POLAR ACTIONS ON THE COMPLEX HYPERBOLIC PLANE

JÜRGEN BERNDT AND JOSÉ CARLOS DÍAZ-RAMOS

Abstract. We classify the polar actions on the complex hyperbolic plane $\mathbb{C}H^2$ up to orbit equivalence. Apart from the trivial and transitive polar actions, there are five polar actions of cohomogeneity one and four polar actions of cohomogeneity two.

1. Introduction

Let M be a Riemannian manifold and denote by $I(M)$ its isometry group. A connected closed subgroup G of $I(M)$ is said to act polarly on M if there exists a connected closed submanifold Σ of M that intersects all the orbits of G orthogonally. Thus, for each $p \in M$ the intersection of Σ and the orbit $G \cdot p$ of G containing p is nonempty, and for all $p \in \Sigma$ the tangent space $T_p\Sigma$ of Σ at p is contained in the normal space $\nu_p(G \cdot p)$ of $G \cdot p$ at p. The submanifold Σ is called a section of the action.

Polar actions on Riemannian symmetric spaces of compact type are understood reasonably well, see [8], [9] and [11] for more details. On the other hand, due to the possible noncompactness of the groups, polar actions on Riemannian symmetric spaces of noncompact type are not understood except for the real hyperbolic spaces. The purpose of this paper is to classify the polar actions on the complex hyperbolic plane $\mathbb{C}H^2$ up to orbit equivalence. This is the first complete such classification on a nontrivial Riemannian symmetric space of noncompact type. We hope that this investigation will provide further insight into the structure theory of polar actions.

The complex hyperbolic plane is a Riemannian symmetric space of noncompact type, namely $\mathbb{C}H^2 = G/K$ with $G = SU(1,2)$ and $K = S(U(1)U(2))$. Denote by $o \in \mathbb{C}H^2$ the unique fixed point of the K-action on $\mathbb{C}H^2$ and by $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ the corresponding Cartan decomposition of the Lie algebra $\mathfrak{g}$ of G. Denote by $\theta \in \mathrm{Aut}(\mathfrak{g})$ the corresponding Cartan involution. Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$ and $\mathfrak{g} = \mathfrak{g}_{-2\alpha} \oplus \mathfrak{g}_{-\alpha} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$ the corresponding restricted root space decomposition of $\mathfrak{g}$. The root space $\mathfrak{g}_0$ decomposes into $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{a}$, where $\mathfrak{k}_0$ is the centralizer of $\mathfrak{k}$ in $\mathfrak{a}$. The complex structure on $\mathbb{C}H^2$ leaves the root space $\mathfrak{g}_\alpha$ invariant, and therefore $\mathfrak{g}_\alpha \cong \mathbb{C}$. By $\mathfrak{g}_\alpha^{\mathbb{R}}$ we denote a real form of $\mathfrak{g}_\alpha$, that is, a real one-dimensional linear subspace of $\mathfrak{g}_\alpha$.

The subalgebra $\mathfrak{n} = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$ is nilpotent and the action of the connected closed subgroup N of G with Lie algebra $\mathfrak{n}$ on $\mathbb{C}H^2$ induces a foliation of $\mathbb{C}H^2$ by horospheres. On a

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horosphere there are two distinguished types of horocycles, those which are generated by a real form $\mathfrak{g}_\alpha^\mathbb{R}$ and those which are generated by $\mathfrak{g}_{2\alpha}$. In the first case the horocycle lies in a totally geodesic real hyperbolic plane $\mathbb{R}H^2 \subset \mathbb{C}H^2$, and in the second case the horocycle lies in a totally geodesic complex hyperbolic line $\mathbb{C}H^1 \subset \mathbb{C}H^2$. We call such horocycles real and complex, respectively. The subalgebra $\mathfrak{n}$ is isomorphic to the Heisenberg algebra, and every horosphere in $\mathbb{C}H^2$ with the induced metric is isometric to the 3-dimensional Heisenberg group with a suitable left-invariant Riemannian metric. The subalgebra $\mathfrak{g}_\alpha^\mathbb{R} \oplus \mathfrak{g}_{2\alpha}$ of $\mathfrak{n}$ is abelian and the orbit through o of the corresponding connected closed subgroup of N is a Euclidean plane $\mathbb{E}^2$ embedded in the horosphere as a non-totally geodesic minimal surface.

Main Theorem. *For each of the subalgebras $\mathfrak{h}$ of $\mathfrak{su}(1,2)$ listed below the connected closed subgroup H of $SU(1,2)$ with Lie algebra $\mathfrak{h}$ acts polarly on $\mathbb{C}H^2$:*

