathbb{C}H^n$ is a fixed point of Q, that is, the isotropy group of H at o is Q. We are now interested in $\mathfrak{h}_{\mathfrak{a}\oplus\mathfrak{n}}$, the orthogonal projection of $\mathfrak{h}$ on $\mathfrak{a}\oplus\mathfrak{n}$. It is clear that $\mathfrak{h}_{\mathfrak{a}\oplus\mathfrak{n}}$ can be written in one of the following forms: $\mathfrak{w}$, $\mathbb{R}(B+X)\oplus\mathfrak{w}$, $\mathbb{R}(B+X+xZ)\oplus\mathfrak{w}$ (with $x\neq 0$), $\mathfrak{w}\oplus\mathbb{R}(Y+Z)$, or $\mathbb{R}(B+X)\oplus\mathfrak{w}\oplus\mathbb{R}(Y+Z)$, where $\mathfrak{w}\subset\mathfrak{g}_\alpha$, and X, $Y\in\mathfrak{g}_\alpha$.

In order to conclude the proof of Theorem 5.1 we deal with these five possibilities separately.

Case 1: $\mathfrak{h}_{\mathfrak{a}\oplus\mathfrak{n}}=\mathfrak{w}$, *with* $\mathfrak{w}$ *a subspace of* $\mathfrak{g}_\alpha$.

Here $\mathfrak{h}$ is in the hypotheses of Proposition 5.3, and it readily follows from Lemma 5.4 that this case is not possible.

Case 2: $\mathfrak{h}_{\mathfrak{a}\oplus\mathfrak{n}}=\mathbb{R}(B+X)\oplus\mathfrak{w}$, *with* $\mathfrak{w}$ *a subspace of* $\mathfrak{g}_\alpha$, *and* $X\in\mathfrak{g}_\alpha\ominus\mathfrak{w}$.

Assume first that $X\neq 0$. Then, $\nu_o(H\cdot o)=\mathbb{R}(-\|X\|^2B+(1-\theta)X)\oplus(1-\theta)(\mathfrak{g}_\alpha\ominus\mathfrak{w})\oplus\mathfrak{p}_{2\alpha}$. Let Σ be a section through o. Since $T_o\Sigma\subset\nu_o(H\cdot o)$, $[\mathfrak{q},-\|X\|^2B+(1-\theta)X]\subset\mathfrak{p}_\alpha$, $[\mathfrak{q},\mathfrak{p}_{2\alpha}]=0$, and $[\mathfrak{q},\mathfrak{p}_\alpha]\subset\mathfrak{p}_\alpha$, we get that $[\mathfrak{q},T_o\Sigma]$ is orthogonal to $\mathfrak{a}$ and $\mathfrak{p}_{2\alpha}$. As $\nu_o(H\cdot o)=T_o\Sigma\oplus[\mathfrak{q},T_o\Sigma]$ (orthogonal direct sum) by Proposition 2.3, we readily get that $\mathfrak{p}_{2\alpha}\subset T_o\Sigma$. Moreover, let $T\in\mathfrak{h}_{\mathfrak{k}_0}$ be such that $T+B+X\in\mathfrak{h}$; then $T+B+X$ is orthogonal to $[\mathfrak{q},T_o\Sigma]$, and since $[\mathfrak{q},T_o\Sigma]\subset\mathfrak{p}_\alpha$ we obtain that X is orthogonal to $[\mathfrak{q},T_o\Sigma]$. The fact that the direct sum $\nu_o(H\cdot o)=T_o\Sigma\oplus[\mathfrak{q},T_o\Sigma]$ is orthogonal implies that $-\|X\|^2B+(1-\theta)X\in T_o\Sigma$. However, since $T_o\Sigma$ is totally real we have

$$0=\langle i(-\|X\|^2B+(1-\theta)X),(1-\theta)Z\rangle=\langle -\frac{1}{2}\|X\|^2(1-\theta)Z+(1-\theta)JX,(1-\theta)Z\rangle=-2\|X\|^2,$$

which is not possible because $X\neq 0$.

Therefore we must have $X=0$, and thus $\mathfrak{h}_{\mathfrak{a}\oplus\mathfrak{n}}=\mathfrak{a}\oplus\mathfrak{w}$. Note that the fact that $\mathfrak{h}$ is a subalgebra of $\mathfrak{k}_0\oplus\mathfrak{a}\oplus\mathfrak{n}$ implies that $\mathfrak{w}$ is a totally real subspace of $\mathfrak{g}_\alpha$. We are now in the hypotheses of Proposition 5.3 and, as shown in the proof of Lemma 5.7, $\mathfrak{w}=0$. We conclude that the action of H is orbit equivalent to the action of the group $\hat{H}$ whose Lie algebra is $\hat{\mathfrak{h}}=\mathfrak{q}\oplus\mathfrak{a}$. This corresponds to Theorem 5.1(a).

Case 3: $\mathfrak{h}_{\mathfrak{a}\oplus\mathfrak{n}}=\mathbb{R}(B+X+xZ)\oplus\mathfrak{w}$, *with* $\mathfrak{w}$ *a subspace of* $\mathfrak{g}_\alpha$, $X\in\mathfrak{g}_\alpha\ominus\mathfrak{w}$, *and* $x\in\mathbb{R}$, $x\neq 0$.

Let $g=\operatorname{Exp}(xZ)\in G$, and let $T+r(B+X+xZ)+V$ be a generic element of $\mathfrak{h}$, with $V\in\mathfrak{w}$, $r\in\mathbb{R}$. Clearly, since $g\in AN$ we have $\operatorname{Ad}(g)(\mathfrak{h})\subset\mathfrak{k}_0\oplus\mathfrak{a}\oplus\mathfrak{n}$. Then, it is easy to obtain

$$\operatorname{Ad}(g)(T+r(B+X+xZ)+V)=T+r(B+X+xZ)+V-rxZ=T+r(B+X)+V.$$

Hence $(\operatorname{Ad}(g)(\mathfrak{h}))_{\mathfrak{a}\oplus\mathfrak{n}}=\mathbb{R}(B+X)\oplus\mathfrak{w}$, and $\operatorname{Ad}(g)(\mathfrak{q})=\mathfrak{q}$. Since Q is a maximal compact subgroup of $I_g(H)=gHg^{-1}$, and the orthogonal projection of the Lie algebra of $I_g(H)$ onto $\mathfrak{a}\oplus\mathfrak{n}$ is $\mathbb{R}(B+X)\oplus\mathfrak{w}$, the new group $I_g(H)$ satisfies the conditions of Case 2.

Therefore, the action of H is orbit equivalent to the action of the group $\hat{H}$ whose Lie algebra is $\hat{\mathfrak{h}} = \mathfrak{q} \oplus \mathfrak{a}$. This also corresponds to Theorem 5.1(a).

