

ISOPARAMETRIC HYPERSURFACES IN DAMEK-RICCI SPACES

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ABSTRACT. We construct uncountably many isoparametric families of hypersurfaces in Damek-Ricci spaces. We characterize those of them that have constant principal curvatures by means of the new concept of generalized Kähler angle. It follows that, in general, these examples are inhomogeneous and have nonconstant principal curvatures.

We also find new cohomogeneity one actions on quaternionic hyperbolic spaces, and an isoparametric family of inhomogeneous hypersurfaces with constant principal curvatures in the Cayley hyperbolic plane.

1. INTRODUCTION

A connected hypersurface of a Riemannian manifold is called an isoparametric hypersurface if its nearby parallel hypersurfaces have constant mean curvature. Cartan characterized isoparametric hypersurfaces in real space forms as those hypersurfaces with constant principal curvatures, and achieved the classification of these objects in real hyperbolic spaces. Segre found the analogous classification for Euclidean spaces. In both cases, every isoparametric hypersurface is an open part of an (extrinsically) homogeneous one, that is, an open part of an orbit of a cohomogeneity one action.

Nevertheless, the problem in spheres turned out to be much more involved and rich. Although the classification of homogeneous hypersurfaces in spheres is known, it is remarkable that not every complete isoparametric hypersurface in a sphere is homogeneous. The known inhomogeneous examples were constructed by Ferus, Karcher and Münzner in [11]. Recently, much progress has been made for spheres [6, 7, 8, 13, 14]. A complete classification has not been achieved yet, but there is only one unsettled case. For a survey on these problems and other related topics, we refer to [15] and [16].

In more general ambient spaces of nonconstant curvature, the equivalence between isoparametric hypersurfaces and hypersurfaces with constant principal curvatures is no longer true. The first examples were found by Wang [17], who constructed some inhomogeneous isoparametric hypersurfaces with nonconstant principal curvatures in the complex projective space, by projecting some of the inhomogeneous hypersurfaces in spheres via the Hopf map.

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Recently, the authors have found a large set of inhomogeneous isoparametric hypersurfaces with nonconstant principal curvatures in complex hyperbolic spaces [10]. To our knowledge, these are the first examples of inhomogeneous isoparametric hypersurfaces in Riemannian symmetric spaces whose construction does not depend on the examples in spheres.

The aim of this article is to provide a construction method of isoparametric hypersurfaces in Damek-Ricci harmonic spaces. These homogeneous spaces are a family that contains the rank one noncompact symmetric spaces as particular cases. They were constructed by Damek and Ricci in [9] and they provide counterexamples to the so-called Lichnerowicz conjecture, stating that every Riemannian harmonic manifold is locally isometric to a two-point homogeneous space. The hypersurfaces that we introduce arise as tubes around certain homogeneous minimal submanifolds whose construction extends the one proposed by Berndt and Brück [1], and by the authors [10]. Although the definition of the new examples is relatively straightforward, the verification that these examples have the desired properties is far from being trivial.

The main concept introduced in this paper is that of *generalized Kähler angle*, which generalizes previous notions of Kähler angle and quaternionic Kähler angle [1]. Among the isoparametric hypersurfaces we construct in this paper, the ones with constant principal curvatures are precisely those whose focal submanifolds have normal spaces of constant generalized Kähler angle (Theorem 4.5). As a consequence, we obtain uncountably many noncongruent isoparametric families of inhomogeneous hypersurfaces with nonconstant principal curvatures in complex and quaternionic hyperbolic spaces. Compare this with the case of spheres, where the known set of inhomogeneous isoparametric families is countable [11].

For the quaternionic hyperbolic spaces, we also provide new examples of cohomogeneity one actions. Recall that quaternionic hyperbolic spaces are the unique Riemannian symmetric spaces of rank one for which a classification of cohomogeneity one actions is still not known.

In the Cayley hyperbolic plane, our method yields an inhomogeneous isoparametric family of hypersurfaces with constant principal curvatures, which is, to our knowledge, the first such example in a Riemannian symmetric space different from a sphere (cf. [12, p. 7]).

This article is organized as follows. In Section 2 we set up the fundamental definitions and results on Damek-Ricci spaces. The definition of generalized Kähler angle is presented in Section 3. In Section 4 the new examples of isoparametric hypersurfaces in Damek-Ricci harmonic spaces are introduced. We start by defining the focal set of the new examples in §4.1, and then in §4.2 we investigate the properties of the tubes around these submanifolds using Jacobi field theory. The main result of this work is stated in Theorem 4.5. Finally, in Section 5 we consider some particular cases in the rank one symmetric spaces of noncompact type. In §5.2 we construct new examples of cohomogeneity one actions on quaternionic hyperbolic spaces (Theorem 5.2), and in §5.3 we give an example of an inhomogeneous isoparametric hypersurface in the Cayley hyperbolic plane (Theorem 5.4).

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2. GENERALIZED HEISENBERG GROUPS AND DAMEK-RICCI SPACES

In this section we recall the construction of Damek-Ricci spaces, presenting some of the properties that we will use later. Since the description of such spaces depends on the so-called generalized Heisenberg algebras, we begin by defining these structures. The main reference for all these notions is [5], where one can find the proofs of the results presented below, as well as further information on Damek-Ricci spaces.

2.1. Generalized Heisenberg algebras and groups. Let \mathfrak{v} and \mathfrak{z} be real vector spaces and $\beta : \mathfrak{v} \times \mathfrak{v} \rightarrow \mathfrak{z}$ a skew-symmetric bilinear map. Define the direct sum $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$ and endow it with an inner product $\langle \cdot, \cdot \rangle_{\mathfrak{n}}$ such that \mathfrak{v} and \mathfrak{z} are perpendicular. Define a linear map $J : \mathfrak{z} \rightarrow \text{End}(\mathfrak{v})$ by

$$\langle J_Z U, V \rangle = \langle \beta(U, V), Z \rangle, \quad \text{for all } U, V \in \mathfrak{v}, Z \in \mathfrak{z},$$

and a Lie algebra structure on \mathfrak{n} by

$$[U + X, V + Y] = \beta(U, V), \quad \text{for all } U, V \in \mathfrak{v}, X, Y \in \mathfrak{z},$$

or equivalently, by

$$\langle [U, V], X \rangle = \langle J_X U, V \rangle, \quad [X, V] = [U, Y] = [X, Y] = 0, \quad \text{for all } U, V \in \mathfrak{v}, X, Y \in \mathfrak{z}.$$

Then, \mathfrak{n} is a two-step nilpotent Lie algebra with center \mathfrak{z} , and, if $J_Z^2 = -\langle Z, Z \rangle I_{\mathfrak{v}}$ for all $Z \in \mathfrak{z}$, \mathfrak{n} is said to be a *generalized Heisenberg algebra* or an *H-type algebra*. (Here and henceforth the identity is denoted by I .) The associated simply connected nilpotent Lie group N , endowed with the induced left-invariant Riemannian metric, is called a *generalized Heisenberg group* or an *H-type group*.

Let $U, V \in \mathfrak{v}$ and $X, Y \in \mathfrak{z}$. In this work, we will make use of the following properties of generalized Heisenberg algebras without explicitly referring to them:

$$\begin{aligned} J_X J_Y + J_Y J_X &= -2\langle X, Y \rangle I_{\mathfrak{v}}, & [J_X U, V] - [U, J_X V] &= -2\langle U, V \rangle X, \\ \langle J_X U, J_X V \rangle &= \langle X, X \rangle \langle U, V \rangle, & \langle J_X U, J_Y U \rangle &= \langle X, Y \rangle \langle U, U \rangle. \end{aligned}$$

In particular, for any unit $Z \in \mathfrak{z}$, J_Z is an almost Hermitian structure on \mathfrak{v} .

The map $J : \mathfrak{z} \rightarrow \text{End}(\mathfrak{v})$ can be extended to the Clifford algebra $Cl(\mathfrak{z}, q)$, where q is the quadratic form given by $q(Z) = -\langle Z, Z \rangle$, in such a way that \mathfrak{v} becomes now a Clifford module over $Cl(\mathfrak{z}, q)$ (see [5, Chapter 3]). The classification of generalized Heisenberg algebras is known (it follows from the classification of representations of Clifford algebras of vector spaces with negative definite quadratic forms). In particular, for each $m \in \mathbb{N}$ there exist an infinite number of non-isomorphic generalized Heisenberg algebras with $\dim \mathfrak{z} = m$.

2.2. Damek-Ricci spaces. Let \mathfrak{a} be a one-dimensional real vector space, B a non-zero vector in \mathfrak{a} and $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$ a generalized Heisenberg algebra, where \mathfrak{z} is the center of \mathfrak{n} . We denote the inner product and the Lie bracket on \mathfrak{n} by $\langle \cdot, \cdot \rangle_{\mathfrak{n}}$ and $[\cdot, \cdot]_{\mathfrak{n}}$, respectively, and consider a new vector space $\mathfrak{a} \oplus \mathfrak{n}$ as the vector space direct sum of \mathfrak{a} and \mathfrak{n} .

From now on in this section, let $s, r \in \mathbb{R}$, $U, V \in \mathfrak{v}$ and $X, Y \in \mathfrak{z}$. We now define an inner product $\langle \cdot, \cdot \rangle$ and a Lie bracket $[\cdot, \cdot]$ on $\mathfrak{a} \oplus \mathfrak{n}$ by

$$\begin{aligned} \langle rB + U + X, sB + V + Y \rangle &= rs + \langle U + X, V + Y \rangle_{\mathfrak{n}}, \quad \text{and} \\ [rB + U + X, sB + V + Y] &= [U, V]_{\mathfrak{n}} + \frac{1}{2}rV - \frac{1}{2}sU + rY - sX. \end{aligned}$$

Thus, $\mathfrak{a} \oplus \mathfrak{n}$ becomes a solvable Lie algebra with an inner product. The corresponding simply connected Lie group AN , equipped with the induced left-invariant Riemannian metric, is a solvable extension of the H-type group N , and is called a *Damek-Ricci space*.