(i) *Actions of cohomogeneity one - the section Σ is a totally geodesic real hyperbolic line $\mathbb{R}H^1 \subset \mathbb{C}H^2$:*
- (a) $\mathfrak{h} = \mathfrak{k} = \mathfrak{s}(\mathfrak{u}(1) \oplus \mathfrak{u}(2)) \cong \mathfrak{u}(2)$*; the orbits are $\{o\}$ and the distance spheres centered at o;*
- (b) $\mathfrak{h} = \mathfrak{g}_{-2\alpha} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{2\alpha} = \mathfrak{s}(\mathfrak{u}(1,1) \oplus \mathfrak{u}(1)) \cong \mathfrak{u}(1,1)$*; the orbits are a totally geodesic complex hyperbolic line $\mathbb{C}H^1 \subset \mathbb{C}H^2$ and the tubes around $\mathbb{C}H^1$;*
- (c) $\mathfrak{h} = \theta(\mathfrak{g}_\alpha^\mathbb{R}) \oplus \mathfrak{a} \oplus \mathfrak{g}_\alpha^\mathbb{R} \cong \mathfrak{so}(1,2)$*; the orbits are a totally geodesic real hyperbolic plane $\mathbb{R}H^2 \subset \mathbb{C}H^2$ and the tubes around $\mathbb{R}H^2$;*
- (d) $\mathfrak{h} = \mathfrak{k}_0 \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$ *or* $\mathfrak{h} = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$*; the orbits form a foliation of $\mathbb{C}H^2$ by horospheres;*
- (e) $\mathfrak{h} = \mathfrak{a} \oplus \mathfrak{g}_\alpha^\mathbb{R} \oplus \mathfrak{g}_{2\alpha}$*; the orbits form a foliation of $\mathbb{C}H^2$; one of its leaves is the minimal ruled real hypersurface of $\mathbb{C}H^2$ generated by a real horocycle in $\mathbb{C}H^2$, and the other leaves are the equidistant hypersurfaces.*

(ii) *Actions of cohomogeneity two - the section Σ is a totally geodesic real hyperbolic plane $\mathbb{R}H^2 \subset \mathbb{C}H^2$:*
- (a) $\mathfrak{h} = \mathfrak{k} \cap (\mathfrak{g}_{-2\alpha} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{2\alpha}) = \mathfrak{s}(\mathfrak{u}(1) \oplus \mathfrak{u}(1) \oplus \mathfrak{u}(1)) \cong \mathfrak{u}(1) \oplus \mathfrak{u}(1)$*; the orbits are obtained by intersecting the orbits of the two cohomogeneity one actions* (a) *and* (b) *in* (i)*: the action has one fixed point o, and on each distance sphere centered at o the orbits are two circles as singular orbits and 2-dimensional tori as principal orbits;*
- (b) $\mathfrak{h} = \mathfrak{g}_0$*; the action leaves a totally geodesic $\mathbb{C}H^1 \subset \mathbb{C}H^2$ invariant. On this $\mathbb{C}H^1$ the action induces a foliation by a totally geodesic real hyperbolic line $\mathbb{R}H^1 \subset \mathbb{C}H^1$ and its equidistant curves in $\mathbb{C}H^1$. The other orbits are 2-dimensional cylinders whose axis is one of the curves in that $\mathbb{C}H^1$;*
- (c) $\mathfrak{h} = \mathfrak{k}_0 \oplus \mathfrak{g}_{2\alpha}$*; the orbits are obtained by intersecting the orbits of the two cohomogeneity one actions* (b) *and* (d) *in* (i)*: the action leaves a horosphere foliation invariant, and on each horosphere the orbits consist of a complex horocycle and the tubes around it;*
- (d) $\mathfrak{h} = \mathfrak{g}_\alpha^\mathbb{R} \oplus \mathfrak{g}_{2\alpha}$*; the orbits are obtained by intersecting the orbits of the two cohomogeneity one actions* (d) *and* (e) *in* (i)*: the action leaves a horosphere foliation*

invariant, and on each horosphere the action induces a foliation for which the minimally embedded Euclidean plane $\mathbb{E}^2$ *and its equidistant surfaces are the leaves.*

Every polar action on $\mathbb{C}H^2$ *is either trivial, transitive, or orbit equivalent to one of the polar actions described above.*

The paper is organized as follows. In Section 2 we summarize some basic material, and in Section 3 we present the proof of the Main Theorem. The only two interesting cases arise for cohomogeneity one and cohomogeneity two. The cohomogeneity one case was settled in [4], and the cohomogeneity two case for actions without singular orbits in [2]. The main contribution of this paper to the classification is the analysis of the cohomogeneity two case with singular orbits.

2. Preliminaries

We refer to [6] for more information. We denote by $\mathbb{C}H^2 = SU(1,2)/S(U(1)U(2))$ the complex hyperbolic plane with constant holomorphic sectional curvature -1. Define $G = SU(1,2)$ and denote by $K \cong S(U(1)U(2))$ the isotropy group of G at some point $o \in \mathbb{C}H^2$. The Cartan decomposition of $\mathfrak{g}$ with respect to o is $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{g}$ and $\mathfrak{k}$ are the Lie algebras of G and K respectively, and $\mathfrak{p}$ is the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to the Killing form B of $\mathfrak{g}$. Let θ be the corresponding Cartan involution. Then $\langle X, Y\rangle = -B(\theta X, Y)$ defines a positive definite inner product on $\mathfrak{g}$ that satisfies $\langle \operatorname{ad}(X)Y, Z\rangle = -\langle Y, \operatorname{ad}(\theta X)Z\rangle$ for all X, Y, $Z \in \mathfrak{g}$. As usual, ad and Ad will denote the adjoint maps of $\mathfrak{g}$ and G, respectively. It is customary to identify $\mathfrak{p}$ with the tangent space $T_o\mathbb{C}H^2$.

A maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$ is 1-dimensional and induces a restricted root space decomposition $\mathfrak{g} = \mathfrak{g}_{-2\alpha} \oplus \mathfrak{g}_{-\alpha} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$, where $\mathfrak{g}_\lambda = \{X \in \mathfrak{g} : \operatorname{ad}(H)X = \lambda(H)X \text{ for all } H \in \mathfrak{a}\}$. Recall that $[\mathfrak{g}_\lambda, \mathfrak{g}_\mu] = \mathfrak{g}_{\lambda+\mu}$, $\theta\mathfrak{g}_\lambda = \mathfrak{g}_{-\lambda}$, and $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{a}$, where $\mathfrak{k}_0 = \mathfrak{g}_0 \cap \mathfrak{k}$. Note that $\mathfrak{k}_0$ is isomorphic to $\mathfrak{u}(1)$ and $\mathfrak{g}_{2\alpha}$ is 1-dimensional. Let $\mathfrak{n} = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$, which is a nilpotent subalgebra of $\mathfrak{g}$ isomorphic to the 3-dimensional Heisenberg algebra. Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ is an Iwasawa decomposition of $\mathfrak{g}$ and the connected subgroup AN of G whose Lie algebra is $\mathfrak{a} \oplus \mathfrak{n}$ acts simply transitively on $\mathbb{C}H^2$. We endow AN, and hence $\mathfrak{a} \oplus \mathfrak{n}$, with the left-invariant metric $\langle\,\cdot\,,\,\cdot\,\rangle_{AN}$, and the complex structure J that make $\mathbb{C}H^2$ and AN isometric. This implies that $\langle X, Y\rangle_{AN} = \langle X_{\mathfrak{a}}, Y_{\mathfrak{a}}\rangle + \frac{1}{2}\langle X_{\mathfrak{n}}, Y_{\mathfrak{n}}\rangle$ for $X, Y \in \mathfrak{a} \oplus \mathfrak{n} \cong T_1AN$, where the subscript means orthogonal projection. The complex structure J on $\mathfrak{a} \oplus \mathfrak{n}$ satisfies that $J\mathfrak{g}_\alpha = \mathfrak{g}_\alpha$ and $J\mathfrak{a} = \mathfrak{g}_{2\alpha}$. Let B be a unit vector in $\mathfrak{a}$ and define $Z = JB \in \mathfrak{g}_{2\alpha}$. Note that $\langle B, B\rangle = \langle B, B\rangle_{AN} = 1$, whereas $\langle Z, Z\rangle = 2\langle Z, Z\rangle_{AN} = 2$. Then

$$[aB + U + xZ, bB + V + yZ] = -\frac{b}{2}U + \frac{a}{2}V + \left(-bx + ay + \frac{1}{2}\langle JU, V\rangle\right) Z,$$

where a, b, x, $y \in \mathbb{R}$, and U, $V \in \mathfrak{g}_\alpha$. Finally, we define $\mathfrak{p}_\lambda = (1-\theta)\mathfrak{g}_\lambda \subset \mathfrak{p}$. Then, $\mathfrak{p} = \mathfrak{a} \oplus \mathfrak{p}_\alpha \oplus \mathfrak{p}_{2\alpha}$, $\mathfrak{p}_\alpha$ is complex, and $\mathfrak{p}_{2\alpha}$ is one-dimensional. If i denotes the complex structure of $\mathfrak{p}$, we have $iB = \frac{1}{2}(1-\theta)Z$, and $i(1-\theta)U = (1-\theta)JU$.

3. Proof of the Main Theorem

For a Riemannian manifold M we denote by $I(M)$ the isometry group of M and by T_pM the tangent space of M at $p \in M$. If Σ is a submanifold of M, we denote by $\nu_p\Sigma$ the normal space of Σ at $p \in \Sigma$. For a subgroup $H \subset I(M)$ we denote by $H \cdot p$ the orbit of the H-action on M containing p. We first recall a result from [7].

Proposition 3.1. *Let M be a complete connected Riemannian manifold and Σ be a connected totally geodesic embedded submanifold of M. A closed subgroup H of $I(M)$ acts polarly on M with section Σ if and only if there exists a point $o \in M$ such that*

(a) $T_o\Sigma \subset \nu_o(H \cdot o)$,
(b) *the slice representation of H_o on $\nu_o(H \cdot o)$ is polar and $T_o\Sigma$ is a section,*
(c) $\nabla_v X^* \in \nu_o\Sigma$ *for all $v \in T_o\Sigma$ and all $X \in \mathfrak{h}$, where X^* denotes the smooth vector field on M defined by $X^*_p = \frac{d}{dt}_{|t=0} \operatorname{Exp}(tX)(p)$ for each $p \in M$.*

We will use a refinement of this result for symmetric spaces of noncompact type. Let $M = G/K$ be a Riemannian symmetric space of noncompact type, where $G = I^o(M)$ is the connected component of $I(M)$ containg the identity transformation of M and K is the isotropy subgroup of G at $o \in M$. Let $\mathfrak{g}$ be the Lie algebra of G, B the Killing form of $\mathfrak{g}$, and θ the Cartan involution of the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. The inner product defined by $\langle X, Y\rangle = -B(X, \theta Y)$ for all $X, Y \in \mathfrak{g}$ is positive definite. We identify T_oM with $\mathfrak{p}$ in the usual way.