Case 4: $\mathfrak{h}_{\mathfrak{a}\oplus\mathfrak{n}} = \mathfrak{w} \oplus \mathbb{R}(Y+Z)$*, with* $\mathfrak{w}$ *a subspace of* $\mathfrak{g}_\alpha$*, and* $Y \in \mathfrak{g}_\alpha \ominus \mathfrak{w}$*.*

Assume that $Y \neq 0$. Then, $\nu_o(H\cdot o) = \mathfrak{a}\oplus(1-\theta)(\mathfrak{g}_\alpha\ominus\mathfrak{w})\oplus\mathbb{R}(2(1-\theta)Y-\|Y\|^2(1-\theta)Z)$. Let Σ be a section through o. Then, by Proposition 2.3 we have $\nu_o(H\cdot o) = T_o\Sigma \oplus [\mathfrak{q}, T_o\Sigma]$ (orthogonal direct sum). Since $[\mathfrak{q}, 2(1-\theta)Y - \|Y\|^2(1-\theta)Z] \subset \mathfrak{p}_\alpha$, $[\mathfrak{q},\mathfrak{a}] = 0$, and $[\mathfrak{q},\mathfrak{p}_\alpha] \subset \mathfrak{p}_\alpha$, we get that $[\mathfrak{q}, T_o\Sigma]$ is orthogonal to $\mathfrak{a}$ and $\mathfrak{p}_{2\alpha}$. Then, $\mathfrak{a} \subset T_o\Sigma$. On the other hand, if $T \in \mathfrak{h}_{\mathfrak{k}_0}$ is such that $T+Y+Z \in \mathfrak{h}$, then $T+Y+Z$ is orthogonal to $[\mathfrak{q}, T_o\Sigma] \subset \nu_o(H\cdot o)$, and since $[\mathfrak{q}, T_o\Sigma] \subset \mathfrak{p}_\alpha$ we also obtain that Y is orthogonal to $[\mathfrak{q}, T_o\Sigma]$. Thus, $2(1-\theta)Y - \|Y\|^2(1-\theta)Z \in T_o\Sigma$. But, since $T_o\Sigma$ is totally real, we get

$$0 = \langle B, i(2(1-\theta)Y - \|Y\|^2(1-\theta)Z)\rangle = \langle B, 2(1-\theta)JY + 2\|Y\|^2 B\rangle = 2\|Y\|^2,$$

which contradicts $Y \neq 0$.

Therefore we have $Y = 0$, and thus, $\mathfrak{h}_{\mathfrak{a}\oplus\mathfrak{n}} = \mathfrak{w} \oplus \mathfrak{g}_{2\alpha}$. We are now in the hypotheses of Proposition 5.3, and we conclude that the action of H is orbit equivalent to the action of the connected subgroup $\hat{H}$ of the isometry group of $\mathbb{C}H^n$ whose Lie algebra is $\hat{\mathfrak{h}} = \mathfrak{q}\oplus\mathfrak{w}\oplus\mathfrak{g}_{2\alpha}$, with $\mathfrak{w}$ a subspace of $\mathfrak{g}_\alpha$. This corresponds to Theorem 5.1(c).

Case 5: $\mathfrak{h}_{\mathfrak{a}\oplus\mathfrak{n}} = \mathbb{R}(B+X)\oplus\mathfrak{w}\oplus\mathbb{R}(Y+Z)$*, with* $\mathfrak{w} \subset \mathfrak{g}_\alpha$*, and* X*,* $Y \in \mathfrak{g}_\alpha \ominus \mathfrak{w}$*.*

This final possibility is more involved.

Our first aim is to show that $Y = 0$. So, assume for the moment that $Y \neq 0$.

Lemma 5.9. *We have* $X = \gamma Y + \frac{2}{\|Y\|^2} JY$*, with* $\gamma \in \mathbb{R}$*.*

Proof. Assume that X and Y are linearly dependent, that is, $X = \lambda Y$, with $\lambda \in \mathbb{R}$. Then, $\mathfrak{h}_{\mathfrak{a}\oplus\mathfrak{n}} = \mathbb{R}(B+\lambda Y)\oplus\mathfrak{w}\oplus\mathbb{R}(Y+Z)$, and there exist T, $S \in \mathfrak{h}_{\mathfrak{k}_0}$ such that $T+B+\lambda Y$, $S+Y+Z \in \mathfrak{h}$. Then,

$$[T,S]+[T,Y]-\lambda[S,Y]+\frac{1}{2}Y+Z = [T+B+\lambda Y, S+Y+Z] \in \mathfrak{h}.$$

Since $[T,Y]-\lambda[S,Y] \in \mathfrak{g}_\alpha \ominus \mathbb{R}Y$ by the skew-symmetry of the elements of $\operatorname{ad}(\mathfrak{k}_0)$, we get $\frac{1}{2}Y + Z \in \mathfrak{h}_{\mathfrak{a}\oplus\mathfrak{n}}$, which is not possible.

Therefore, we can assume that X and Y are linearly independent vectors of $\mathfrak{g}_\alpha$. In particular, $X \neq 0$. Take and fix for the rest of the calculations T, $S \in \mathfrak{h}_{\mathfrak{k}_0}$ such that $T+B+X$, $S+Y+Z \in \mathfrak{h}$.

In this case, the normal space to the orbit through the origin o can be written as

$$\begin{aligned}\nu_o(H\cdot o) = {}&\mathbb{R}(-\|X\|^2 B + (1-\theta)X - \frac{1}{2}\langle X,Y\rangle(1-\theta)Z) \oplus (\mathfrak{p}_\alpha \ominus (1-\theta)(\mathfrak{w}\oplus\mathbb{R}X\oplus\mathbb{R}Y))\\ &\oplus \mathbb{R}(-\langle X,Y\rangle B + (1-\theta)Y - \frac{1}{2}\|Y\|^2(1-\theta)Z).\end{aligned}$$

Let Σ be a section of the action of H on $\mathbb{C}H^n$ through the point $o \in \mathbb{C}H^n$. By Proposition 2.3 we have $\nu_o(H\cdot o) = T_o\Sigma \oplus [\mathfrak{q}, T_o\Sigma]$ (orthogonal direct sum). In particular the vectors $T+B+X$ and $S+Y+Z$ are orthogonal to $[\mathfrak{q}, T_o\Sigma] \subset \mathfrak{p}_\alpha$ (because $[\mathfrak{k}_0,\mathfrak{a}] = [\mathfrak{k}_0,\mathfrak{g}_{2\alpha}] = 0$). This implies that X and Y are already orthogonal to $[\mathfrak{q}, T_o\Sigma]$, and thus, so are $-\|X\|^2B+(1-\theta)X-\frac{1}{2}\langle X,Y\rangle(1-\theta)Z$ and $-\langle X,Y\rangle B+(1-\theta)Y-\frac{1}{2}\|Y\|^2(1-\theta)Z$. Hence, they are in $T_o\Sigma$ and we can write

$$\begin{aligned}T_o\Sigma=\mathbb{R}(-\|X\|^2B+(1-\theta)X-\frac{1}{2}\langle X,Y\rangle(1-\theta)Z)\\ \oplus(1-\theta)\mathfrak{s}\oplus\mathbb{R}(-\langle X,Y\rangle B+(1-\theta)Y-\frac{1}{2}\|Y\|^2(1-\theta)Z),\end{aligned}$$

where $\mathfrak{s}\subset\mathfrak{g}_\alpha\ominus\mathfrak{w}$ is totally real, and $\mathbb{C}X\oplus\mathbb{C}Y$ is orthogonal to $\mathfrak{s}$ (because sections are totally real). The fact that $T_o\Sigma$ is totally real also implies

$$\begin{aligned}0&=\langle i(-\|X\|^2B+(1-\theta)(X-\frac{1}{2}\langle X,Y\rangle Z)),-\langle X,Y\rangle B+(1-\theta)(Y-\frac{1}{2}\|Y\|^2Z)\rangle\\ &=\langle(1-\theta)(-\frac{1}{2}\|X\|^2Z+JX)+\langle X,Y\rangle B),-\langle X,Y\rangle B+(1-\theta)(Y-\frac{1}{2}\|Y\|^2Z)\rangle\\ &=\|X\|^2\|Y\|^2-\langle X,Y\rangle^2+2\langle JX,Y\rangle.\end{aligned}\tag{4}$$