The Levi-Civita connection ∇ of a Damek-Ricci space is given by

$$\nabla_{sB+V+Y}(rB+U+X) = -\frac{1}{2}J_XV - \frac{1}{2}J_YU - \frac{1}{2}rV - \frac{1}{2}[U, V] - rY + \frac{1}{2}\langle U, V \rangle B + \langle X, Y \rangle B.$$

From this expression, one can obtain the curvature tensor R of AN , where we agree to take the convention $R(W_1, W_2) = [\nabla_{W_1}, \nabla_{W_2}] - \nabla_{[W_1, W_2]}$.

A Damek-Ricci space AN is a symmetric space if and only if AN is isometric to a rank one symmetric space. In this case, AN is either isometric to a complex hyperbolic space $\mathbb{C}H^n$ with constant holomorphic sectional curvature -1 (in this case, $\dim \mathfrak{z} = 1$), or to a quaternionic hyperbolic space $\mathbb{H}H^n$ with constant quaternionic sectional curvature -1 (here $\dim \mathfrak{z} = 3$), or to the Cayley hyperbolic plane $\mathbb{O}H^2$ with minimal sectional curvature -1 ($\dim \mathfrak{z} = 7$). As a limit case, which we will disregard in what follows, one would obtain the real hyperbolic space $\mathbb{R}H^n$ if one puts $\mathfrak{z} = 0$.

The non-symmetric Damek-Ricci spaces are counterexamples to the so-called Lichnerowicz conjecture, stating that every Riemannian harmonic manifold is locally isometric to a two-point homogeneous space. There are several equivalent conditions for a manifold to be harmonic; see [5, §2.6]. One of them is the following: a manifold is harmonic if and only if its sufficiently small geodesic spheres are isoparametric. However, while geodesic spheres in symmetric Damek-Ricci spaces are homogeneous isoparametric hypersurfaces with constant principal curvatures, geodesic spheres in non-symmetric Damek-Ricci spaces are inhomogeneous isoparametric hypersurfaces with nonconstant principal curvatures (see [9] and [5, §4.4 and §4.5]).

3. GENERALIZED KÄHLER ANGLE

In this section we introduce the new notion of generalized Kähler angle of a vector of a subspace of a Clifford module with respect to that subspace. This notion will be crucial for the rest of the work.

Let \mathfrak{v} be a Clifford module over $Cl(\mathfrak{z}, q)$ and denote by $J: \mathfrak{z} \rightarrow \text{End}(\mathfrak{v})$ the restriction to \mathfrak{z} of the Clifford algebra representation. We equip \mathfrak{z} with the inner product induced by polarization of $-q$, and extend it to an inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$, so that \mathfrak{v} and \mathfrak{z} are perpendicular, and J_Z is an orthogonal map for each unit $Z \in \mathfrak{z}$. Then, \mathfrak{n} has the structure of a generalized Heisenberg algebra as defined above.

Let \mathfrak{w} be a subspace of \mathfrak{v} . We denote by $\mathfrak{w}^\perp = \mathfrak{v} \ominus \mathfrak{w}$ the orthogonal complement of \mathfrak{w} in \mathfrak{v} . For each $Z \in \mathfrak{z}$ and $\xi \in \mathfrak{w}^\perp$, we write $J_Z\xi = P_Z\xi + F_Z\xi$, where $P_Z\xi$ is the orthogonal

projection of $J_Z\xi$ onto \mathfrak{w} , and $F_Z\xi$ is the orthogonal projection of $J_Z\xi$ onto \mathfrak{w}^\perp . We define the Kähler angle of $\xi \in \mathfrak{w}^\perp$ with respect to the element $Z \in \mathfrak{z}$ (or, equivalently, with respect to J_Z) and the subspace $\mathfrak{w}^\perp \subset \mathfrak{v}$ as the angle $\varphi \in [0, \pi/2]$ between $J_Z\xi$ and \mathfrak{w}^\perp [1]; thus φ satisfies $\langle F_Z\xi, F_Z\xi \rangle = \cos^2(\varphi)\langle Z, Z \rangle \langle \xi, \xi \rangle$. It readily follows from $J_Z^2 = -\langle Z, Z \rangle I_{\mathfrak{v}}$ that $\langle P_Z\xi, P_Z\xi \rangle = \sin^2(\varphi)\langle Z, Z \rangle \langle \xi, \xi \rangle$. Hence, if Z and ξ have unit length, φ is determined by the fact that $\cos(\varphi)$ is the length of the orthogonal projection of $J_Z\xi$ onto \mathfrak{w}^\perp .

The following theorem is a generalization of [1, Lemma 3] (which concerned only the case of the quaternionic hyperbolic space $\mathbb{H}H^n$). The proof is new and simpler than in [1]. This result will be fundamental for the calculations we will carry out later.

Theorem 3.1. *Let \mathfrak{w}^\perp be some vector subspace of \mathfrak{v} and let $\xi \in \mathfrak{w}^\perp$ be a nonzero vector. Then there exists an orthonormal basis $\{Z_1, \dots, Z_m\}$ of \mathfrak{z} and a uniquely defined m -tuple $(\varphi_1, \dots, \varphi_m)$ such that:*

- (a) φ_i is the Kähler angle of ξ with respect to J_{Z_i} , for each $i = 1, \dots, m$.
- (b) $\langle P_{Z_i}\xi, P_{Z_j}\xi \rangle = \langle F_{Z_i}\xi, F_{Z_j}\xi \rangle = 0$ whenever $i \neq j$.
- (c) $0 \leq \varphi_1 \leq \varphi_2 \leq \dots \leq \varphi_m \leq \pi/2$.
- (d) φ_1 is minimal and φ_m is maximal among the Kähler angles of ξ with respect to all the elements of \mathfrak{z} .

Proof. Since the map $Z \in \mathfrak{z} \mapsto F_Z\xi \in \mathfrak{w}^\perp$ is linear, we can define the quadratic form

$$Q_\xi: Z \in \mathfrak{z} \mapsto \langle F_Z\xi, F_Z\xi \rangle \in \mathbb{R}.$$

Observe that φ is the Kähler angle of ξ with respect to $Z \in \mathfrak{z}$ ($Z \neq 0$) and the subspace $\mathfrak{w}^\perp \subset \mathfrak{v}$ if and only if $Q_\xi(Z) = \cos^2(\varphi)\langle Z, Z \rangle \langle \xi, \xi \rangle$.

Let $\{Z_1, \dots, Z_m\}$ be an orthonormal basis of \mathfrak{z} for which the quadratic form Q_ξ assumes a diagonal form. Define the real numbers $\varphi_1, \dots, \varphi_m \in [0, \pi/2]$ by the expression $Q_\xi(Z_i) = \cos^2(\varphi_i)\langle \xi, \xi \rangle$, for every $i = 1, \dots, m$. We can further assume that $\varphi_1 \leq \dots \leq \varphi_m$, by reordering the elements of the basis in a suitable way.

If L is the symmetric bilinear form associated with Q_ξ , then $L_\xi(X, Y) = \langle F_X\xi, F_Y\xi \rangle$, for each $X, Y \in \mathfrak{z}$. But then the fact that $\{Z_1, \dots, Z_m\}$ is an orthonormal basis for which Q_ξ assumes a diagonal form is equivalent to $0 = L_\xi(Z_i, Z_j) = \langle F_{Z_i}\xi, F_{Z_j}\xi \rangle$ for all $i \neq j$. This, together with the ordering of $(\varphi_1, \dots, \varphi_m)$ and the fact that $\{Z_1, \dots, Z_m\}$ is an orthonormal basis, implies that the m -tuple $(\varphi_1, \dots, \varphi_m)$ is uniquely defined for a fixed \mathfrak{w}^\perp and a fixed $\xi \in \mathfrak{w}^\perp$. Moreover, due to the bilinearity of L_ξ , it is clear that φ_1 is minimal and φ_m is maximal among the Kähler angles of ξ with respect to all the elements of \mathfrak{z} . Finally, we also have that $\langle P_{Z_i}\xi, P_{Z_j}\xi \rangle = \langle J_{Z_i}\xi, J_{Z_j}\xi \rangle - \langle F_{Z_i}\xi, F_{Z_j}\xi \rangle = 0$, whenever $i \neq j$. \square

Motivated by Theorem 3.1, we define the *generalized Kähler angle* of ξ with respect to \mathfrak{w}^\perp as the m -tuple $(\varphi_1, \dots, \varphi_m)$ satisfying properties (a)-(d) of Theorem 3.1.

Remark 3.2. Observe that the Kähler angles $\varphi_1, \dots, \varphi_m$ depend, not only on the subspace \mathfrak{w}^\perp of \mathfrak{v} , but also on the vector $\xi \in \mathfrak{w}^\perp$.

Assuming the notation of the previous theorem, we will say that the subspace \mathfrak{w}^\perp of \mathfrak{v} has *constant generalized Kähler angle* $(\varphi_1, \dots, \varphi_m)$ if the m -tuple $(\varphi_1, \dots, \varphi_m)$ is independent of the unit vector $\xi \in \mathfrak{w}^\perp$.