Corollary 3.2. *Let $M = G/K$ be a Riemannian symmetric space of noncompact type, and let Σ be a connected totally geodesic submanifold of M with $o \in \Sigma$. A connected closed subgroup H of $I(M)$ acts polarly on M with section Σ if and only if $T_o\Sigma \subset \nu_o(H \cdot o)$, $T_o\Sigma$ is a section of the slice representation of H_o on $\nu_o(H \cdot o)$, and*

$$\langle [v, w], X\rangle = -B([v, w], \theta X) = 0 \text{ for all } v,\ w \in T_o\Sigma \subset \mathfrak{p} \text{ and all } X \in \mathfrak{h}.$$

Proof. Every totally geodesic submanifold in G/K is embedded. Conditions (a) and (b) of Proposition 3.1 are satisfied by hypothesis, so we only have to check condition (c). Let $v \in T_o\Sigma$ and $X \in \mathfrak{h}$. Then, v can be considered as a vector in $\mathfrak{p} \subset \mathfrak{g}$, and hence we have $\nabla_v X^* = [v^*, X^*]_o = -[v, X]^*_o = -[v, X]_{\mathfrak{p}}$, where the subscript means orthogonal projection onto $\mathfrak{p}$ (see for example [12, § IV.6]). Therefore, $\nabla_v X^* \in \nu_o\Sigma$ if and only if for each $w \in T_o\Sigma \subset \mathfrak{p}$ we have $0 = \langle \nabla_v X^*, w\rangle = -\langle [v, X]_{\mathfrak{p}}, w\rangle = \langle X, [v, w]\rangle$. ☐

Assume that H is a connected closed subgroup of $SU(1, 2)$ acting polarly on $\mathbb{C}H^2$, and let Σ be a section of the action of H. Since Σ is totally geodesic, it is congruent to a point, a geodesic which we can view as a totally geodesic $\mathbb{R}H^1$, a totally geodesic complex hyperbolic line $\mathbb{C}H^1$, a totally geodesic real hyperbolic plane $\mathbb{R}H^2$, or the whole complex hyperbolic plane $\mathbb{C}H^2$. Clearly, if Σ is a point, then the action of H is transitive, and if Σ is the entire space, then the action is trivial. So the only possibilities left are $\mathbb{R}H^1$, $\mathbb{R}H^2$, and $\mathbb{C}H^1$.

If $\Sigma = \mathbb{R}H^1$, then the action of H is of cohomogeneity one (and also hyperpolar). Cohomogeneity one actions on complex hyperbolic spaces were classified in [4]. A more geometric classification in terms of the constancy of the principal curvatures of a real hypersurface in $\mathbb{C}H^2$ can be found in [1]. This corresponds to item (i) of the Main Theorem.

Therefore, the only remaining possibility for Σ is to be an $\mathbb{R}H^2$ or a $\mathbb{C}H^1$, which both have dimension 2. Hence, from now on we assume that H acts on $\mathbb{C}H^2$ with cohomogeneity 2.

If all the orbits of the action of H have the same dimension, then there are no exceptional orbits and H induces a homogeneous polar foliation of $\mathbb{C}H^2$ [3]. Homogeneous polar foliations of complex hyperbolic spaces were classified by the authors in [2]. This corresponds to case (iid) of the Main Theorem.

Thus we can assume from now on that the action of H has a singular orbit. For dimension reasons, this orbit can only be 0-dimensional or 1-dimensional. Assume first that there is a 0-dimensional orbit, that is, there is a point $o \in \mathbb{C}H^2$ that is fixed by the action of H. In this case the group H has to be compact. Indeed, let $\{h_n\}$ be a sequence contained in H. Since H fixes o, we have that $\{h_n(o)\}$ converges to o. Since the group is closed in $SU(1,2)$, the action of H is proper and hence, by definition of proper action, $\{h_n\}$ has a convergent subsequence. This shows that H is compact. In any case, polar actions with a fixed point on $\mathbb{C}H^2$ have been classified in [7, Proposition 12 (ii)]. There are exactly three possibilities up to orbit equivalence: the trivial action, the isotropy action of $S(U(1)U(2))$ (which is of cohomogeneity one), and the action of $S(U(1)U(1)U(1)) \cong U(1) \cdot U(1)$, which is of cohomogeneity two and corresponds to case (iia) of the Main Theorem. It is worthwhile to point out at this stage that polar actions with a fixed point in $\mathbb{C}H^2$ correspond to polar actions in $\mathbb{C}P^1$. The only nontrivial and nontransitive polar action on $\mathbb{C}P^1$ up to orbit equivalence is the isotropy action of $U(1) \cong S(U(1)U(1))$, which has two fixed points as singular orbits; the rest of the orbits are principal, and in particular one of its orbits is a totally geodesic $\mathbb{R}P^1$ in $\mathbb{C}P^1$. This action is orbit equivalent to the action of $SO(2)$ on $\mathbb{C}P^1$.