Now, using Lemma 2.1(a), and (4), we compute

$$\begin{aligned}&[-\|X\|^2B+(1-\theta)(X-\frac{1}{2}\langle X,Y\rangle Z),-\langle X,Y\rangle B+(1-\theta)(Y-\frac{1}{2}\|Y\|^2Z)]\\ &\quad=\frac{1}{2}(1+\theta)\big(-2[\theta X,Y]+\langle X,Y\rangle X-\|X\|^2Y-\|Y\|^2JX+\langle X,Y\rangle JY-\langle JX,Y\rangle Z\big).\end{aligned}$$

This vector is in $[T_o\Sigma,T_o\Sigma]$, which is orthogonal to $\mathfrak{h}$ by Proposition 2.3, so taking inner product with $S+Y+Z$, and using Lemma 2.1(b) and (4), we get $0=-2\langle[S,X],Y\rangle-\frac{1}{2}\|Y\|^2\langle JX,Y\rangle$, which implies

$$\langle[S,X],Y\rangle=-\frac{1}{4}\|Y\|^2\langle JX,Y\rangle.\tag{5}$$

We also have

$$[T+B+X,S+Y+Z]=[T,S]+[T,Y]-[S,X]+\frac{1}{2}Y+\big(1+\frac{1}{2}\langle JX,Y\rangle\big)Z,$$

which is in $\mathfrak{h}$, so taking inner product with $-\langle X,Y\rangle B+(1-\theta)(Y-\frac{1}{2}\|Y\|^2Z)$, and using (5), we obtain

$$0=-\langle[S,X],Y\rangle+\frac{1}{2}\|Y\|^2-\|Y\|^2\big(1+\frac{1}{2}\langle JX,Y\rangle\big)=-\frac{1}{2}\|Y\|^2\big(1+\frac{1}{2}\langle JX,Y\rangle\big).$$

Since $Y\neq 0$, we get $\langle JX,Y\rangle=-2$ and thus (4) can be written as

$$\|X\|^2\|Y\|^2-\langle X,Y\rangle^2=4=\langle JX,Y\rangle^2.$$

Now put $X=\gamma Y+\delta JY+E$ with E orthogonal to $\mathbb{C}Y$, and $\gamma,\delta\in\mathbb{R}$. Then, the previous equation reads $\|E\|^2\|Y\|^2=0$, which yields $E=0$. This implies the result. ☐

Therefore the situation now is $\mathfrak{h}_{\mathfrak{a}\oplus\mathfrak{n}}=\mathbb{R}(B+\gamma Y+\frac{2}{\|Y\|^2}JY)\oplus\mathfrak{w}\oplus\mathbb{R}(Y+Z)$, with $\mathbb{C}Y\subset\mathfrak{g}_\alpha\ominus\mathfrak{w}$. The normal space can be rewritten as

$$\begin{aligned}\nu_o(H\cdot o)=\mathbb{R}(-2B+(1-\theta)JY)\oplus(\mathfrak{p}_\alpha\ominus(1-\theta)(\mathfrak{w}\oplus\mathbb{C}Y))\\ \oplus\mathbb{R}(-\gamma\|Y\|^2B+(1-\theta)Y-\frac{1}{2}\|Y\|^2(1-\theta)Z),\end{aligned}$$

and arguing as above, if Σ is a section through o, then

$$T_o\Sigma = \mathbb{R}(-2B+(1-\theta)JY)\oplus(1-\theta)\mathfrak{s}\oplus\mathbb{R}(-\gamma\|Y\|^2B+(1-\theta)Y-\frac{1}{2}\|Y\|^2(1-\theta)Z), \tag{6}$$

where $\mathfrak{s}\subset\mathfrak{g}_\alpha\ominus(\mathfrak{w}\oplus\mathbb{C}Y)$ is a totally real subspace of $\mathfrak{g}_\alpha$.

Lemma 5.10. *If $S\in\mathfrak{h}_{\mathfrak{k}_0}$ is such that $S+Y+Z\in\mathfrak{h}$ then $[S,JY]=\frac{1}{4}\|Y\|^2Y$.*

Proof. First of all, by the properties of root systems and the skew-symmetry of the elements of $\operatorname{ad}(\mathfrak{k}_0)$, we have $[S,JY]\in\mathfrak{g}_\alpha\ominus\mathbb{R}JY$.

Lemma 2.1(a) yields

$$\begin{aligned}&[-2B+(1-\theta)JY,-\gamma\|Y\|^2B+(1-\theta)(Y-\frac{1}{2}\|Y\|^2Z)]\\&\qquad=(1+\theta)\Big(-[\theta JY,Y]+\Big(\frac{1}{2}\|Y\|^2-1\Big)Y+\frac{\gamma}{2}\|Y\|^2JY+\frac{1}{2}\|Y\|^2Z\Big),\end{aligned}\tag{7}$$

which is a vector in $[T_o\Sigma,T_o\Sigma]$.

Take $U\in\mathfrak{w}$, and let $T_U\in\mathfrak{h}_{\mathfrak{k}_0}$ be such that $T_U+U\in\mathfrak{h}$. Taking inner product with (7) and using Lemma 2.1(b) we get $0=2\langle[T_U,JY],Y\rangle$. Using this equality and since $\mathfrak{h}$ is a Lie subalgebra, we now have

$$0=\langle[S+Y+Z,T_U+U],-2B+(1-\theta)JY\rangle=\langle[S,T_U]+[S,U]-[T_U,Y],JY\rangle=\langle[S,U],JY\rangle,$$

and since $U\in\mathfrak{w}$ is arbitrary, $[S,JY]\in\mathfrak{g}_\alpha\ominus(\mathfrak{w}\oplus\mathbb{R}JY)$.

Let $\xi\in\mathfrak{s}$. Proposition 2.3 implies

$$0=\langle S+Y+Z,[-2B+(1-\theta)JY,(1-\theta)\xi]\rangle=-\langle S,(1+\theta)[\theta JY,\xi]\rangle=-2\langle[S,JY],\xi\rangle.$$

Let $\eta\in\mathfrak{g}_\alpha\ominus(\mathfrak{w}\oplus\mathbb{C}Y)$ be an arbitrary vector. Since $\operatorname{Ad}(Q)(T_o\Sigma)=\nu_o(H\cdot o)$ by Proposition 2.3, we can conjugate the section Σ in such a way that $\eta\in\mathfrak{s}$. (Note that $-2B+(1-\theta)JY$ and $-\gamma\|Y\|^2B+(1-\theta)Y-\frac{1}{2}\|Y\|^2(1-\theta)Z$ always belong to $T_o\Sigma$ by (6).) Hence, the equation above shows that $[S,JY]$ is orthogonal to $\mathfrak{g}_\alpha\ominus(\mathfrak{w}\oplus\mathbb{C}Y)$. Altogether this implies $[S,JY]\in\mathbb{R}Y$.

Finally, taking inner product of (7) with $S+Y+Z\in\mathfrak{h}$ we get, using Lemma 2.1(a), $0=2\langle[S,Y],JY\rangle+\frac{1}{2}\|Y\|^4$, and hence $[S,JY]=\frac{1}{4}\|Y\|^2Y$ as we wanted. ∎

We define $g=\operatorname{Exp}(-4JY/\|Y\|^2)$. Recall that the Lie algebra of the isotropy group of H at $g(o)$ is $\mathfrak{h}_{g(o)}=\operatorname{Ad}(g)(\mathfrak{k})\cap\mathfrak{h}=\mathfrak{q}\cap\ker\operatorname{ad}(JY)$, according to Lemma 5.5. Let $S\in\mathfrak{h}_{\mathfrak{k}_0}$ be such that $S+Y+Z\in\mathfrak{h}$. Then, Lemma 5.10 yields

$$\operatorname{Ad}(g)(S)=S-\frac{4}{\|Y\|^2}[JY,S]+\frac{8}{\|Y\|^4}[JY,[JY,S]]=S+Y+Z\in\operatorname{Ad}(g)(\mathfrak{k})\cap\mathfrak{h}.$$

However, it is clear that $S+Y+Z\notin\mathfrak{q}\cap\ker\operatorname{ad}(JY)$, which gives a contradiction.