If $\mathfrak{v} = \mathbb{C}^n$ and $\mathfrak{z} = \mathbb{R}$, then the complex structure of \mathbb{C}^n is $J = J_1$. For a given subspace \mathfrak{w} of \mathbb{C}^n , we denote $F = F_1$ and $P = P_1$, and we define $\bar{F}\xi = F\xi/\|F\xi\|$ if $F\xi \neq 0$. We will need the following result from [1, Lemma 2]:

Lemma 3.3. *Let \mathfrak{w}^\perp be some linear subspace of \mathbb{C}^n , and $\xi \in \mathfrak{w}^\perp$ a unit vector with Kähler angle $\varphi \in (0, \pi/2)$. Then, there exists a unique vector $\eta \in \mathbb{C}^n \ominus \mathbb{C}\xi$ such that $\bar{F}\xi = \cos(\varphi)J\xi + \sin(\varphi)J\eta$.*

4. THE NEW EXAMPLES

The new isoparametric hypersurfaces will be tubes around certain homogeneous submanifolds of a Damek-Ricci space. Thus, in this section, we proceed first with the construction of these submanifolds and then determine their extrinsic geometry. This is done in Subsection 4.1. The geometry of the tubes around these focal submanifolds is studied in Subsection 4.2, where their main properties are given.

4.1. The focal manifold of the new examples. As we explained above, the new examples are constructed as tubes around certain homogeneous submanifolds. Each isoparametric family will have at most one submanifold that is not a hypersurface. This is the focal submanifold of the family, and we define it in this subsection.

Let AN be a Damek-Ricci space with Lie algebra $\mathfrak{a} \oplus \mathfrak{n} = \mathfrak{a} \oplus \mathfrak{v} \oplus \mathfrak{z}$, where $\dim \mathfrak{z} = m$. Let \mathfrak{w} be a proper subspace of \mathfrak{v} and define $\mathfrak{w}^\perp = \mathfrak{v} \ominus \mathfrak{w}$, the orthogonal complement of \mathfrak{w} in \mathfrak{v} . Then,

$$\mathfrak{s}_{\mathfrak{w}} = \mathfrak{a} \oplus \mathfrak{w} \oplus \mathfrak{z}$$

is a solvable Lie subalgebra of $\mathfrak{a} \oplus \mathfrak{n}$, as one can easily check from the bracket relations in §2.2. Let $S_{\mathfrak{w}}$ be the corresponding connected subgroup of AN whose Lie algebra is $\mathfrak{s}_{\mathfrak{w}}$. Since AN acts by isometries on itself and $S_{\mathfrak{w}}$ is a subgroup of AN , $S_{\mathfrak{w}}$ is also a homogeneous submanifold of AN .

Let $\xi \in \mathfrak{w}^\perp$ be a unit normal vector field along the submanifold $S_{\mathfrak{w}}$. Let $\{Z_1, \dots, Z_m\}$ be an orthonormal basis of \mathfrak{z} satisfying the properties of Theorem 3.1. In order to simplify the notation, for each $i \in \{1, \dots, m\}$, we set J_i , P_i and F_i instead of J_{Z_i} , P_{Z_i} and F_{Z_i} , respectively. It is convenient to define

$$m_0 = \max\{i : \varphi_i = 0\} + 1 \quad \text{and} \quad m_{\pi/2} = \min\{i : \varphi_i = \pi/2\} - 1,$$

where φ_i is the Kähler angle of ξ with respect to $Z_i \in \mathfrak{z}$ (set $m_0 = 1$ if $\varphi_i > 0$ for all i , and $m_{\pi/2} = m$ if $\varphi_i < \pi/2$ for all i). Thus, m_0 is the first index i for which $\varphi_i > 0$, and $m_{\pi/2}$ is the last index i for which $\varphi_i < \pi/2$. It might of course happen that $m_0 > m$ if $\varphi_i = 0$ for all i , or $m_{\pi/2} < 1$ if $\varphi_i = \pi/2$ for all i , in which case some of the equations that follow are just disregarded.

With this notation we can now define

$$\bar{P}_i\xi = \frac{1}{\sin(\varphi_i)}P_i\xi, \quad \text{for } i = m_0, \dots, m, \quad \text{and} \quad \bar{F}_i\xi = \frac{1}{\cos(\varphi_i)}F_i\xi, \quad \text{for } i = 1, \dots, m_{\pi/2}.$$

Since ξ is of unit length, so are $\bar{P}_i\xi$ and $\bar{F}_i\xi$ whenever they exist. Moreover, by Theorem 3.1, the set $\{\bar{P}_{m_0}\xi, \dots, \bar{P}_m\xi, \bar{F}_1\xi, \dots, \bar{F}_{m_{\pi/2}}\xi\}$ constitutes an orthonormal system of vector fields along $S_{\mathfrak{w}}$, the first $m - m_0 + 1$ of which being tangent, and the rest normal to $S_{\mathfrak{w}}$.

We are now interested in calculating the shape operator \mathcal{S} of $S_{\mathfrak{w}}$. Recall that the shape operator \mathcal{S}_ξ of $S_{\mathfrak{w}}$ with respect to a unit normal $\xi \in \nu S_{\mathfrak{w}}$ is defined by $\mathcal{S}_\xi X = -(\nabla_X \xi)^\top$, for any $X \in TS_{\mathfrak{w}}$, and where $(\cdot)^\top$ denotes orthogonal projection onto the tangent space. The expression for the Levi-Civita connection of the Damek-Ricci space AN allows us to calculate the shape operator of $S_{\mathfrak{w}}$ for left-invariant vector fields:

$$\begin{aligned} \mathcal{S}_\xi B &= 0, \\ \mathcal{S}_\xi Z_i &= \frac{1}{2} P_i \xi = 0, \quad \text{if } i = 1, \dots, m_0 - 1, \\ \mathcal{S}_\xi Z_i &= \frac{1}{2} P_i \xi = \frac{1}{2} \sin(\varphi_i) \bar{P}_i \xi, \quad \text{if } i = m_0, \dots, m, \\ \mathcal{S}_\xi \bar{P}_i \xi &= \frac{1}{2} [\xi, \bar{P}_i \xi]^\top = \frac{1}{2} \sum_{j=1}^m \langle J_j \xi, \bar{P}_i \xi \rangle Z_j = \frac{1}{2} \sin(\varphi_i) Z_i, \quad \text{if } i = m_0, \dots, m, \\ \mathcal{S}_\xi U &= \frac{1}{2} [\xi, U]^\top = \frac{1}{2} \sum_{j=1}^m \langle J_j \xi, U \rangle Z_j = 0, \quad \text{if } U \in \mathfrak{w} \ominus \left(\bigoplus_{i=m_0}^m \mathbb{R} \bar{P}_i \xi \right). \end{aligned}$$

From the expressions above, we obtain that the principal curvatures of $S_{\mathfrak{w}}$ with respect to the unit normal vector ξ are

$$0, \quad \frac{1}{2} \sin \varphi_i, \quad \text{and} \quad -\frac{1}{2} \sin \varphi_i,$$

and their corresponding principal spaces are, respectively,

$$\mathfrak{a} \oplus \left(\mathfrak{w} \ominus \left(\bigoplus_{j=m_0}^m \mathbb{R} \bar{P}_j \xi \right) \right) \oplus \left(\bigoplus_{j=1}^{m_0-1} Z_j \right), \quad \mathbb{R}(Z_i + \bar{P}_i \xi), \quad \text{and} \quad \mathbb{R}(Z_i - \bar{P}_i \xi),$$

where $i = m_0, \dots, m$. In any case, the submanifold $S_{\mathfrak{w}}$ is minimal (even austere) and, if $\dim \mathfrak{w}^\perp = 1$, then $S_{\mathfrak{w}}$ is a minimal hypersurface of AN .

Remark 4.1. We emphasize that, although the dependance on ξ is not made explicit in the notation, $(\varphi_1, \dots, \varphi_m)$, $\{Z_1, \dots, Z_m\}$, m_0 , and $m_{\pi/2}$ do depend on ξ .

4.2. Solving the Jacobi equation. Denote by M^r the tube of radius r around the submanifold $S_{\mathfrak{w}}$ that was described in the previous section. We claim that, for every $r > 0$, M^r is an isoparametric hypersurface which has, in general, nonconstant principal curvatures.

In order to show that M^r has the properties mentioned above, we will make use of Jacobi field theory. The main step of our approach is to write down the Jacobi equation along a geodesic normal to $S_{\mathfrak{w}}$ and to solve some initial value problems for this equation. We emphasize that the method used in [10] to write down the Jacobi equation corresponding to complex hyperbolic space $\mathbb{C}H^n$ (which consisted in expressing the Jacobi fields in terms

of parallel translations) is not feasible here. Thus we will express the Jacobi fields in terms of left-invariant vector fields. The relevance of Theorem 3.1 will become clear with this method.

Given a unit speed geodesic γ in the Damek-Ricci space AN , a vector field ζ along γ is called a Jacobi vector field if it satisfies the Jacobi equation in AN along γ , namely

$$\zeta'' + R(\zeta, \dot{\gamma})\dot{\gamma} = 0,$$

where $\dot{\gamma}$ is the tangent vector of γ , and $'$ stands for covariant differentiation along the geodesic γ .