Finally, let us assume that H has a singular orbit of dimension 1 and no fixed points. Let $\mathfrak{h}$ be the Lie algebra of H. Let $\mathfrak{l}$ be a proper maximal subalgebra of $\mathfrak{su}(1,2)$ containing $\mathfrak{h}$. It is known that $\mathfrak{l}$ is either reductive or parabolic (see [10] or [5, Theorem 3.2] for a more detailed proof).

Assume first that $\mathfrak{l}$ is reductive. Then, up to conjugation, $\mathfrak{l}$ is $\mathfrak{s}(\mathfrak{u}(1,1) \oplus \mathfrak{u}(1)) \cong \mathfrak{su}(1,1)$, $\mathfrak{so}(1,2)$, or $\mathfrak{s}(\mathfrak{u}(1) \oplus \mathfrak{u}(2)) \cong \mathfrak{u}(2)$. The last possibility corresponds to a compact group and hence $H \subset S(U(1)U(2))$ would have a fixed point by Cartan's fixed point theorem, contradicting our assumption. Then $\mathfrak{l} = \mathfrak{su}(1,1)$ or $\mathfrak{l} = \mathfrak{so}(1,2)$. In both cases $\mathfrak{l}$ has dimension 3, and the action of L, the connected Lie subgroup of $SU(1,2)$ whose Lie algebra is $\mathfrak{l}$, is of cohomogeneity one. Thus, $\dim \mathfrak{h} < 3$. By the classification of Lie algebras of low dimension, this implies that $\mathfrak{h}$ is solvable, and hence it is contained in a Borel subalgebra $\mathfrak{b}$, that is, a maximal solvable subalgebra of $\mathfrak{su}(1,2)$. There are, up to conjugation, exactly two types of Borel subalgebras in $\mathfrak{su}(1,2)$: of maximally compact type, and of maximally noncompact type. Again, $\mathfrak{h}$ cannot be contained in a Borel subalgebra of maximally compact type, because such a subalgebra $\mathfrak{b}$ is compact and hence H would have a fixed point by Cartan's fixed point theorem. Hence $\mathfrak{h}$ is contained in a Borel subalgebra of maximally noncompact type. Then, with respect to a suitable Cartan decomposition $\mathfrak{su}(1,2) = \mathfrak{k} \oplus \mathfrak{p}$, and a suitable maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$, we have $\mathfrak{b} = \mathfrak{t} \oplus \mathfrak{a} \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$.

Here $\mathfrak{t}\oplus\mathfrak{a}$ is a Cartan subalgebra of $\mathfrak{su}(1,2)$; $\mathfrak{a}\subset\mathfrak{p}$ is called is vector part, and $\mathfrak{t}\subset\mathfrak{k}$ is called the toroidal part. It is easy to see in this case that $\mathfrak{t}=\mathfrak{k}_0$. Hence, $\mathfrak{b}=\mathfrak{k}_0\oplus\mathfrak{a}\oplus\mathfrak{g}_\alpha\oplus\mathfrak{g}_{2\alpha}$ turns out to be a parabolic subalgebra. Thus, we may assume from now on that the maximal subalgebra $\mathfrak{l}$ containing $\mathfrak{h}$ is parabolic. Write, as before, this parabolic subalgebra as $\mathfrak{l}=\mathfrak{k}_0\oplus\mathfrak{a}\oplus\mathfrak{g}_\alpha\oplus\mathfrak{g}_{2\alpha}$.

Since a subgroup of $SU(1,2)$ whose Lie algebra is contained in $\mathfrak{a}\oplus\mathfrak{g}_\alpha\oplus\mathfrak{g}_{2\alpha}$ induces a foliation on $\mathbb{C}H^2$, we conclude that the orthogonal projection of $\mathfrak{h}$ onto $\mathfrak{k}_0$ is nonzero. Moreover, we know that $\mathfrak{k}_0$ is 1-dimensional, and that the orbit of H through the origin o is at most 2-dimensional, which implies that the orthogonal projection of $\mathfrak{h}$ onto $\mathfrak{a}\oplus\mathfrak{g}_\alpha\oplus\mathfrak{g}_{2\alpha}$ is at most 2-dimensional. Therefore, $\mathfrak{h}$ can be written as $\mathfrak{h}=\mathfrak{k}_0\oplus\mathbb{R}\xi\oplus\mathbb{R}\eta$, where ξ, $\eta\in\mathfrak{a}\oplus\mathfrak{g}_\alpha\oplus\mathfrak{g}_{2\alpha}$ are linearly independent vectors, or $\mathfrak{h}=\mathbb{R}(T+\xi)\oplus\mathbb{R}\eta$, with $T\in\mathfrak{k}_0$, $T\neq 0$, and ξ, $\eta\in\mathfrak{a}\oplus\mathfrak{g}_\alpha\oplus\mathfrak{g}_{2\alpha}$. We analyze both possibilities.