Therefore we have proved that $Y=0$. Thus $\mathfrak{h}_{\mathfrak{a}\oplus\mathfrak{n}}=\mathbb{R}(B+X)\oplus\mathfrak{w}\oplus\mathfrak{g}_{2\alpha}$. If $X=0$ then $\mathfrak{h}_{\mathfrak{a}\oplus\mathfrak{n}}=\mathfrak{a}\oplus\mathfrak{w}\oplus\mathfrak{g}_{2\alpha}$, and we are under the hypotheses of Proposition 5.3, which implies that the action of H is orbit equivalent to the action of the group $\hat{H}$ whose Lie algebra is $\hat{\mathfrak{h}}=\mathfrak{q}\oplus\mathfrak{a}\oplus\mathfrak{w}\oplus\mathfrak{g}_{2\alpha}$. This corresponds to Theorem 5.1(b).

For the rest of this case we assume $X\neq0$. Note that the normal space to the orbit through o is $\nu_o(H\cdot o)=\mathbb{R}(-\|X\|^2B+(1-\theta)X)\oplus(\mathfrak{p}_\alpha\ominus(1-\theta)(\mathfrak{w}\oplus\mathbb{R}X))$. If Σ is a section through o, since $\nu_o(H\cdot o) = T_o\Sigma \oplus [\mathfrak{q}, T_o\Sigma]$ (orthogonal direct sum), and $[\mathfrak{q}, T_o\Sigma] \subset \mathfrak{p}_\alpha$, it is easy to deduce, as in previous cases, that

$$T_o\Sigma = \mathbb{R}(-\|X\|^2 B + (1-\theta)X) \oplus (1-\theta)\mathfrak{s},$$

where $\mathbb{R}X \oplus \mathfrak{s}$ is a real subspace of $\mathfrak{g}_\alpha$.

We define $g = \operatorname{Exp}(2X)$. We will show $(\operatorname{Ad}(g)(\mathfrak{h}))_{\mathfrak{a}\oplus\mathfrak{n}} = \mathfrak{a} \oplus \mathfrak{w} \oplus \mathfrak{g}_{2\alpha}$ and $\operatorname{Ad}(g)(\mathfrak{q}) = \mathfrak{q}$, which will allow us to apply Proposition 5.3. From now on we take $T \in \mathfrak{h}_{\mathfrak{k}_0}$ such that $T + B + X \in \mathfrak{h}$.

Let $S \in \mathfrak{q}$. Then $[S, T] + [S, X] = [S, T + B + X] \in \mathfrak{h}$, and thus $[S, X] \in \mathfrak{w}$. Now let $U \in \mathfrak{w}$ be an arbitrary vector, and let $S_U \in \mathfrak{h}_{\mathfrak{k}_0}$ such that $S_U + U \in \mathfrak{h}$. We have $0 = \langle [S, S_U + U], -\|X\|^2 B + (1-\theta)X\rangle = -\langle [S, X], U\rangle$, which together with the previous assertion implies $[S, X] = 0$. Then $\operatorname{Ad}(g)(\mathfrak{q}) = \mathfrak{q}$. In particular this implies that Q is a maximal compact subgroup of $I_g(H) = gHg^{-1}$.

Now we calculate $[T, X]$. Let $U \in \mathfrak{w}$ and $S_U \in \mathfrak{h}_{\mathfrak{k}_0}$ such that $S_U + U \in \mathfrak{h}$. Then, by the skew-symmetry of the elements of $\operatorname{ad}(\mathfrak{k}_0)$ we have $0 = \langle [T+B+X, S_U+U], -\|X\|^2 B + (1-\theta)X\rangle = -\langle [T, X], U\rangle$, so $[T, X] \in \mathfrak{g}_\alpha \ominus \mathfrak{w}$. Let now $\xi \in \mathfrak{s}$. By Proposition 2.3 we get, using Lemma 2.1(b), $0 = \langle T + B + X, [-\|X\|^2 B + (1-\theta)X, (1-\theta)\xi]\rangle = -\langle T, (1+\theta)[\theta X, \xi]\rangle = -2\langle [T, X], \xi\rangle$. Using again Proposition 2.3 we have $\nu_o(H\cdot o) = \operatorname{Ad}(Q)(T_o\Sigma)$, and thus, for any $\eta \in \mathfrak{g}_\alpha \ominus (\mathfrak{w} \oplus \mathbb{R}X)$ we can find a section through o such that $(1-\theta)\eta \in T_o\Sigma$ (note that $-\|X\|^2 B + (1-\theta)X \in T_o\Sigma$ for any section). Hence the previous argument shows $\langle [T, X], \eta\rangle = 0$, and altogether this means $[T, X] = 0$. Therefore, $\operatorname{Ad}(g)(T+B+X) = T+B$, so the projection of this vector onto $\mathfrak{a} \oplus \mathfrak{n}$ is in $\mathfrak{a} \subset \mathfrak{a} \oplus \mathfrak{w} \oplus \mathfrak{g}_{2\alpha}$.

Fix $U \in \mathfrak{w}$ and $S_U \in \mathfrak{h}_{\mathfrak{k}_0}$ such that $S_U + U \in \mathfrak{h}$. We calculate $[S_U, X]$. For any $\xi \in \mathfrak{s}$, by Proposition 2.3 and Lemma 2.1(b), we get $0 = \langle S_U + U, [-\|X\|^2 B + (1-\theta)X, (1-\theta)\xi]\rangle = -2\langle [S_U, X], \xi\rangle$. As in the previous paragraph, one can argue that ξ can be taken arbitrarily in $\mathfrak{g}_\alpha \ominus (\mathfrak{w} \oplus \mathbb{R}X)$ by changing the tangent space to the section, if necessary, by an element of $\operatorname{Ad}(Q)$. Hence $[S_U, X] \in \mathfrak{w}$, which yields $\operatorname{Ad}(g)(S_U+U) = S_U + U - 2[S_U, X] + \frac{1}{2}(\langle JX, U\rangle - 2\langle JX, [S_U, X]\rangle)Z$, and thus, its projection onto $\mathfrak{a} \oplus \mathfrak{n}$ belongs to $\mathfrak{a} \oplus \mathfrak{w} \oplus \mathfrak{g}_{2\alpha}$.

Finally, let $S_Z \in \mathfrak{h}_{\mathfrak{k}_0}$ such that $S_Z + Z \in \mathfrak{h}$. For each $\xi \in \mathfrak{s}$ we obtain $0 = \langle S_Z + Z, [-\|X\|^2 B + (1-\theta)X, (1-\theta)\xi]\rangle = -2\langle [S_Z, X], \xi\rangle$, and since ξ can be taken to be in $\mathfrak{g}_\alpha \ominus (\mathfrak{w} \oplus \mathbb{R}X)$ by a suitable conjugation of the section by an element in $\operatorname{Ad}(Q)$, we deduce $[S_Z, X] \in \mathfrak{w}$. Hence, $\operatorname{Ad}(g)(S_Z + Z) = S_Z - 2[S_Z, X] + (1 - \langle JX, [S_Z, X]\rangle)Z$, and the orthogonal projection of this vector onto $\mathfrak{a} \oplus \mathfrak{n}$ belongs to $\mathfrak{a} \oplus \mathfrak{w} \oplus \mathfrak{g}_{2\alpha}$.