Let $p \in S_{\mathfrak{w}}$ be an element of the submanifold, and $\xi \in \nu_p S_{\mathfrak{w}}$ a unit normal vector at p . Let γ be the geodesic of AN such that $\gamma(0) = p$ and $\dot{\gamma}(0) = \xi$. Denote by $\dot{\gamma}(t)^\perp$ the orthogonal complement of $\dot{\gamma}(t)$ in $T_{\gamma(t)}AN$ and by \mathcal{S} the shape operator of the submanifold $S_{\mathfrak{w}}$. We denote by ζ_v the Jacobi vector field with initial conditions

$$(1) \quad \zeta_v(0) = v^\top, \quad \zeta'_v(0) = -\mathcal{S}_\xi v^\top + v^\perp, \quad \text{where } v = v^\top + v^\perp, \quad v^\top \in \mathfrak{s}_{\mathfrak{w}}, \quad \text{and } v^\perp \in \mathfrak{w}^\perp \ominus \mathbb{R}\xi.$$

We define the $\text{Hom}((\mathfrak{a} \oplus \mathfrak{n}) \ominus \mathbb{R}\xi, \dot{\gamma}^\perp)$ -valued tensor fields C and E along γ satisfying $C(r)v = \zeta_v(r)$ and $E(r)v = (\zeta'_v(r))^\top$, for every $r \in \mathbb{R}$ and every left-invariant vector field $v \in (\mathfrak{a} \oplus \mathfrak{n}) \ominus \mathbb{R}\xi$, where now $(\cdot)^\top$ denotes the projection onto $\dot{\gamma}^\perp$. Standard Jacobi field theory ensures that if $C(r)$ is nonsingular for every unit $\xi \in \nu S_{\mathfrak{w}}$, then the tube M^r of radius r around $S_{\mathfrak{w}}$ is a hypersurface of AN . Moreover, in this case, the shape operator \mathcal{S}^r of M^r at the point $\gamma(r)$ with respect to the unit vector $-\dot{\gamma}(r)$ is given by $\mathcal{S}^r \zeta_v(r) = (\zeta'_v(r))^\top$, for every $v \in (\mathfrak{a} \oplus \mathfrak{n}) \ominus \mathbb{R}\xi$, that is, $\mathcal{S}^r = E(r)C(r)^{-1}$.

Therefore, our objective in what follows is to determine an explicit expression for the Jacobi fields whose initial conditions are given by (1). In order to achieve this goal, we fix here and henceforth an orthonormal basis $\{Z_1, \dots, Z_m\}$ of \mathfrak{z} satisfying the properties of Theorem 3.1, and let $(\varphi_1, \dots, \varphi_m)$ be the corresponding generalized Kähler angle of ξ with respect to \mathfrak{w}^\perp . Recall that $(\varphi_1, \dots, \varphi_m)$ and $\{Z_1, \dots, Z_m\}$ depend on ξ , but we remove this dependence from the notation for the sake of simplicity. Let $\{U_1, \dots, U_l\}$ be an orthonormal basis of $\mathfrak{w} \ominus (\oplus_{j=m_0}^m \mathbb{R}\bar{P}_j\xi)$, and let $\{\eta_1, \dots, \eta_h\}$ be an orthonormal basis of $\mathfrak{w}^\perp \ominus (\mathbb{R}\xi \oplus (\oplus_{j=1}^{m_{\pi/2}} \bar{F}_j\xi))$. Then the set

$$(2) \quad \left\{ \xi, B, U_1, \dots, U_l, \eta_1, \dots, \eta_h, \bar{P}_{m_0}\xi, \dots, \bar{P}_m\xi, \bar{F}_1\xi, \dots, \bar{F}_{m_{\pi/2}}\xi, Z_1, \dots, Z_m \right\}$$

constitutes an orthonormal basis of left-invariant vector fields of $\mathfrak{a} \oplus \mathfrak{n}$.

The main step of the proof of Theorem 4.5 is the following

Proposition 4.2. *With the notation as above we have*

$$\begin{aligned}
\zeta_B(t) &= B + \sinh\left(\frac{t}{2}\right)\xi, \\
\zeta_{U_i}(t) &= \cosh\left(\frac{t}{2}\right)U_i, \quad i = 1, \dots, l, \\
\zeta_{\eta_i}(t) &= 2 \sinh\left(\frac{t}{2}\right)\eta_i, \quad i = 1, \dots, h, \\
\zeta_{\bar{P}_i\xi}(t) &= \cosh\left(\frac{t}{2}\right)\bar{P}_i\xi - \sin(\varphi_i) \sinh(t)Z_i, \quad i = m_0, \dots, m, \\
\zeta_{\bar{F}_i\xi}(t) &= 2 \sinh\left(\frac{t}{2}\right)\bar{F}_i\xi - 2 \cos(\varphi_i) \sinh^2\left(\frac{t}{2}\right)Z_i, \quad i = 1, \dots, m_{\pi/2}, \\
\zeta_{Z_i}(t) &= \sinh\left(\frac{t}{2}\right)F_i\xi + \left(1 + \sin^2(\varphi_i) \sinh^2\left(\frac{t}{2}\right)\right)Z_i, \quad i = 1, \dots, m.
\end{aligned}$$

Proof. In order to prove this result it suffices to take the expressions above and show that they satisfy the Jacobi equation and the initial conditions (1). The calculations are long so we will just show an example of how they are performed for ζ_{Z_i} .

First of all, recall that $p \in S_{\mathfrak{w}}$, and $\xi \in \nu_p S_{\mathfrak{w}}$ is a unit normal vector at p . The geodesic γ satisfies $\gamma(0) = p$ and $\dot{\gamma}(0) = \xi$. By [5, §4.1.11, Theorem 2] we know that $\dot{\gamma}(t) = \operatorname{sech}\left(\frac{t}{2}\right)\xi - \tanh\left(\frac{t}{2}\right)B$, for every $t \in \mathbb{R}$, where ξ and B are considered as left-invariant vector fields on AN . Actually, in [5] this result is stated only for the case when $p = e$ is the identity element of AN . However, since $\gamma_p = L_p \circ \gamma_e$, the homogeneity of $S_{\mathfrak{w}}$ implies that

$$(3) \quad \dot{\gamma}_p(t) = L_{p*}\dot{\gamma}_e(t) = \operatorname{sech}\left(\frac{t}{2}\right)\xi - \tanh\left(\frac{t}{2}\right)B,$$

for every $t \in \mathbb{R}$, where L_p denotes the left multiplication by p in the group AN , γ_e is the normal geodesic through the identity element e with initial velocity ξ , and γ_p is the normal geodesic through the point $p \in AN$ with initial velocity $L_{p*}\xi = \xi$.

It is easy to check that $\zeta_{Z_i}(0) = Z_i$, which is a tangent vector to $\mathfrak{s}_{\mathfrak{w}}$. Now we have to calculate ζ'_{Z_i} . By the Leibniz rule we get

$$(4) \quad \begin{aligned} \zeta'_{Z_i}(t) &= \nabla_{\dot{\gamma}(t)}\zeta_{Z_i} = \frac{1}{2} \cosh\left(\frac{t}{2}\right)F_i\xi + \sinh\left(\frac{t}{2}\right)\nabla_{\dot{\gamma}(t)}F_i\xi + \sin^2(\varphi_i) \sinh\left(\frac{t}{2}\right) \cosh\left(\frac{t}{2}\right)Z_i \\ &\quad + \left(1 + \sin^2(\varphi_i) \sinh^2\left(\frac{t}{2}\right)\right)\nabla_{\dot{\gamma}(t)}Z_i. \end{aligned}$$

Using (3), and the formula for the Levi-Civita connection in §2.2, we obtain

$$\begin{aligned}
\nabla_{\dot{\gamma}(t)}F_i\xi &= \operatorname{sech}\left(\frac{t}{2}\right)\nabla_{\xi}F_i\xi - \tanh\left(\frac{t}{2}\right)\nabla_B F_i\xi = \frac{1}{2} \operatorname{sech}\left(\frac{t}{2}\right)[\xi, F_i\xi], \quad \text{and} \\
\nabla_{\dot{\gamma}(t)}Z_i &= \operatorname{sech}\left(\frac{t}{2}\right)\nabla_{\xi}Z_i - \tanh\left(\frac{t}{2}\right)\nabla_B Z_i = -\frac{1}{2} \operatorname{sech}\left(\frac{t}{2}\right)J_{Z_i}\xi.
\end{aligned}$$

Now, for $j \in \{1, \dots, m\}$, the bracket relations from §2.1 yield $\langle [\xi, F_i\xi], Z_j \rangle = \langle J_{Z_j}\xi, F_i\xi \rangle = \langle F_j\xi, F_i\xi \rangle = \cos^2(\varphi_i)\delta_{ij}$, where δ is the Kronecker delta. Thus, $[\xi, F_i\xi] = \cos^2(\varphi_i)Z_i$. Using

$J_{Z_i}\xi = P_i\xi + F_i\xi$, and inserting these expressions into (4) we obtain

$$\begin{aligned} \zeta'_{Z_i}(t) &= -\frac{1}{2} \operatorname{sech}\left(\frac{t}{2}\right) \left(1 + \sin^2(\varphi_i) \sinh^2\left(\frac{t}{2}\right)\right) P_i\xi + \frac{1}{2} \cos^2(\varphi_i) \sinh\left(\frac{t}{2}\right) \tanh\left(\frac{t}{2}\right) F_i\xi \\ &\quad + \frac{1}{2} \left(\sin^2(\varphi_i) \sinh(t) + \cos^2(\varphi_i) \tanh\left(\frac{t}{2}\right)\right) Z_i, \quad i = 1, \dots, m. \end{aligned}$$

From this expression, and the shape operator of $S_{\mathfrak{w}}$ obtained in §4.1 we easily get $\zeta'_{Z_i}(0) = -\frac{1}{2}P_i\xi = -\frac{1}{2}\sin\varphi_i\bar{P}_i\xi = -\mathcal{S}_\xi Z_i$, so the initial conditions (1) are satisfied.