Assume first that $\mathfrak{h}=\mathfrak{k}_0\oplus\mathbb{R}\xi\oplus\mathbb{R}\eta$, where ξ, $\eta\in\mathfrak{a}\oplus\mathfrak{g}_\alpha\oplus\mathfrak{g}_{2\alpha}$ are linearly independent vectors. It follows from the properties of root spaces, the fact that $\mathfrak{h}$ is a Lie algebra, and the skew-symmetry of the elements of $\operatorname{ad}(\mathfrak{k}_0)$, that $\operatorname{ad}(\mathfrak{k}_0)\xi\in\mathfrak{g}_\alpha\cap(\mathfrak{h}\ominus\mathbb{R}\xi)=\mathbb{R}\eta$, $\operatorname{ad}(\mathfrak{k}_0)\eta\in\mathfrak{g}_\alpha\cap(\mathfrak{h}\ominus\mathbb{R}\eta)=\mathbb{R}\xi$. Since $\langle\operatorname{ad}(T)\xi,\eta\rangle=-\langle\operatorname{ad}(T)\eta,\xi\rangle$ for each $T\in\mathfrak{k}_0$, $\operatorname{ad}(\mathfrak{k}_0)\xi$ and $\operatorname{ad}(\mathfrak{k}_0)\eta$ are both zero, or both nonzero. If $\operatorname{ad}(\mathfrak{k}_0)\xi=\operatorname{ad}(\mathfrak{k}_0)\eta=0$, we conclude that ξ, $\eta\in\mathfrak{a}\oplus\mathfrak{g}_{2\alpha}$, so $\mathfrak{h}=\mathfrak{k}_0\oplus\mathfrak{a}\oplus\mathfrak{g}_{2\alpha}$. This is not possible because the corresponding group H would act with cohomogeneity one. Let us assume then that $\operatorname{ad}(\mathfrak{k}_0)\xi$ and $\operatorname{ad}(\mathfrak{k}_0)\eta$ are both nonzero. Hence, $\mathbb{R}\xi\oplus\mathbb{R}\eta\subset\mathfrak{g}_\alpha$, and since they are linearly independent and $\mathfrak{g}_\alpha$ is 2-dimensional, it follows that $\mathfrak{h}=\mathfrak{k}_0\oplus\mathfrak{g}_\alpha$. This is not possible because $\mathfrak{k}_0\oplus\mathfrak{g}_\alpha$ is not a Lie algebra.

In order to deal with the second possibility we start first with

Lemma 3.3. *Assume $\mathfrak{h}=\mathbb{R}(T+\xi)\oplus\mathbb{R}\eta$, with $0\neq T\in\mathfrak{k}_0$ and ξ, $\eta\in\mathfrak{a}\oplus\mathfrak{g}_\alpha\oplus\mathfrak{g}_{2\alpha}$. Then $\mathfrak{h}$ can be written in one of the following forms:*

(a) *$0\neq\xi\in\mathfrak{g}_\alpha$ and $0\neq\eta\in\mathfrak{g}_{2\alpha}$, or*
(b) *$\xi=0$ and $0\neq\eta\in\mathfrak{a}\oplus\mathfrak{g}_{2\alpha}$, or*
(c) *$\xi=[T,Y]+Z$ and $\eta=2B+Y+dZ$, where $d\in\mathbb{R}$ and $0\neq Y\in\mathfrak{g}_\alpha$ such that $[[T,Y],Y]=2Z$.*

Proof. Write $\xi=aB+X+bZ$, and $\eta=cB+Y+dZ$, with a, b, c, $d\in\mathbb{R}$, and X, $Y\in\mathfrak{g}_\alpha$. We may assume that $\langle X,Y\rangle=0$.

First of all, by the algebraic properties of root spaces, $[T+\xi,\eta]\in(\mathfrak{g}_\alpha\oplus\mathfrak{g}_{2\alpha})\cap\mathfrak{h}\subset\mathbb{R}\eta$. We can therefore write $\lambda\eta=[T+\xi,\eta]$ for some $\lambda\in\mathbb{R}$. Inserting the above expressions for ξ and η, and taking the components of the resulting expression in $\mathfrak{a}$, $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{2\alpha}$ we get

$$\lambda c=0, \tag{1}$$

$$\lambda Y=\frac{a}{2}Y-\frac{c}{2}X+[T,Y], \tag{2}$$

$$\lambda d=ad-bc+\frac{1}{2}\langle[X,Y],Z\rangle. \tag{3}$$

We consider the two cases $Y=0$ and $Y\neq 0$ separately.

Case 1: $Y=0$.

If $\lambda = 0$, Equation (3) with $Y = 0$ says that $ad - bc = 0$, and hence the vectors $aB + bZ$ and $cB + dZ$ are linearly dependent, so we can write $\mathfrak{h} = \mathbb{R}(T + X) \oplus \mathbb{R}(cB + dZ)$. Now, from (2) we get $cX = 0$. If $c = 0$ then $\mathfrak{h} = \mathbb{R}(T + X) \oplus \mathfrak{g}_{2\alpha}$ and we are in case (a), whereas if $X = 0$ we are in case (b).