These last calculations show that $(\operatorname{Ad}(g)(\mathfrak{h}))_{\mathfrak{a}\oplus\mathfrak{n}} \subset \mathfrak{a} \oplus \mathfrak{w} \oplus \mathfrak{g}_{2\alpha}$. Since $g \in AN$ normalizes $\mathfrak{k}_0 \oplus \mathfrak{a} \oplus \mathfrak{n}$, we have that $\operatorname{Ad}(g)(\mathfrak{h}) \subset \mathfrak{k}_0 \oplus \mathfrak{a} \oplus \mathfrak{n}$. Then the kernel of the projection of $\operatorname{Ad}(g)(\mathfrak{h})$ onto $\mathfrak{a} \oplus \mathfrak{n}$ is precisely $\operatorname{Ad}(g)(\mathfrak{h}) \cap \mathfrak{k}_0$, which is a compact subalgebra of $\operatorname{Ad}(g)(\mathfrak{h})$ containing $\mathfrak{q} = \operatorname{Ad}(g)(\mathfrak{q})$. By the maximality of $\mathfrak{q}$ we get that $\operatorname{Ad}(g)(\mathfrak{h}) \cap \mathfrak{k}_0 = \mathfrak{q}$. But then by elementary linear algebra

$$\begin{aligned}\dim(\operatorname{Ad}(g)\mathfrak{h})_{\mathfrak{a}\oplus\mathfrak{n}} &= \dim \operatorname{Ad}(g)(\mathfrak{h}) - \dim(\operatorname{Ad}(g)(\mathfrak{h}) \cap \mathfrak{k}_0) \\ &= \dim \mathfrak{h} - \dim \mathfrak{q} = \dim \mathfrak{h}_{\mathfrak{a}\oplus\mathfrak{n}} = \dim(\mathfrak{a} \oplus \mathfrak{w} \oplus \mathfrak{g}_{2\alpha}).\end{aligned}$$

All in all we have shown that the Lie algebra $\operatorname{Ad}(g)(\mathfrak{h})$ of $I_g(H) = gHg^{-1}$ satisfies $(\operatorname{Ad}(g)(\mathfrak{h}))_{\mathfrak{a}\oplus\mathfrak{n}} = \mathfrak{a} \oplus \mathfrak{w} \oplus \mathfrak{g}_{2\alpha}$, and that Q is a maximal compact subgroup of $I_g(H)$.

Therefore, we can apply Proposition 5.3 to $I_g(H)$. This implies that the action of H on $\mathbb{C}H^n$ is orbit equivalent to the action of the group $\hat{H}$ whose Lie algebra is $\hat{\mathfrak{h}} = \mathfrak{q}\oplus\mathfrak{a}\oplus\mathfrak{w}\oplus\mathfrak{g}_{2\alpha}$. This corresponds to Theorem 5.1(b).

Altogether, we have concluded the proof of Theorem 5.1.

6. Proof of the main results

In this section we conclude the proof of Theorems A and B using the results of Sections 4 and 5.

Proof of Theorem A. The actions described in part (i) are polar by virtue of Lemma 4.1 and Theorem 4.2, whereas the polarity of the actions in part (ii) follows from Theorem 3.1.

An action of a subgroup H of the isometry group $I(M)$ of a Riemannian manifold M is proper if and only if H is a closed subgroup of $I(M)$. Hence we may assume $H \subset SU(1,n)$ is closed. Since the polarity of the action depends only on the Lie algebra of H by Proposition 2.3, we may assume that H is connected.

Thus, let H be a connected closed subgroup of $SU(1,n)$ acting polarly on $\mathbb{C}H^n$. The Lie algebra $\mathfrak{h}$ of H is contained in a maximal subalgebra of $\mathfrak{su}(1,n)$. By [29, Theorem 1.9, Ch. 6], the maximal nonsemisimple subalgebras of a semisimple real Lie algebra are parabolic or coincide with the centralizer of a pseudotoric subalgebra. (A subalgebra $\mathfrak{t} \subset \mathfrak{g}$ is called *pseudotoric* if $\operatorname{Exp}\operatorname{ad}\mathfrak{t} \subset \operatorname{Int}\mathfrak{g}$ is a torus.) The maximal subalgebras of simple real Lie algebras which are centralizers of pseudotoric subalgebras have been classified in [32]. However, it is easy to determine them in the case of $\mathfrak{su}(1,n)$. Indeed, it follows from [29, Theorem 3.3, Ch. 4] that for all pseudotoric subalgebras $\mathfrak{t}$ of $\mathfrak{su}(1,n)$ there is an element $g \in SU(1,n)$ such that $\operatorname{Ad}(g)\mathfrak{t}$ is contained in the subalgebra comprised of all diagonal matrices in $\mathfrak{su}(1,n)$. Since we are interested in maximal subalgebras which are centralizers of pseudotoric subalgebras $\mathfrak{t}$ we may restrict ourselves to one-dimensional pseudotoric subalgebras $\mathfrak{t}$. For such a subalgebra we have $\mathfrak{t} = \mathbb{R}\operatorname{diag}(it_0,\dots,it_n)$, for $t_0,\dots,t_n \in \mathbb{R}$ such that $t_0+\dots+t_n = 0$. The centralizers of such $\mathfrak{t}$ are the subalgebras of the form $\mathfrak{s}(\mathfrak{u}(1,n_1)\oplus\mathfrak{u}(n_2)\oplus\dots\oplus\mathfrak{u}(n_\ell))$ where $n_1+\dots+n_\ell = n$. In particular, any maximal connected subgroup of $SU(1,n)$ whose Lie algebra is the centralizer of a pseudotoric subalgebra is conjugate to one of the maximal subgroups $S(U(1,k)U(n-k))$, $k = 0,\dots,n-1$.

First, let us assume H is contained (after conjugation) in a maximal subgroup of the form $S(U(1,k)U(n-k))$ or in a semisimple maximal subgroup of $SU(1,n)$. In both cases, the action of H on $\mathbb{C}H^n$ leaves a totally geodesic submanifold invariant; this follows from the Karpelevich-Mostow Theorem [19], [27] (it is obvious in the first case). This situation has been studied in Section 4.

If the action of H leaves a totally geodesic $\mathbb{R}H^n$ invariant, then Theorem 4.2 applies and the H-action is orbit equivalent to the cohomogeneity one action of $SO(1,n)$. This corresponds to case (i) with $k = n$ in Theorem A. If the action of H leaves a totally geodesic $\mathbb{R}H^k$ invariant, with $k < n$, then it also leaves a totally geodesic $\mathbb{C}H^k$ invariant.

Let then k be the smallest complex dimension of a totally geodesic complex hyperbolic subspace left invariant by the H-action. If $k = 0$, then the H-action has a fixed point. In this case, it follows from [16] that H is a subgroup of $S(U(1)U(n)) \cong U(n)$ that corresponds to a polar action on $\mathbb{C}P^{n-1}$, and therefore is induced by the isotropy representation of a Hermitian symmetric space. This corresponds to case (i) with $k=0$ in Theorem A.