The very same approach can be used to calculate ζ''_{Z_i} . We omit the explicit calculations here, which are very similar to those shown above, and give the result:

$$\begin{aligned} \zeta''_{Z_i}(t) &= -\frac{3}{4} \sin^2(\varphi_i) \sinh\left(\frac{t}{2}\right) P_i\xi + \frac{1}{16} (6 \cos(2\varphi_i) - 2) \sinh\left(\frac{t}{2}\right) F_i\xi \\ &\quad + \frac{1}{4} \left(\cosh(t) - 2 \cos(2\varphi_i) \sinh^2\left(\frac{t}{2}\right)\right) Z_i. \end{aligned}$$

Finally, we need to calculate $R(\zeta_{Z_i}(t), \dot{\gamma}(t))\dot{\gamma}(t)$. We have the following identities for the curvature tensor, where $U, V \in \mathfrak{v}$, and $Z \in \mathfrak{z}$ are of unit length (for the complete formula, see [5, §4.1.7]):

$$\begin{aligned} R(U, Z)B &= \frac{1}{4} J_Z U, & R(B, Z)B &= Z, & R(U, V)V &= \frac{1}{4} (\langle U, V \rangle V - U + 3J_{[U, V]}V), \\ R(U, V)B &= \frac{1}{2} [U, V], & R(B, U)B &= \frac{1}{4} U, & R(U, B)V &= \frac{1}{4} \langle U, V \rangle B + \frac{1}{4} [U, V], \\ R(B, Z)U &= \frac{1}{2} J_Z U, & R(U, Z)U &= \frac{1}{4} Z, & R(U, B)U &= \frac{1}{4} B. \end{aligned}$$

Using the properties of the curvature tensor and the formulas above, we get after some calculations $\zeta''_{Z_i} + R(\zeta_{Z_i}, \dot{\gamma})\dot{\gamma} = 0$ as we wanted to show. \square

Our aim in what follows is to finish the calculation of the shape operator \mathcal{S}^r of M^r at $\gamma(r)$. Recall that we first need to calculate $C(r): (\mathfrak{a} \oplus \mathfrak{n}) \ominus \mathbb{R}\xi \rightarrow \dot{\gamma}^\perp = T_{\gamma(r)}M^r$, $v \mapsto \zeta_v(r)$. In order to describe this operator we consider the following distributions on $\mathfrak{a} \oplus \mathfrak{n}$:

$$\begin{aligned} \mathfrak{U} &= \bigoplus_{j=1}^l \mathbb{R}U_j, & \mathfrak{F}_i &= \mathbb{R}\bar{F}_i\xi \oplus \mathbb{R}Z_i, & i &= 1, \dots, m_0 - 1, \\ \mathfrak{H} &= \bigoplus_{j=1}^h \mathbb{R}\eta_j, & \mathfrak{M}_i &= \mathbb{R}\bar{P}_i\xi \oplus \mathbb{R}\bar{F}_i\xi \oplus \mathbb{R}Z_i, & i &= m_0, \dots, m_{\pi/2}, \\ & & \mathfrak{P}_i &= \mathbb{R}\bar{P}_i\xi \oplus \mathbb{R}Z_i, & i &= m_{\pi/2} + 1, \dots, m. \end{aligned}$$

Then, we can decompose

$$\begin{aligned} (\mathfrak{a} \oplus \mathfrak{n}) \ominus \mathbb{R}\xi &= \mathbb{R}B \oplus \mathfrak{U} \oplus \mathfrak{H} \oplus \left(\bigoplus_{i=1}^{m_0-1} \mathfrak{F}_i \right) \oplus \left(\bigoplus_{i=m_0}^{m_{\pi/2}} \mathfrak{M}_i \right) \oplus \left(\bigoplus_{i=m_{\pi/2}+1}^m \mathfrak{P}_i \right), \text{ and} \\ T_{\gamma(r)}M^r &= \mathbb{R} \left(\operatorname{sech}\left(\frac{r}{2}\right) B + \tanh\left(\frac{r}{2}\right) \xi \right) \oplus \mathfrak{U} \oplus \mathfrak{H} \oplus \left(\bigoplus_{i=1}^{m_0-1} \mathfrak{F}_i \right) \oplus \left(\bigoplus_{i=m_0}^{m_{\pi/2}} \mathfrak{M}_i \right) \oplus \left(\bigoplus_{i=m_{\pi/2}+1}^m \mathfrak{P}_i \right). \end{aligned}$$

With respect to these decompositions, a direct application of Proposition 4.2 shows that the operator $C(r)$ can be written as

$$\begin{aligned} C(r) = & \cosh\left(\frac{r}{2}\right) I_{l+1} \oplus \left(2 \sinh\left(\frac{r}{2}\right) I_h\right) \oplus \left(\bigoplus_{i=1}^{m_0-1} \begin{pmatrix} 2 \sinh\left(\frac{r}{2}\right) & \sinh\left(\frac{r}{2}\right) \\ -2 \sinh^2\left(\frac{r}{2}\right) & 1 \end{pmatrix}\right) \\ & \oplus \left(\bigoplus_{i=m_0}^{m_{\pi/2}} \begin{pmatrix} \cosh\left(\frac{r}{2}\right) & 0 & 0 \\ 0 & 2 \sinh\left(\frac{r}{2}\right) & \cos(\varphi_i) \sinh\left(\frac{r}{2}\right) \\ -\sin(\varphi_i) \sinh(r) & -2 \cos(\varphi_i) \sinh^2\left(\frac{r}{2}\right) & 1 + \sin^2(\varphi_i) \sinh^2\left(\frac{r}{2}\right) \end{pmatrix}\right) \\ & \oplus \left(\bigoplus_{i=m_{\pi/2}+1}^m \begin{pmatrix} \cosh\left(\frac{r}{2}\right) & 0 \\ -\sinh(r) & \cosh^2\left(\frac{r}{2}\right) \end{pmatrix}\right). \end{aligned}$$

In particular, the determinant of $C(r)$ is

$$\det(C(r)) = 2^{h+m_{\pi/2}} \left(\cosh\left(\frac{r}{2}\right)\right)^{2+l+3m-m_0} \left(\sinh\left(\frac{r}{2}\right)\right)^{h+m_{\pi/2}},$$

which is nonzero for every $r > 0$, and hence, the tubes M^r around $S_{\mathfrak{w}}$ are hypersurfaces for every $r > 0$.

The next step is to consider the operator $E(r): (\mathfrak{a} \oplus \mathfrak{n}) \ominus \mathbb{R}\xi \rightarrow T_{\gamma(r)}M^r$, $v \mapsto (\zeta'_v(r))^\top$. To that end, we need to calculate the covariant derivative along γ of the Jacobi vector fields given in Proposition 4.2. We omit the explicit calculations, which follow the procedure described in the proof of Proposition 4.2, and give $E(r)$ with respect to the same decomposition as above:

$$\begin{aligned} E(r) = & \frac{1}{2} \sinh\left(\frac{r}{2}\right) I_{l+1} \oplus \left(\cosh\left(\frac{r}{2}\right) I_h\right) \oplus \left(\bigoplus_{i=1}^{m_0-1} \begin{pmatrix} \cosh(r) \operatorname{sech}\left(\frac{r}{2}\right) & \frac{1}{2} \sinh\left(\frac{r}{2}\right) \tanh\left(\frac{r}{2}\right) \\ \tanh\left(\frac{r}{2}\right) - \sinh(r) & \frac{1}{2} \tanh\left(\frac{r}{2}\right) \end{pmatrix}\right) \\ & \oplus \left(\bigoplus_{i=m_0}^{m_{\pi/2}} A_i(r)\right) \oplus \left(\bigoplus_{i=m_{\pi/2}+1}^m \begin{pmatrix} \frac{3}{2} \sinh\left(\frac{r}{2}\right) & -\frac{1}{2} \cosh\left(\frac{r}{2}\right) \\ \frac{1}{2} - \cosh(r) & \frac{1}{2} \sinh(r) \end{pmatrix}\right), \end{aligned}$$

where $A_i(r)$ is

$$\frac{1}{2} \begin{pmatrix} (2 - \cos(2\varphi_i)) \sinh\left(\frac{r}{2}\right) & 2 \sin(2\varphi_i) \operatorname{csch}(r) \sinh^3\left(\frac{r}{2}\right) & -\sin(\varphi_i) \operatorname{sech}\left(\frac{r}{2}\right) (1 + \sin^2(\varphi_i) \sinh^2\left(\frac{r}{2}\right)) \\ \sin(2\varphi_i) \sinh\left(\frac{r}{2}\right) & 2 \cosh\left(\frac{r}{2}\right) (1 + \cos^2(\varphi_i) \tanh^2\left(\frac{r}{2}\right)) & \cos^3(\varphi_i) \sinh\left(\frac{r}{2}\right) \tanh\left(\frac{r}{2}\right) \\ \sin(\varphi_i) (1 - 2 \cosh(r)) & 2 \cos(\varphi_i) (\tanh\left(\frac{r}{2}\right) - \sinh(r)) & \sin^2(\varphi_i) \sinh(r) + \cos^2(\varphi_i) \tanh\left(\frac{r}{2}\right) \end{pmatrix}.$$

Using the expression $\mathcal{S}^r = E(r)C(r)^{-1}$ and some tedious but elementary calculations we get to the main results of this section.

