POLAR ACTIONS WITH A FIXED POINT

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ABSTRACT. We prove a criterion for an isometric action of a Lie group on a Riemannian manifold to be polar. From this criterion, it follows that an action with a fixed point is polar if and only if the slice representation at the fixed point is polar and the section is the tangent space of an embedded totally geodesic submanifold. We apply this to obtain a classification of polar actions with a fixed point on symmetric spaces.

1. INTRODUCTION

A proper isometric action of a Lie group on a Riemannian manifold is called *polar* if there exists a connected embedded submanifold Σ which meets all orbits and which intersects the group orbits orthogonally in each of its points. Such a submanifold Σ is called a *section* of the group action. In the special case where the section is flat in the induced metric, the action is called *hyperpolar*.

In this note, we prove some infinitesimal criteria for isometric actions to be polar and apply them to obtain a classification of polar actions on symmetric spaces with a fixed point. We neither assume that the symmetric space acted upon is compact nor that it is irreducible. The result is the following.

Theorem 1. *Let M be a symmetric space. Let H be a connected closed subgroup of the isometry group of M acting on M with a fixed point. Then the H-action on M lifts to an isometric action with a fixed point on the universal cover $\tilde{M}$ which is polar if and only if the H-action on M is. In this case, the H-action on $\tilde{M}$ is orbit equivalent to an action defined as follows. Let $\tilde{M} = M_0 \times M_1 \times \ldots \times M_\ell$ be a decomposition of the symmetric space $\tilde{M}$ into one Euclidean and ℓ irreducible factors. Let $H_0, \ldots, H_\ell$ be compact Lie groups such that each factor H_i acts trivially or polarly with a fixed point on M_i as described below.*

(i) *If M_i is Euclidean, then the H_i-action on M_i is orbit equivalent to a polar representation.*

(ii) *If M_i is irreducible of rank greater than one, then the action of H_i is orbit equivalent to the isotropy action of M_i.*

(iii) *If M_i is isometric to S^n or H^n, the action is given by a polar representation on $\mathbb{R}^n$.*

(iv) *If M_i is isometric to $\mathbb{C}\mathsf{P}^n$ or $\mathbb{C}\mathsf{H}^n$, then the action is given by a polar action on $\mathbb{C}\mathsf{P}^{n-1}$, see Section 4.*

(v) *If M_i is isometric to $\mathbb{H}P^n$ or $\mathbb{H}H^n$, then the action is given by a polar action on $\mathbb{H}P^{n-1}$, see Section 4.*

(vi) *If M_i is isometric to $\mathbb{O}P^2$ or $\mathbb{O}H^2$, then the action is given by the isotropy group $\mathrm{Spin}(9)$ of M_i or one of its subgroups $\mathrm{Spin}(8)$, $\mathrm{Spin}(7) \cdot \mathrm{SO}(2)$, $\mathrm{Spin}(6) \cdot \mathrm{Spin}(3)$.*

Note that an action as described in Theorem 1 is hyperpolar if and only if the actions of H_i on M_i are orbit equivalent to the isotropy action of M_i for $i = 1, \ldots, n$. Hyperpolar actions without fixed point on compact irreducible symmetric spaces have been classified in [10].

We remark that the result was known for irreducible spaces of higher rank [4] and for the compact symmetric spaces of rank one [14]. It follows from Corollary 9 that the actions described in parts (i) to (vi) of the Theorem exhaust all orbit equivalence classes of polar actions with a fixed point on strongly isotropy irreducible Riemannian homogeneous spaces.

Using the results of Bergmann [1] on reducible polar representations, all proper polar actions with a fixed point for each symmetric space (not only up to orbit equivalence) can be described, although there is no convenient way of writing down a complete list of all these actions due to the nature of the result.

Our result is of interest for several reasons. One new aspect is that we use here the duality between symmetric spaces of the compact and those of the non-compact type. In particular, we obtain new results on polar actions on the non-compact symmetric spaces of rank one. It is an intriguing question whether duality can be used to study polar actions without fixed points.

It follows immediately from the main result of Dadok [6] that a polar action with a fixed point on a simply connected symmetric space is orbit equivalent to the product of polar actions on the individual factors of the decomposition of the symmetric space into Euclidean and irreducible factors. However, it is known that such a splitting into a product of actions does not hold in general, i.e. for actions without fixed points, see the following examples.

Examples 2.