Let us assume from now on that $k \geq 1$. Lemma 4.1 guarantees that the H-action is orbit equivalent to the product action of a closed subgroup H_1 of $SU(1,k)$ acting polarly on $\mathbb{C}H^k$ times a closed subgroup H_2 of $U(n-k)$ acting polarly (and with a fixed point) on $\mathbb{C}H^{n-k}$. By assumption, the H_1-action on $\mathbb{C}H^k$ does not leave any totally geodesic $\mathbb{C}H^l$ or $\mathbb{R}H^l$ with $l<k$ invariant. Hence, either the H_1-action on $\mathbb{C}H^k$ is orbit equivalent to the $SO(1,k)$-action on $\mathbb{C}H^k$, or H_1 is contained in a maximal parabolic subgroup of $SU(1,k)$. The first case corresponds to part (i) with $k \in \{1,\dots,n\}$. Note that for $Q=H_2$, the Q-action on $\mathbb{C}H^{n-k}$ is determined by its slice representation at the fixed point, so Q acts polarly with a totally real section on $T_o\mathbb{C}H^{n-k} \cong \mathbb{C}^{n-k}$.

Let us consider the second case, that is, H_1 is contained in a maximal parabolic subgroup of $SU(1,k)$, $k \in \{1,\dots,n\}$. As explained at the beginning of Section 5, we may assume $\mathfrak{h}_1 \subset \mathfrak{k}_0^1 \oplus \mathfrak{a} \oplus \mathfrak{g}_\alpha^1 \oplus \mathfrak{g}_{2\alpha}$, where now $\mathfrak{g}_\alpha^1$ is a complex subspace of $\mathfrak{g}_\alpha$ with complex dimension $k-1$, and $\mathfrak{k}_0^1 \cong \mathfrak{u}(k-1)$ is the normalizer of $\mathfrak{a}$ in $\mathfrak{k}\cap\mathfrak{su}(1,k)$. It follows that the H_1-action is orbit equivalent to the action of a closed subgroup of $SU(1,k)$ with one of the Lie algebras described in Theorem 5.1: (a) $\mathfrak{q}^1 \oplus \mathfrak{a}$, (b) $\mathfrak{q}^1 \oplus \mathfrak{a} \oplus \mathfrak{w} \oplus \mathfrak{g}_{2\alpha}$, or (c) $\mathfrak{q}^1 \oplus \mathfrak{w} \oplus \mathfrak{g}_{2\alpha}$, where $\mathfrak{w}$ is a real subspace of $\mathfrak{g}_\alpha^1$, and $\mathfrak{q}^1 \subset \mathfrak{k}_0^1$ normalizes $\mathfrak{w}$. Since $H_2 \subset U(n-k)$ acts on $\mathbb{C}H^{n-k}$, we can define $\mathfrak{q}=\mathfrak{q}^1 \oplus \mathfrak{h}_2$, which is a subalgebra of $\mathfrak{k}_0$. Part (a) of Theorem 5.1 is then a particular case of Theorem A(i) for $k=1$, while parts (b) and (c) of Theorem 5.1 correspond to Theorem A(ii), where $\mathfrak{b}=\mathfrak{a}$ and $\mathfrak{b}=0$, respectively. Lemma 2.2, Proposition 2.4 and the fact that the slice representation of a polar action is also polar, guarantee that the action of $\mathfrak{q}$ on the orthogonal complement of $\mathfrak{w}$ in $\mathfrak{g}_\alpha$ is polar with a totally real section. □

Before beginning the proof of Theorem B, we need to calculate the mean curvature vector of the orbits of minimum orbit type.

Lemma 6.1. *Let H be the connected Lie subgroup of $SU(1,n)$ whose Lie algebra is $\mathfrak{h} = \mathbb{R}(aB+X) \oplus \mathfrak{w} \oplus \mathfrak{g}_{2\alpha}$, for some $a \in \mathbb{R}$, $\mathfrak{w}$ subspace of $\mathfrak{g}_\alpha$, and $X \in \mathfrak{g}_\alpha \ominus \mathfrak{w}$, $a \neq 0$, $X \neq 0$. Then, the mean curvature vector of $H \cdot o$ is*

$$\mathcal{H} = \frac{3+\dim\mathfrak{w}}{2(a^2+\|X\|^2)}(\|X\|^2 B - aX).$$

Proof. In order to shorten the notation, let us denote by $\langle\,\cdot\,,\,\cdot\,\rangle$ the metric on AN defined in Section 2. Then, it is well known that the Levi-Civita connection of AN is given by (see for example [4] or [10])

$$\nabla_{aB+U+xZ}(bB+V+yZ) = \left(\frac{1}{2}\langle U,V\rangle + xy\right)B - \frac{1}{2}\left(bU+yJU+xJV\right) + \left(\frac{1}{2}\langle JU,V\rangle - bx\right)Z,$$

where a, b, x, $y \in \mathbb{R}$, U, $V \in \mathfrak{g}_\alpha$, and all vector fields are considered to be left-invariant. The normal space to the orbit $H\cdot o$ is given by the left-invariant distribution $\mathbb{R}\xi\oplus(\mathfrak{g}_\alpha\ominus(\mathfrak{w}\oplus\mathbb{R}X))$, where ξ is the unit vector

$$\xi = \frac{1}{\|X\|\sqrt{a^2+\|X\|^2}}(\|X\|^2 B - aX).$$

Then, the shape operator $\mathcal{S}_\eta$ with respect to a left-invariant normal vector η is given by the equation $\mathcal{S}_\eta V = -(\nabla_V \eta)^\top$, where $(\cdot)^\top$ means orthogonal projection onto $\mathfrak{h}$.

Bearing all this in mind and applying the formula for the Levi-Civita connection above we get:

$$\begin{aligned}
\mathcal{S}_\xi(aB+X) &= \frac{\|X\|}{2\sqrt{a^2+\|X\|^2}}(aB+X),\\
\mathcal{S}_\xi(W) &= \frac{\|X\|}{2\sqrt{a^2+\|X\|^2}}\Big(W + a\frac{\langle JW, X\rangle}{\|X\|^2}Z\Big), \text{ for each } W\in\mathfrak{w},\\
\mathcal{S}_\xi(Z) &= \frac{\|X\|}{2\sqrt{a^2+\|X\|^2}}\Big(-\frac{a}{\|X\|^2}(JX)^\top + 2Z\Big).
\end{aligned}$$

This implies

$$\operatorname{tr}\mathcal{S}_\xi = \frac{(3+\dim\mathfrak{w})\|X\|}{2\sqrt{a^2+\|X\|^2}}.$$

On the other hand, if $U\in\mathfrak{g}_\alpha\ominus(\mathfrak{w}\oplus\mathbb{R}X)$ we have

$$\begin{aligned}
\mathcal{S}_U(aB+X) &= \frac{1}{2}\langle JU, X\rangle Z,\\
\mathcal{S}_U(W) &= \frac{1}{2}\langle JU, W\rangle Z, \text{ for each } W\in\mathfrak{w},\\
\mathcal{S}_U(Z) &= \frac{1}{2}(JU)^\top.
\end{aligned}$$

Hence $\operatorname{tr}\mathcal{S}_U = 0$.

Altogether we have proved the lemma. □

We can also obtain easily the following result.

Corollary 6.2. *If* $\mathfrak{h} = \mathfrak{b}\oplus\mathfrak{w}\oplus\mathfrak{g}_{2\alpha}$, *with* $\mathfrak{b}\in\{0,\mathfrak{a}\}$ *and* $\mathfrak{w}$ *a subspace of* $\mathfrak{g}_\alpha$, *then the mean curvature vector of* $H\cdot o$ *is*

$$\mathcal{H} = \begin{cases} 0, & \text{if } \mathfrak{b}=\mathfrak{a},\\ \frac{1}{2}(2+\dim\mathfrak{w})B, & \text{if } \mathfrak{b}=0. \end{cases}$$

Now we finish the proof of Theorem B.