Proposition 4.3. *The shape operator \mathcal{S}^r of the tube M^r around the homogeneous submanifold $S_{\mathfrak{w}}$ of AN with respect to the decomposition $T_{\gamma(r)}M^r = \mathbb{R}(\operatorname{sech}\left(\frac{r}{2}\right)B + \tanh\left(\frac{r}{2}\right)\xi) \oplus \mathfrak{U} \oplus$*

$\mathfrak{H} \oplus (\oplus_{i=1}^{m_0-1} \mathfrak{F}_i) \oplus (\oplus_{i=m_0}^{m_{\pi/2}} \mathfrak{M}_i) \oplus (\oplus_{i=m_{\pi/2}+1}^m \mathfrak{P}_i)$ is given by

$$\begin{aligned} \mathcal{S}^r = & \left(\frac{1}{2} \tanh\left(\frac{r}{2}\right) I_{l+1} \right) \oplus \left(\frac{1}{2} \coth\left(\frac{r}{2}\right) I_h \right) \oplus \left(\bigoplus_{i=1}^{m_0-1} \begin{pmatrix} \frac{1}{2} \coth\left(\frac{r}{2}\right) & -\frac{1}{2} \operatorname{sech}\left(\frac{r}{2}\right) \\ -\frac{1}{2} \operatorname{sech}\left(\frac{r}{2}\right) & \tanh\left(\frac{r}{2}\right) \end{pmatrix} \right) \\ & \oplus \left(\bigoplus_{i=m_0}^{m_{\pi/2}} \begin{pmatrix} \frac{1}{2} \tanh\left(\frac{r}{2}\right) & 0 & -\frac{1}{2} \sin(\varphi_i) \operatorname{sech}\left(\frac{r}{2}\right) \\ 0 & \frac{1}{2} \coth\left(\frac{r}{2}\right) & -\frac{1}{2} \cos(\varphi_i) \operatorname{sech}\left(\frac{r}{2}\right) \\ -\frac{1}{2} \sin(\varphi_i) \operatorname{sech}\left(\frac{r}{2}\right) & -\frac{1}{2} \cos(\varphi_i) \operatorname{sech}\left(\frac{r}{2}\right) & \tanh\left(\frac{r}{2}\right) \end{pmatrix} \right) \\ & \oplus \left(\bigoplus_{i=m_{\pi/2}+1}^m \begin{pmatrix} \frac{1}{2} \tanh\left(\frac{r}{2}\right) & -\frac{1}{2} \operatorname{sech}\left(\frac{r}{2}\right) \\ -\frac{1}{2} \operatorname{sech}\left(\frac{r}{2}\right) & \tanh\left(\frac{r}{2}\right) \end{pmatrix} \right). \end{aligned}$$

As a consequence, we immediately get

Corollary 4.4. *The mean curvature \mathcal{H}^r of the tube M^r at the point $\gamma(r)$ is*

$$\begin{aligned} \mathcal{H}^r(\gamma(r)) &= \frac{1}{2} \left((h + m_{\pi/2}) \coth\left(\frac{r}{2}\right) + (2 + l + 3m - m_0) \tanh\left(\frac{r}{2}\right) \right) \\ &= \frac{1}{2} \left((\operatorname{codim} S_{\mathfrak{w}} - 1) \coth\left(\frac{r}{2}\right) + (\dim S_{\mathfrak{w}} + \dim \mathfrak{z}) \tanh\left(\frac{r}{2}\right) \right). \end{aligned}$$

Therefore, for every $r > 0$, the tube M^r around $S_{\mathfrak{w}}$ is a hypersurface with constant mean curvature, and hence, tubes around the submanifold $S_{\mathfrak{w}}$ constitute an isoparametric family of hypersurfaces in AN , that is, every tube M^r is an isoparametric hypersurface.

We can also give the characteristic polynomial of \mathcal{S}^r , which can be written as

$$p_{r,\xi}(x) = (\lambda - x)^{l+1} \left(\frac{1}{4\lambda} - x \right)^h \prod_{i=1}^m q_{r,\xi}^i(x),$$

where $\lambda = \frac{1}{2} \tanh\left(\frac{r}{2}\right)$, and

$$q_{r,\xi}^i(x) = x^2 - \left(2\lambda + \frac{1}{4\lambda} \right) x + \frac{1}{4} + \lambda^2, \quad \text{if } i = 1, \dots, m_0 - 1,$$

$$q_{r,\xi}^i(x) = -x^3 + \left(3\lambda + \frac{1}{4\lambda} \right) x^2 - \frac{1}{2} (6\lambda^2 + 1) x + \frac{16\lambda^4 + 16\lambda^2 - 1 + (4\lambda^2 - 1)^2 \cos 2\varphi_i}{32\lambda},$$

if $i = m_0, \dots, m_{\pi/2}$,

$$q_{r,\xi}^i(x) = x^2 - 3\lambda x - \frac{1}{4} + 3\lambda^2, \quad \text{if } i = m_{\pi/2} + 1, \dots, m.$$

The zeroes of $p_{r,\xi}$ are the principal curvatures of the tube M^r at the point $\gamma(r)$. Notice that the zeroes of $q_{r,\xi}^i$ for $i \in \{1, \dots, m_0 - 1\}$ are $\lambda = \frac{1}{2} \tanh\left(\frac{r}{2}\right)$ and $\lambda + \frac{1}{4\lambda} = \coth(r)$, while for $i \in \{m_{\pi/2} + 1, \dots, m\}$ they are $\frac{1}{2} (3\lambda \pm \sqrt{1 - 3\lambda^2})$. If $i \in \{m_0, \dots, m_{\pi/2}\}$ the zeroes of $q_{r,\xi}^i$ are given by complicated expressions, because they are solutions of a cubic polynomial. This polynomial coincides with the one in [2, p. 146] (where an analysis of its zeroes is carried out) and in [10, p. 5].

From these results we deduce that, in general, the principal curvatures of M^r , and even the number of principal curvatures of M^r , may vary from point to point, which implies that in general, M^r is an inhomogeneous hypersurface. Actually, *the principal curvatures of M^r are constant if and only if \mathfrak{w}^\perp has constant generalized Kähler angle*, that is, if the m -tuple $(\varphi_1, \dots, \varphi_m)$ does not depend on ξ .

We summarize the main results obtained so far.

Theorem 4.5. *Let AN be a Damek-Ricci space with Lie algebra $\mathfrak{a} \oplus \mathfrak{n}$, where \mathfrak{a} is one-dimensional and $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$ is a generalized Heisenberg algebra with center \mathfrak{z} . Let $S_{\mathfrak{w}}$ be the connected subgroup of AN whose Lie algebra is $\mathfrak{s}_{\mathfrak{w}} = \mathfrak{a} \oplus \mathfrak{w} \oplus \mathfrak{z}$, where \mathfrak{w} is any proper subspace of \mathfrak{v} .*

Then, the tubes around the submanifold $S_{\mathfrak{w}}$ are isoparametric hypersurfaces of AN , and have constant principal curvatures if and only if $\mathfrak{w}^\perp = \mathfrak{v} \ominus \mathfrak{w}$ has constant generalized Kähler angle.

5. RANK-ONE SYMMETRIC SPACES OF NONCOMPACT TYPE

In this section we present some particular examples of isoparametric hypersurfaces in the noncompact rank one symmetric spaces of nonconstant curvature. Note that, in the case of real hyperbolic spaces, our method only gives rise to tubes around totally geodesic real hyperbolic subspaces, which are well-known examples.

5.1. Complex hyperbolic spaces $\mathbb{C}H^n$. The study of this case was the aim of [10]. As explained there, isoparametric hypersurfaces arising from our method are homogeneous if and only if they have constant principal curvatures (and hence, if and only if \mathfrak{w}^\perp has constant Kähler angle). It follows that for every $n \geq 3$ one obtains inhomogeneous isoparametric hypersurfaces with nonconstant principal curvatures in $\mathbb{C}H^n$.

5.2. Quaternionic hyperbolic spaces $\mathbb{H}H^n$. Our definition of generalized Kähler angle includes as a particular case the notion of quaternionic Kähler angle introduced in [1]. The construction of several examples of subspaces of \mathbb{H}^{n-1} with constant quaternionic Kähler angle led Berndt and Brück to some examples of cohomogeneity one actions on $\mathbb{H}H^n$. In [4], Berndt and Tamaru proved that these examples exhaust all cohomogeneity one actions on $\mathbb{H}H^n$ with a non-totally geodesic singular orbit whenever $n = 2$ or the codimension of the singular orbit is two.

Moreover, they reduced the problem of classifying cohomogeneity one actions on $\mathbb{H}H^n$ to the following one: *find all subspaces \mathfrak{w}^\perp of $\mathfrak{v} = \mathbb{H}^{n-1}$ with constant quaternionic Kähler angle and determine for which of them there exists a subgroup of $Sp(n-1)Sp(1)$ that acts transitively on the unit sphere of \mathfrak{w}^\perp (via the standard representation on \mathbb{H}^{n-1}).* However, a complete classification of cohomogeneity one actions on quaternionic hyperbolic spaces is not yet known, and neither is a classification of the subspaces of \mathbb{H}^{n-1} with constant quaternionic Kähler angle, which seems to be a difficult linear algebra problem. Furthermore, it is not clear whether an answer to this latter problem would directly lead to the answer of the former. In fact, in view of Theorem 4.5 a subspace \mathfrak{w}^\perp of \mathfrak{v} with constant

quaternionic Kähler angle gives rise to an isoparametric hypersurface in $\mathbb{H}H^n$ with constant principal curvatures, but then one would have to decide whether this hypersurface is homogeneous or not. Nonetheless, what Theorem 4.5 guarantees, as well as in the case of complex hyperbolic spaces, is the existence of inhomogeneous isoparametric hypersurfaces with nonconstant principal curvatures in $\mathbb{H}H^n$, for every $n \geq 3$.