If $\lambda \neq 0$, we get $c = 0$ from (1) and therefore we can write $\mathfrak{h} = \mathbb{R}(T + aB + X) \oplus \mathfrak{g}_{2\alpha}$. It is obvious that in this case the orbit $H \cdot o$ is 2-dimensional. Hence, if Σ is a section of the action with $o \in \Sigma$, we must have $T_o\Sigma = \{v \in \mathfrak{p} : \langle v, \xi\rangle = \langle v, \eta\rangle = 0\}$. For $X = 0$ we have $T_o\Sigma = \mathfrak{p}_\alpha$, and Corollary 3.2 implies $0 = \langle Z, [(1-\theta)U, (1-\theta)JU]\rangle = \langle Z, [U, JU]\rangle = \|U\|^2$ for all $U \in \mathfrak{g}_\alpha$, which is impossible. Therefore we must have $X \neq 0$, and then $T_o\Sigma = \mathbb{R}((1-\theta)JX) \oplus \mathbb{R}(-\|X\|^2 B + a(1-\theta)X)$. Since Σ is totally geodesic, $T_o\Sigma$ is either real or complex, and this can happen only if $a = 0$, which implies case (a).

Case 2: $Y \neq 0$.

As X and Y are orthogonal and $[T, Y]$ is orthogonal to Y, Equation (2) implies $\lambda = \frac{a}{2}$ and $[T, Y] = \frac{c}{2}X$. Since the connected subgroup $K_0 \cong U(1)$ of $S(U(1)U(2))$ with Lie algebra $\mathfrak{k}_0$ acts transitively on the unit circle in $\mathfrak{g}_\alpha$, it follows that $[T, Y] \neq 0$ and hence also $c \neq 0$ and $X \neq 0$. From (1) we get $\lambda = 0$ (which implies that $\mathfrak{h}$ is abelian) and thus also $a = 0$, and from (3) we then get $\langle [X, Y], Z\rangle = 2bc$. Since $X, Y \neq 0$ and $\dim \mathfrak{g}_\alpha = 2$ we also get $b \neq 0$. Finally, since $b, c \neq 0$ we can renormalize T and Y so that $b = 1$ and $c = 2$, thus getting (c). $\square$

The next step is to show that the actions arising from Lemma 3.3 are orbit equivalent to the actions described in items (iib) or (iic) of the Main Theorem. We have three different possibilities:

(a) $\mathfrak{h} = \mathbb{R}(T + X) \oplus \mathfrak{g}_{2\alpha}$ *with* $0 \neq T \in \mathfrak{k}_0$ *and* $0 \neq X \in \mathfrak{g}_\alpha$.

Since $T \neq 0$ and $\mathrm{ad}(T)$ is skewsymmetric, we have $[T, [T, X]] = -\rho X$ for some $\rho > 0$. We define $g = \mathrm{Exp}(-\frac{1}{\rho}[T, X]) \in G$. Then we get $\mathrm{Ad}(g)Z = Z$ and, since $[[T, X], T + X] = \rho X + [[T, X], X]$,

$$\mathrm{Ad}(g)(T + X) = T + X - X - \frac{1}{\rho}[[T, X], X] + \frac{1}{2\rho}[[T, X], X] = T - \frac{1}{2\rho}[[T, X], X].$$

Since $[[T, X], X] \in \mathfrak{g}_{2\alpha}$ this implies $\mathrm{Ad}(g)(\mathfrak{h}) = \mathfrak{k}_0 \oplus \mathfrak{g}_{2\alpha}$. It follows that the action is conjugate to the one in (iic) of the Main Theorem.

(b) $\mathfrak{h} = \mathfrak{k}_0 \oplus \mathbb{R}(aB + bZ)$ *with* $a, b \in \mathbb{R}$, $a \neq 0$ *or* $b \neq 0$.

If $a = 0$ we get $\mathfrak{h} = \mathfrak{k}_0 \oplus \mathfrak{g}_{2\alpha}$, which is case (iic) of the Main Theorem. Thus we can assume $a \neq 0$. In this case we define $g = \mathrm{Exp}(\frac{b}{a}Z)$. Since $[\mathfrak{k}_0, \mathfrak{g}_{2\alpha}] = 0$ we get $\mathrm{Ad}(g)\mathfrak{k}_0 = \mathfrak{k}_0$, and since $[B, Z] = Z$ we get $\mathrm{Ad}(g)(aB + bZ) = aB$. Altogether this implies $\mathrm{Ad}(g)\mathfrak{h} = \mathfrak{k}_0 \oplus \mathfrak{a} = \mathfrak{g}_0$, and therefore the action is conjugate to the one in (iib) of the Main Theorem.

(c) $\mathfrak{h} = \mathbb{R}(T + [T, Y] + Z) \oplus \mathbb{R}(2B + Y + dZ)$ *with* $d \in \mathbb{R}$, $0 \neq T \in \mathfrak{k}_0$ *and* $0 \neq Y \in \mathfrak{g}_\alpha$ *such that* $[[T, Y], Y] = 2Z$.