Proof of Theorem B. In the following we consider some orbits of *minimum orbit type*, as in Proposition 5.2; some of them are also *minimal submanifolds* in the sense that their mean curvature vector vanishes. If this is the case, we will say that they have *vanishing mean curvature* in order to avoid confusion.

Let H_1 and H_2 be two subgroups of $U(1,n)$ acting polarly on $\mathbb{C}H^n$, and assume that these two actions are orbit equivalent. Let us denote by $\mathfrak{h}_1$ and $\mathfrak{h}_2$ the Lie algebras of H_1 and H_2. We distinguish three main cases.

Case 1. First of all, assume that the actions of H_1 and H_2 are given by Theorem A(i), that is, $\mathfrak{h}_i = \mathfrak{q}_i \oplus \mathfrak{so}(1, k_i)$, $i \in \{1, 2\}$. The group H_i has a totally real, totally geodesic $\mathbb{R}H^{k_i}$ as a singular orbit of minimum orbit type. This immediately implies $k_1 = k_2$. If $k_1 = k_2 = n$ then both actions are orbit equivalent to the action of $SO(1, n)$, according to Theorem 4.2, and thus the isotropy actions of Q_1 and Q_2 are both trivial. Assume $k_1 = k_2 < n$. It follows from Section 4 that H_i restricts to a cohomogeneity one action on the corresponding totally geodesic $\mathbb{C}H^{k_i}$ that contains this $\mathbb{R}H^{k_i}$. It follows from Theorem 4.1 that the slice representation of H_i at o is polar and a section of this action is of the form $\ell_i \oplus \Sigma_i$, where ℓ_i is a line in $T_o\mathbb{C}H^{k_i}$ and Σ_i is totally real in the complex subspace $T_o\mathbb{C}H^n \ominus T_o\mathbb{C}H^{k_i}$. The unitary representation of Q_i on the complex vector space $T_o\mathbb{C}H^n \ominus T_o\mathbb{C}H^{k_i}$ cannot have trivial factors, since otherwise a maximal trivial factor would be complex, and therefore the section would not be totally real. Hence, the only orbit of minimal dimension must be the totally geodesic $\mathbb{R}H^{k_i}$. Since the actions of H_1 and H_2 are orbit equivalent, we conclude that $H_1 \cdot o$ must be mapped to $H_2 \cdot o$. Now it is easy to deduce that the actions of Q_1 and Q_2 must be orbit equivalent.

Case 2. Assume now that H_1 is given by Theorem A(i), and H_2 is given by Theorem A(ii), that is, $\mathfrak{h}_1 = \mathfrak{q}_1 \oplus \mathfrak{so}(1, k)$ and $\mathfrak{h}_2 = \mathfrak{q}_2 \oplus \mathfrak{b} \oplus \mathfrak{w} \oplus \mathfrak{g}_{2\alpha}$. By assumption, there is an isometry ϕ such that $\phi(H_1 \cdot p) = H_2 \cdot \phi(p)$ for any $p \in \mathbb{C}H^n$. We know that H_1 has a totally real, totally geodesic $\mathbb{R}H^k$ as a singular orbit of minimum orbit type. Let $g \in AN$ be such that $\phi(o) = g(o)$ and define $\tilde{H}_2$ to be the connected Lie subgroup of $SU(1, n)$ whose Lie algebra is $\tilde{\mathfrak{h}}_2 = \mathfrak{b} \oplus \mathfrak{w} \oplus \mathfrak{g}_{2\alpha}$. As $\phi(H_1 \cdot o)$ must also be an orbit of H_2 of minimal dimension, it follows that $\tilde{H}_2 \cdot g(o) = H_2 \cdot g(o)$ since $H_2 \cdot g(o)$ has minimal dimension and contains $\tilde{H}_2 \cdot g(o)$. We have $\tilde{H}_2 \cdot g(o) = g(g^{-1}\tilde{H}_2 g) \cdot o = g(I_{g^{-1}}(\tilde{H}_2) \cdot o)$, from where it follows that $\tilde{H}_2 \cdot g(o)$ is congruent to the orbit of $I_{g^{-1}}(\tilde{H}_2)$ through o. Since $g \in AN$, it is not difficult to deduce from the bracket relations of AN that $\operatorname{Ad}(g^{-1})\tilde{\mathfrak{h}}_2 = \mathbb{R}(aB + X) \oplus \mathfrak{w} \oplus \mathfrak{g}_{2\alpha}$ for some $a \in \mathbb{R}$ and $X \in \mathfrak{g}_\alpha \ominus \mathfrak{w}$. The fact that $\operatorname{Ad}(g^{-1})\tilde{\mathfrak{h}}_2$ is totally real immediately implies $a = 0$, and thus $\mathfrak{b} = 0$. Moreover, $X = 0$ for dimension reasons, and $\mathfrak{w}$ is totally real. Then, Corollary 6.2 implies that $\operatorname{Ad}(g^{-1})\tilde{\mathfrak{h}}_2 = \mathfrak{w} \oplus \mathfrak{g}_{2\alpha}$ has mean curvature vector $\mathcal{H} = \frac{1}{2}(2 + \dim \mathfrak{w})B \neq 0$. In particular, $H_2 \cdot g(o)$ cannot be totally geodesic. Therefore, polar actions given by Theorem A(i) cannot be orbit equivalent to polar actions given by Theorem A(ii).

Case 3. Finally, assume that H_1 and H_2 are given by Theorem A(ii), that is, $\mathfrak{h}_i = \mathfrak{q}_i \oplus \mathfrak{b}_i \oplus \mathfrak{w}_i \oplus \mathfrak{g}_{2\alpha}$, $i \in \{1, 2\}$.

Let $\mathfrak{b}_1 = \mathfrak{a}$, $\mathfrak{b}_2 = 0$. The orbit $H_1 \cdot o$ is of minimum orbit type, and it has vanishing mean curvature as Corollary 6.2 shows. This orbit must be mapped to an orbit of H_2 of minimal dimension. Let $\tilde{H}_2$ be the Lie subgroup of $SU(1, n)$ whose Lie algebra is $\tilde{\mathfrak{h}}_2 = \mathfrak{w}_2 \oplus \mathfrak{g}_{2\alpha}$. Assume $H_1 \cdot o$ is mapped to $H_2 \cdot g(o)$ with $g \in AN$. Since $H_2 \cdot o$ has minimal dimension, we must have $H_2 \cdot g(o) = \tilde{H}_2 \cdot g(o)$, as $H_2 \cdot g(o)$ has minimal dimension and $\tilde{H}_2 \cdot g(o) \subset H_2 \cdot g(o)$. We have $\tilde{H}_2 \cdot g(o) = g(g^{-1}\tilde{H}_2 g \cdot o)$, and it is easy to show using the bracket relations of AN that $\operatorname{Ad}(g^{-1})\tilde{\mathfrak{h}}_2 = \tilde{\mathfrak{h}}_2 = \mathfrak{w}_2 \oplus \mathfrak{g}_{2\alpha}$. Corollary 6.2 then implies that $H_2 \cdot g(o) = \tilde{H}_2 \cdot g(o)$ has non-vanishing mean curvature. This contradicts the fact that the mean curvature vector of $H_1 \cdot o$ is zero. Therefore $\mathfrak{b}_1 = \mathfrak{b}_2$.