The subspaces of \mathbb{H}^{n-1} with constant quaternionic Kähler angle known up to now can take the following values of quaternionic Kähler angles [4]: $(0, 0, 0)$, $(0, \pi/2, \pi/2)$, $(\pi/2, \pi/2, \pi/2)$, $(0, 0, \pi/2)$, $(\varphi, \pi/2, \pi/2)$ and $(0, \varphi, \varphi)$. In this subsection, we will give new examples of subspaces of \mathbb{H}^{n-1} , $n \geq 5$, with constant quaternionic Kähler angle $(\varphi_1, \varphi_2, \varphi_3)$, with $0 < \varphi_1 \leq \varphi_2 \leq \varphi_3 \leq \pi/2$, and $\cos(\varphi_1) + \cos(\varphi_2) < 1 + \cos(\varphi_3)$. This includes, for example, the cases $(\varphi, \varphi, \varphi)$, with $0 < \varphi < \pi/2$, and $(\varphi_1, \varphi_2, \pi/2)$, with $\cos(\varphi_1) + \cos(\varphi_2) < 1$. Theorem 4.5 ensures that these new subspaces yield new examples of isoparametric hypersurfaces with constant principal curvatures in $\mathbb{H}H^n$. In fact, these hypersurfaces are homogeneous, as shown in Theorem 5.2. This provides a large new family of cohomogeneity one actions on quaternionic hyperbolic spaces.

From now on, $(i, i+1, i+2)$ will always be a cyclic permutation of $(1, 2, 3)$. Fix a canonical basis $\{J_1, J_2, J_3\}$ of the quaternionic structure of \mathbb{H}^{n-1} , that is, $J_i^2 = -I$ and $J_i J_{i+1} = J_{i+2} = -J_{i+1} J_i$, with $i \in \{1, 2, 3\}$.

Let $0 < \varphi_1 \leq \varphi_2 \leq \varphi_3 \leq \pi/2$ with $\cos(\varphi_1) + \cos(\varphi_2) < 1 + \cos(\varphi_3)$, and consider a four dimensional totally real subspace of \mathbb{H}^{n-1} and a basis of unit vectors $\{e_0, e_1, e_2, e_3\}$ of it, where $\langle e_0, e_i \rangle = 0$, for $i = 1, 2, 3$, and

$$(5) \quad \langle e_i, e_{i+1} \rangle = \frac{\cos(\varphi_{i+2}) - \cos(\varphi_i) \cos(\varphi_{i+1})}{\sin(\varphi_i) \sin(\varphi_{i+1})}, \quad i = 1, 2, 3.$$

The existence of these vectors is ensured by the following

Lemma 5.1. *We have:*

- (a) *Let $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$. Then, there exists a basis of unit vectors $\{e_1, e_2, e_3\}$ of \mathbb{R}^3 such that $\langle e_i, e_{i+1} \rangle = \alpha_{i+2}$ if and only if $|\alpha_i| < 1$ for all i , and $\alpha_1^2 + \alpha_2^2 + \alpha_3^2 < 1 + 2\alpha_1\alpha_2\alpha_3$.*
- (b) *Assume $0 < \varphi_1 \leq \varphi_2 \leq \varphi_3 \leq \pi/2$. Then, there exists a basis of unit vectors $\{e_1, e_2, e_3\}$ of \mathbb{R}^3 with the inner products as in (5) if and only if $\cos(\varphi_1) + \cos(\varphi_2) < 1 + \cos(\varphi_3)$.*

Proof. The proof of (a) is elementary so we omit it. For the proof of (b) we only give a few basic indications and leave the details to the reader.

We define $x_i = \cos(\varphi_i)$. With this notation, the conditions $|\alpha_i| < 1$ in part (a) are equivalent to $x_1^2 + x_2^2 + x_3^2 < 1 + 2x_1x_2x_3$, whereas the condition $\alpha_1^2 + \alpha_2^2 + \alpha_3^2 < 1 + 2\alpha_1\alpha_2\alpha_3$ turns out to be equivalent to

$$\frac{(x_1 - x_2 - x_3 + 1)(x_1 + x_2 - x_3 - 1)(x_1 - x_2 + x_3 - 1)(x_1 + x_2 + x_3 + 1)}{(x_1^2 - 1)(x_2^2 - 1)(x_3^2 - 1)} < 0.$$

Since $0 < \varphi_1 \leq \varphi_2 \leq \varphi_3 \leq \pi/2$, we have $0 \leq x_3 \leq x_2 \leq x_1 < 1$, and thus the equation above is equivalent to $x_1 + x_2 - x_3 - 1 < 0$. Finally, it is not hard to show that $0 \leq x_3 \leq x_2 \leq x_1 < 1$, and $x_1 + x_2 - x_3 - 1 < 0$ imply $x_1^2 + x_2^2 + x_3^2 < 1 + 2x_1x_2x_3$. \square

For the sake of simplicity let us define $\varphi_0 = 0$ and $J_0 = I$. Notice that $\langle J_j e_k, e_l \rangle = 0$ for $j \in \{1, 2, 3\}$ and $k, l \in \{0, 1, 2, 3\}$, because $\text{span}\{e_0, e_1, e_2, e_3\}$ is a totally real subspace of \mathbb{H}^{n-1} . Then we can define

$$\xi_k = \cos(\varphi_k)J_k e_0 + \sin(\varphi_k)J_k e_k, \quad k \in \{0, 1, 2, 3\}.$$

(Note that $\xi_0 = e_0$.) We consider the subspace \mathfrak{w}^\perp generated by these four vectors, for which $\{\xi_0, \xi_1, \xi_2, \xi_3\}$ is an orthonormal basis. Now, taking a generic unit vector $\xi = a_0\xi_0 + a_1\xi_1 + a_2\xi_2 + a_3\xi_3 \in \mathfrak{w}^\perp$, some straightforward calculations show that the matrix of the quadratic form Q_ξ defined in Theorem 3.1 with respect to the basis $\{J_1, J_2, J_3\}$ (i.e. the matrix whose entries are $\langle F_j \xi, F_k \xi \rangle = \sum_{l=0}^3 \langle J_j \xi, \xi_l \rangle \langle J_k \xi, \xi_l \rangle$, for $j, k \in \{1, 2, 3\}$) is the diagonal matrix with entries $\cos^2(\varphi_1)$, $\cos^2(\varphi_2)$, $\cos^2(\varphi_3)$. Therefore, \mathfrak{w}^\perp has constant quaternionic Kähler angle $(\varphi_1, \varphi_2, \varphi_3)$.

Next, we show that the submanifold $S_{\mathfrak{w}}$ is the singular orbit of a cohomogeneity one action on $\mathbb{H}H^n$, and hence, the tubes around $S_{\mathfrak{w}}$ are homogeneous isoparametric hypersurfaces. Let $G = Sp(n, 1)$ be the connected component of the identity of the isometry group of $\mathbb{H}H^n$, and let $K = Sp(n)Sp(1)$ be the isotropy group of G at the identity element of $AN = \mathbb{H}H^n$. Denote by $N_K(S_{\mathfrak{w}}) = \{k \in K : kS_{\mathfrak{w}}k^{-1} \subset S_{\mathfrak{w}}\}$ the normalizer of $S_{\mathfrak{w}}$ in K , and by $N_K^0(S_{\mathfrak{w}})$ the connected component of the identity. Notice that $S_{\mathfrak{w}}$ can be seen as a submanifold of $AN = \mathbb{H}H^n$, and also as a subgroup of $AN \subset G$. We have:

Theorem 5.2. *Let \mathfrak{w}^\perp be the subspace of $\mathfrak{v} = \mathbb{H}^{n-1}$ of constant quaternionic Kähler angle $(\varphi_1, \varphi_2, \varphi_3)$, with $0 < \varphi_1 \leq \varphi_2 \leq \varphi_3 \leq \pi/2$, and $\cos(\varphi_1) + \cos(\varphi_2) < 1 + \cos(\varphi_3)$, as defined above, and consider $\mathfrak{w} = \mathfrak{v} \ominus \mathfrak{w}^\perp$. Then:*

- (a) *The tubes around the submanifold $S_{\mathfrak{w}}$ are isoparametric hypersurfaces with constant principal curvatures.*
- (b) *There is a subgroup of $Sp(n-1)Sp(1)$ that acts transitively on the unit sphere of \mathfrak{w}^\perp (via the standard representation on $\mathfrak{v} = \mathbb{H}^{n-1}$).*
- (c) *The subgroup $N_K^0(S_{\mathfrak{w}})S_{\mathfrak{w}}$ of G acts isometrically with cohomogeneity one on $\mathbb{H}H^n$, $S_{\mathfrak{w}}$ is a singular orbit of this action, and the other orbits are tubes around $S_{\mathfrak{w}}$.*

Proof. Assertion (a) follows from previous calculations in this section. Part (c) follows from part (b) using [1, p. 220] (cf. [4, Theorem 4.1(i)]). Let us then prove (b). For the case $(\varphi, \pi/2, \pi/2)$, with $0 < \varphi \leq \pi/2$, assertion (b) is already known to be true [1]. If $0 < \varphi_1 \leq \varphi_2 < \varphi_3 = \pi/2$, then the proof given below needs to be adapted: we would get $F_3 = 0$, and hence \bar{F}_3 would not be defined; in that case we would explicitly define $\bar{F}_3 = \bar{F}_1 \bar{F}_2$. Thus, we will assume $\varphi_3 < \pi/2$ in what follows.