We define $g = \operatorname{Exp}(Y + \frac{d}{2}Z)$. Then

$$\begin{aligned}\operatorname{Ad}(g)(T + [T,Y] + Z) &= T + [T,Y] + Z + [Y,T] + [Y,[T,Y]] + \frac{1}{2}[Y,[Y,T]] = T,\\ \operatorname{Ad}(g)(B + Y + dZ) &= 2B + Y + dZ + 2[Y,B] + d[Z,B] = 2B,\end{aligned}$$

and therefore $\operatorname{Ad}(g)\mathfrak{h} = \mathfrak{k}_0 \oplus \mathfrak{a} = \mathfrak{g}_0$. Consequently the action is conjugate to the one in (iib) of the Main Theorem.

Altogether we have proved

Proposition 3.4. *Assume that H acts polarly and without fixed points on $\mathbb{C}H^2$ with cohomogeneity 2 and with a 1-dimensional singular orbit. Then the Lie algebra of H is conjugate to $\mathfrak{g}_0$ or $\mathfrak{k}_0 \oplus \mathfrak{g}_{2\alpha}$.*

In order to finish the proof of the Main Theorem if remains to show that the actions of the groups whose Lie algebras are $\mathfrak{g}_0$ or $\mathfrak{k}_0 \oplus \mathfrak{g}_{2\alpha}$ are indeed polar. We use the criterion given in Corollary 3.2.

Case 1: H is the connected Lie subgroup of $SU(1,2)$ whose Lie algebra is $\mathfrak{h} = \mathfrak{g}_0$.

We consider the submanifold $\Sigma = \exp_o(\mathfrak{s})$ with $\mathfrak{s} = (1-\theta)(\mathfrak{g}_\alpha^{\mathbb{R}} \oplus \mathfrak{g}_{2\alpha})$. Here, $\exp_o$ denotes the exponential map $T_o\mathbb{C}H^2 \to \mathbb{C}H^2$, and we are identifying $T_o\mathbb{C}H^2$ with $\mathfrak{p}$ as usual. It is clear that $\mathfrak{s}$ is a real subspace of $\mathfrak{p}$, and hence Σ is a totally geodesic real hyperbolic plane $\mathbb{R}H^2 \subset \mathbb{C}H^2$.

Obviously, $T_o\Sigma = \mathfrak{s} \subset \mathfrak{p}_\alpha \oplus \mathfrak{p}_{2\alpha} = \nu_o(H \cdot o)$. If $K_0 \cong U(1)$ denotes the connected Lie group of $SU(1,2)$ whose Lie algebra is $\mathfrak{k}_0 \cong \mathfrak{u}(1)$, the slice representation of H at o is the representation of K_0 on $\mathfrak{p}_\alpha \oplus \mathfrak{p}_{2\alpha}$, which is equivalent to the sum of the standard representation of $U(1)$ on $\mathfrak{p}_\alpha \cong \mathbb{C}$, and the trivial representation on $\mathfrak{p}_{2\alpha} \cong \mathbb{R}$. Thus $\mathfrak{s}$ is a section of the slice representation. Since $[\mathfrak{s},\mathfrak{s}] = (1+\theta)[\theta\mathfrak{g}_\alpha^{\mathbb{R}}, \mathfrak{g}_{2\alpha}] \subset \mathfrak{g}_{-\alpha} \oplus \mathfrak{g}_\alpha$, which obviously is perpendicular to $\mathfrak{g}_0 = \mathfrak{h}$, it now follows from Corollary 3.2 that the action of H on $\mathbb{C}H^2$ is polar.

Case 2: H is the connected Lie subgroup of $SU(1,2)$ whose Lie algebra is $\mathfrak{h} = \mathfrak{k}_0 \oplus \mathfrak{g}_{2\alpha}$.

In this case we consider $\Sigma = \exp_o(\mathfrak{s})$ with $\mathfrak{s} = \mathfrak{a} \oplus (1-\theta)(\mathfrak{g}_\alpha^{\mathbb{R}})$. Again, $\mathfrak{s}$ is a real subspace of $\mathfrak{p}$ and Σ is a totally geodesic $\mathbb{R}H^2 \subset \mathbb{C}H^2$. Moreover, we have $T_o\Sigma = \mathfrak{s} \subset \mathfrak{a} \oplus \mathfrak{p}_\alpha = \nu_o(H \cdot o)$, and the slice representation of H at o is the representation of K_0 on $\mathfrak{a} \oplus \mathfrak{p}_\alpha$, which is equivalent to the sum of the standard representation of $U(1)$ on $\mathfrak{p}_\alpha \cong \mathbb{C}$, and the trivial representation on $\mathfrak{a} \cong \mathbb{R}$. Therefore $\mathfrak{s}$ is a section of the slice representation. Finally, $[\mathfrak{s},\mathfrak{s}] = (1+\theta)\mathfrak{g}_\alpha^{\mathbb{R}} \subset \mathfrak{g}_{-\alpha} \oplus \mathfrak{g}_\alpha$ is orthogonal to $\mathfrak{h} = \mathfrak{k}_0 \oplus \mathfrak{g}_{2\alpha}$, and thus it follows from Corollary 3.2 that the action of H on $\mathbb{C}H^2$ is polar.

Altogether we have proved the Main Theorem.

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