Assume that $\mathfrak{b}_1 = \mathfrak{b}_2 = 0$. Let ϕ be an isometry such that $\phi(H_1 \cdot o) = H_2 \cdot \phi(o)$, and take $g \in AN$ such that $\phi(o) = g(o)$. Let $\tilde{H}_2$ be the Lie subgroup of $SU(1,n)$ whose Lie algebra is $\tilde{\mathfrak{h}}_2 = \mathfrak{w}_2 \oplus \mathfrak{g}_{2\alpha}$, and recall that $\operatorname{Ad}(g^{-1})\tilde{\mathfrak{h}}_2 = \tilde{\mathfrak{h}}_2$, and thus $I_{g^{-1}}(\tilde{H}_2) = \tilde{H}_2$. Since $H_2 \cdot g(o)$ must be of minimal dimension, we have $H_2 \cdot g(o) = \tilde{H}_2 \cdot g(o) = g(g^{-1}\tilde{H}_2 g \cdot o) = g(\tilde{H}_2 \cdot o) = g(H_2 \cdot o)$, and thus, $g^{-1}\phi(H_1 \cdot o) = g^{-1}H_2 \cdot \phi(o) = g^{-1}H_2 g \cdot o = H_2 \cdot o$. By composing with an element of H_2 we can further assume that $\phi(H_1 \cdot o) = H_2 \cdot o$, and $\phi(o) = o$. In particular, we have $\phi_*(T_o(H_1 \cdot o)) = T_o(H_2 \cdot o)$, that is, $\phi_*((1-\theta)(\mathfrak{w}_1 \oplus \mathfrak{g}_{2\alpha})) = (1-\theta)(\mathfrak{w}_2 \oplus \mathfrak{g}_{2\alpha})$. We have the Kähler angle decompositions (see Subsection 2.3), $\mathfrak{w}_i = \oplus_{\varphi \in \Phi_i} \mathfrak{w}_{i,\varphi}$, and thus, $(1-\theta)\mathfrak{w}_i = \oplus_{\varphi \in \Phi_i}(1-\theta)\mathfrak{w}_{i,\varphi}$. Since ϕ_* maps real subspaces of constant Kähler angle φ to real subspaces of constant Kähler angle φ, we must have $\Phi := \Phi_1 = \Phi_2$ and

$$\phi_*(1-\theta)(\mathfrak{w}_{1,\pi/2} \oplus \mathfrak{g}_{2\alpha}) = (1-\theta)(\mathfrak{w}_{2,\pi/2} \oplus \mathfrak{g}_{2\alpha}), \quad \phi_*(1-\theta)(\mathfrak{w}_{1,\varphi}) = (1-\theta)(\mathfrak{w}_{2,\varphi}),$$

for all $\varphi \in \Phi \setminus \{\pi/2\}$. As a consequence, $\dim \mathfrak{w}_{1,\varphi} = \dim \mathfrak{w}_{2,\varphi}$ for all $\varphi \in \Phi$. It follows from Remark 2.9 that there exists $k \in K_0$ such that $\operatorname{Ad}(k)\mathfrak{w}_{1,\varphi} = \mathfrak{w}_{2,\varphi}$ for all $\varphi \in \Phi$, and thus $\operatorname{Ad}(k)(\mathfrak{w}_1 \oplus \mathfrak{g}_{2\alpha}) = \mathfrak{w}_2 \oplus \mathfrak{g}_{2\alpha}$. Let $\hat{H}_2 = k^{-1}H_2 k$. Obviously, the actions of H_2 and $\hat{H}_2$ on $\mathbb{C}H^n$ are orbit equivalent. Indeed, $\hat{\mathfrak{h}}_2 = \hat{\mathfrak{q}}_2 \oplus \mathfrak{w}_1 \oplus \mathfrak{g}_{2\alpha}$ for some subalgebra $\hat{\mathfrak{q}}_2$ of $\mathfrak{k}_0$. Since the actions of H_1 and $\hat{H}_2$ are orbit equivalent, their slice representations are orbit equivalent and so are the actions of Q_1 and $\hat{Q}_2$ on the normal space of $H_1 \cdot o = \hat{H}_2 \cdot o$ (note that the action of K_0 on $\mathfrak{a}$ is trivial). Therefore, H_1 and H_2 are orbit equivalent if and only if there exists $k \in K_0$ such that $\operatorname{Ad}(k)\mathfrak{w}_1 = \mathfrak{w}_2$, and the slice representations $Q_i \times (1-\theta)\mathfrak{w}_i^\perp \to (1-\theta)\mathfrak{w}_i^\perp$, $i \in \{1,2\}$, are orbit equivalent.

We now deal with the last possibility: $\mathfrak{b}_1 = \mathfrak{b}_2 = \mathfrak{a}$. The proof goes along the lines of the previous subcase, with some differences that we will point out. Let ϕ be an isometry such that $\phi(H_1 \cdot o) = H_2 \cdot \phi(o)$ and take $g \in AN$ such that $\phi(o) = g(o)$. As before, we consider $\tilde{H}_2$ the subgroup of $SU(1,n)$ whose Lie algebra is $\tilde{\mathfrak{h}}_2 = \mathfrak{a} \oplus \mathfrak{w}_2 \oplus \mathfrak{g}_{2\alpha}$. Since $H_2 \cdot g(o)$ is of minimal dimension, $H_2 \cdot g(o) = \tilde{H}_2 \cdot g(o) = g(g^{-1}\tilde{H}_2 g \cdot o)$, so $H_2 \cdot g(o)$ is congruent to the orbit through o of the Lie subgroup of $SU(1,n)$ whose Lie algebra is of the form $\operatorname{Ad}(g)\tilde{\mathfrak{h}}_2 = \mathbb{R}(aB + X) \oplus \mathfrak{w}_2 \oplus \mathfrak{g}_{2\alpha}$, for some $a \in \mathbb{R}$ and $X \in \mathfrak{g}_\alpha \ominus \mathfrak{w}$. Since $H_1 \cdot o$ has vanishing mean curvature by Corollary 6.2, according to Lemma 6.1 we must have that $X = 0$. In this case we get $\operatorname{Ad}(g)\tilde{\mathfrak{h}}_2 = \mathfrak{h}_2$. By composing with an element of H_2 we can further assume that $\phi(o) = o$. Arguing as in the previous case, we have the Kähler angle decompositions $\mathfrak{w}_i = \oplus_{\varphi \in \Phi_i} \mathfrak{w}_{i,\varphi}$, and thus, it follows that $\Phi := \Phi_1 = \Phi_2$ and

$$\phi_*(1-\theta)(\mathfrak{a} \oplus \mathfrak{w}_{1,0} \oplus \mathfrak{g}_{2\alpha}) = (1-\theta)(\mathfrak{a} \oplus \mathfrak{w}_{2,0} \oplus \mathfrak{g}_{2\alpha}), \quad \phi_*(1-\theta)(\mathfrak{w}_{1,\varphi}) = (1-\theta)\mathfrak{w}_{2,\varphi},$$

for all $\varphi \in \Phi \setminus \{0\}$. Again, by Remark 2.9, it follows that there exists $k \in K_0$ such that $\operatorname{Ad}(k)\mathfrak{w}_{1,\varphi} = \mathfrak{w}_{2,\varphi}$ for all $\varphi \in \Phi$, and therefore, H_1 and H_2 are orbit equivalent if and only if there exists $k \in K_0$ such that $\operatorname{Ad}(k)\mathfrak{w}_1 = \mathfrak{w}_2$, and the slice representations $Q_i \times (1-\theta)\mathfrak{w}_i^\perp \to (1-\theta)\mathfrak{w}_i^\perp$, $i \in \{1,2\}$, are orbit equivalent. □

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