Let $i \in \{1, 2, 3\}$. As above, henceforth $(i, i+1, i+2)$ will be a cyclic permutation of $(1, 2, 3)$. First note that $F_i \xi_0 = \langle J_i \xi_0, \xi_i \rangle \xi_i = \cos(\varphi_i) \xi_i$, and thus $\bar{F}_i \xi_0 = \xi_i$. This implies $\langle F_i \xi_{i+1}, \xi_0 \rangle = -\langle \xi_{i+1}, F_i \xi_0 \rangle = -\cos(\varphi_i) \langle \xi_{i+1}, \xi_i \rangle = 0$. The skew-symmetry of J_i yields $\langle F_i \xi_{i+1}, \xi_{i+1} \rangle = 0$, and from $J_i \xi_{i+1} = \cos(\varphi_{i+1})J_{i+2}e_0 + \sin(\varphi_{i+1})J_{i+2}e_{i+1}$ we get $\langle F_i \xi_{i+1}, \xi_i \rangle = 0$ using $\langle J_j e_k, e_l \rangle = 0$. Altogether this implies that $F_i \xi_{i+1}$ must be a multiple of ξ_{i+2} , and hence one readily gets $\bar{F}_i \xi_{i+1} = \xi_{i+2} = -\bar{F}_{i+1} \xi_i$. Applying these results twice, we get $\bar{F}_i \bar{F}_{i+1} = \bar{F}_{i+2} = -\bar{F}_{i+1} \bar{F}_i$, and $\bar{F}_i^2 = -I$, so $\{\bar{F}_1, \bar{F}_2, \bar{F}_3\}$ is a quaternionic structure on \mathfrak{w}^\perp .

Let $\eta_0 \in \mathfrak{w}^\perp$ be an arbitrary unit vector. We define $f_0 = \eta_0$ and apply Lemma 3.3 to find unit vectors $f_1, f_2, f_3 \in \mathbb{H}^{n-1}$ orthogonal to f_0 , such that $\eta_i = \bar{F}_i f_0 = \cos(\varphi_i) J_i f_0 + \sin(\varphi_i) J_i f_i$. Then $f_i = -(J_i \bar{F}_i f_0 + \cos(\varphi_i) f_0) / \sin(\varphi_i)$. We easily obtain $\langle J_i \bar{F}_i f_0, f_0 \rangle = -\langle \bar{F}_i f_0, \bar{F}_i f_0 \rangle = -\cos(\varphi_i)$, and using $\bar{F}_{i+2} \bar{F}_{i+1} = -\bar{F}_i$ we get

$$\begin{aligned} \langle J_i \bar{F}_i f_0, J_{i+1} \bar{F}_{i+1} f_0 \rangle &= -\langle \bar{F}_i f_0, J_{i+2} \bar{F}_{i+1} f_0 \rangle = -\cos(\varphi_{i+2}) \langle \bar{F}_i f_0, \bar{F}_{i+2} \bar{F}_{i+1} f_0 \rangle \\ &= \cos(\varphi_{i+2}) \langle \bar{F}_i f_0, \bar{F}_i f_0 \rangle = \cos(\varphi_{i+2}). \end{aligned}$$

Altogether this implies

$$\langle f_i, f_{i+1} \rangle = \frac{\cos(\varphi_{i+2}) - \cos(\varphi_i) \cos(\varphi_{i+1})}{\sin(\varphi_i) \sin(\varphi_{i+1})} = \langle e_i, e_{i+1} \rangle.$$

We show that $\langle J_j f_k, f_l \rangle = 0$ for all $j \in \{1, 2, 3\}$, and $k, l \in \{0, 1, 2, 3\}$. For example,

$$\langle J_i f_{i+1}, f_{i+2} \rangle = \frac{1}{\sin(\varphi_{i+1}) \sin(\varphi_{i+2})} \langle J_{i+2} \bar{F}_{i+1} f_0 + \cos(\varphi_{i+1}) J_i f_0, J_{i+2} \bar{F}_{i+2} f_0 + \cos(\varphi_{i+2}) f_0 \rangle.$$

Using the properties of the generalized Kähler angle (see Theorem 3.1), and the definition of \bar{F}_i , we obtain $\langle J_{i+2} \bar{F}_{i+1} f_0, J_{i+2} \bar{F}_{i+2} f_0 \rangle = \langle \bar{F}_{i+1} f_0, \bar{F}_{i+2} f_0 \rangle = 0$. Similarly, one gets $\langle J_{i+2} \bar{F}_{i+1} f_0, f_0 \rangle = \langle J_i f_0, J_{i+2} \bar{F}_{i+2} f_0 \rangle = \langle J_i f_0, f_0 \rangle = 0$, and hence $\langle J_i f_{i+1}, f_{i+2} \rangle = 0$. Other combinations of indices can be handled analogously to obtain $\langle J_j f_k, f_l \rangle = 0$.

Now, one can apply the Gram-Schmidt process to $\{e_0, e_1, e_2, e_3\}$ to obtain an \mathbb{H} -orthonormal set $\{e'_0, e'_1, e'_2, e'_3\}$, and similarly with $\{f_0, f_1, f_2, f_3\}$, to obtain an \mathbb{H} -orthonormal set $\{f'_0, f'_1, f'_2, f'_3\}$. Then there exists an element $T \in Sp(4) \subset Sp(n-1) \subset Sp(n-1)Sp(1)$ such that $T e'_i = f'_i$ for $i = 0, 1, 2, 3$, and $T J_l = J_l T$ for $l = 1, 2, 3$. Since the transition matrices from $\{e'_i\}$ to $\{e_i\}$, and from $\{f'_i\}$ to $\{f_i\}$ coincide, we get $T e_i = f_i$, and hence $T \xi_i = \eta_i$ for $i = 0, 1, 2, 3$. Therefore, $T \xi_0 = \eta_0$ and $T \mathfrak{w}^\perp = \mathfrak{w}^\perp$. Since $\eta_0 \in \mathfrak{w}^\perp$ is arbitrary, (b) follows. \square

Remark 5.3. This construction can be extended to subspaces of \mathbb{H}^{n-1} (for n sufficiently high) with real dimension multiple of four and with constant quaternionic Kähler angle $(\varphi_1, \varphi_2, \varphi_3)$ as before, just by considering orthogonal sums of subspaces \mathfrak{w}^\perp like the one constructed above. Theorem 5.2 can easily be extended to show the homogeneity of the corresponding isoparametric hypersurfaces.

5.3. The Cayley hyperbolic plane $\mathbb{O}H^2$. Based on some results of [1], Berndt and Tamaru achieved the classification of homogeneous hypersurfaces in the Cayley hyperbolic plane $\mathbb{O}H^2$ [4]. Some of these homogeneous examples appear as particular cases of the construction we have developed. If we put $k = \dim \mathfrak{w}^\perp$, then for $k \in \{1, 2, 3, 4, 6, 7, 8\}$ the tubes M^r around $S_{\mathfrak{w}}$ are homogeneous hypersurfaces for every $r > 0$ and, together with $S_{\mathfrak{w}}$, constitute the orbits of a cohomogeneity one action; if $k = 5$, none of the tubes around $S_{\mathfrak{w}}$ is homogeneous [1, p. 233].

Therefore, our method yields a family of inhomogeneous isoparametric hypersurfaces if and only if the codimension of $S_{\mathfrak{w}}$ is $k = 5$. But for this case, something unexpected happens: these hypersurfaces have constant principal curvatures.

Let ξ be a unit vector in \mathfrak{w}^\perp . Taking into account that $\mathfrak{v} = \mathbb{R}^8$ is an irreducible Clifford module of $\mathfrak{z} = \mathbb{R}^7$, the properties of generalized Heisenberg algebras imply that the linear map $Z \in \mathfrak{z} \mapsto J_Z \xi \in \mathfrak{v} \ominus \mathbb{R}\xi$ is an isometry. Hence, we can find an orthonormal basis $\{Z_1, \dots, Z_7\}$ of the vector space \mathfrak{z} such that $\mathfrak{w} = \text{span}\{J_{Z_5}\xi, J_{Z_6}\xi, J_{Z_7}\xi\}$ and $\mathfrak{w}^\perp = \text{span}\{\xi, J_{Z_1}\xi, J_{Z_2}\xi, J_{Z_3}\xi, J_{Z_4}\xi\}$. It is then clear that $J_{Z_5}, J_{Z_6}, J_{Z_7}$ map ξ into \mathfrak{w} and $J_{Z_1}, J_{Z_2}, J_{Z_3}, J_{Z_4}$ map ξ into \mathfrak{w}^\perp . By definition, the generalized Kähler angle of ξ with respect to \mathfrak{w}^\perp is $(0, 0, 0, 0, \pi/2, \pi/2, \pi/2)$.

As the above argument is valid for every unit $\xi \in \mathfrak{w}^\perp$, we conclude that \mathfrak{w}^\perp has constant Kähler angle and so, by Theorem 4.5 and the inhomogeneity result in [1], we obtain:

Theorem 5.4. *The tubes around the homogeneous submanifolds $S_{\mathfrak{w}}$ with $\dim \mathfrak{w}^\perp = 5$ are inhomogeneous isoparametric hypersurfaces with constant principal curvatures in the Cayley hyperbolic plane.*

Let us consider now the case $k = 4$. A similar argument as above can be used to show that the generalized Kähler angle of \mathfrak{w}^\perp is $(0, 0, 0, \pi/2, \pi/2, \pi/2, \pi/2)$. Then, the calculations before Theorem 4.5 show that for any choice of \mathfrak{w}^\perp the principal curvatures, and their corresponding multiplicities, of the tube of radius r around $S_{\mathfrak{w}}$, depend only on r . By [4, Theorem 4.7], it follows that there are uncountably many orbit equivalence classes of cohomogeneity one actions on $\mathbb{O}H^2$ arising from this method with $k = 4$. Therefore, we obtain an *uncountable family of noncongruent homogeneous isoparametric systems with the same constant principal curvatures, counted with multiplicities*. This phenomenon was known in the inhomogeneous case for spheres [11], and in the homogeneous case for noncompact symmetric spaces of rank higher than two [3].

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