(1) Consider the action of $H = \mathrm{SO}(n)$ on $\mathbb{R}^{2n} = \mathbb{R}^n \oplus \mathbb{R}^n$ where H acts on each summand by the standard representation. Then the principal isotropy subgroup is $\mathrm{SO}(n-2)$ and H acts with cohomogeneity one on the product $\mathsf{S}^{n-1} \times \mathsf{S}^{n-1}$ of the unit spheres in each summand $\mathbb{R}^n$. This action is hyperpolar.

(2) Let G be a semisimple compact Lie group with a biinvariant metric and let $K_1, K_2 \subset G$ be two closed subgroups such that (G, K_1) and (G, K_2) are *symmetric pairs*, i.e. there is an involutive automorphism σ of G such that $(G^\sigma)_0 \subseteq K_i \subseteq G^\sigma$. The action of $K_1 \times K_2$ on G given by

$$(k_1, k_2) \cdot g = k_1 \, g \, k_2^{-1}$$

is hyperpolar [9] (a so-called *Hermann action*). By [9, Proposition 2.11], it follows that the action of $K_1 \times K_2 \times G$ on $G \times G$ given by

$$(k_1, k_2, g) \cdot (g_1, g_2) = (k_1 \, g_1 \, g^{-1}, k_2 \, g_2 \, g^{-1})$$

is hyperpolar as well. Using [9, Proposition 2.11] once more, we see that the action of $\Delta G = \{(g, g) \mid g \in G\}$ on $G/K_1 \times G/K_2$ is hyperpolar, but not a product. Example (1) is the special case of this construction where $G = \mathrm{SO}(n)$, $K_1 = K_2 = \mathrm{SO}(n-1)$.

(3) The action of $H = \mathrm{Sp}(2) \cong \mathrm{Spin}(5)$ on $\mathsf{S}^7 \times \mathsf{S}^4$, induced by the direct sum of the standard $\mathrm{Sp}(2)$-representation on $\mathbb{H}^2 = \mathbb{R}^8$ and the standard $\mathrm{SO}(5)$-representation on $\mathbb{R}^5$ is of cohomogeneity one, but does not split.

(4) Similarly, the action of $\mathrm{Spin}(9)$ on $\mathsf{S}^8 \times \mathsf{S}^{15}$ by the standard plus the spin representation is also of cohomogeneity one and non-splitting.

(5) The action of $\mathrm{Spin}(8)$ on $\mathsf{S}^7 \times \mathsf{S}^7 \times \mathsf{S}^7$ given by the three inequivalent representations on $\mathbb{R}^8$ of $\mathrm{Spin}(8)$ is of cohomogeneity one, but does not split.

It would be desirable to have structural and classification results on (hyper)polar actions on reducible symmetric spaces. Using the construction in Example (2), one may produce non-splitting actions with arbitrarily many factors.

After the completion of the classification of polar and coisotropic actions on the compact irreducible Hermitian symmetric spaces in his article [3], which showed in particular that polar actions are hyperpolar on the irreducible Hermitian spaces of *higher rank*, i.e. of rank greater than one, Biliotti stated the following conjecture:

Conjecture 3. [3] *A polar action on an irreducible compact symmetric space of rank greater than one is hyperpolar.*

The conjecture has been proven to hold for the symmetric spaces with simple compact isometry group [11] and, more recently, for the exceptional simple compact Lie groups [12] with biinvariant metric. (It is still open for the classical compact Lie groups.) However, in the noncompact setting it is not true that polar actions on irreducible symmetric spaces of higher rank are hyperpolar, since there are polar homogeneous foliations with non-flat sections by recent results of [2]. But our result shows the conjecture of Biliotti remains true also for spaces of noncompact type if we restrict ourselves to polar actions of compact Lie groups, since these actions will have a fixed point by Cartan's fixed point theorem. On the other hand, there are no nontrival polar homogeneous foliations on irreducible compact symmetric spaces:

Proposition 4. *Let N be a symmetric space of compact type with a homogeneous polar foliation. Let Σ be a section and let M be a principal orbit of the homogeneous foliation. Then $\tilde{N}$ is isometric to the Riemannian product $\tilde{M} \times \tilde{\Sigma}$, where the tilde denotes universal covers.*

Proof. By [11, Proposition 4.1], the polar homogeneous foliation on N lifts to a homogeneous polar foliation on the universal cover $\tilde{N}$. It follows from [14, Lemma 1A.3] that the corresponding isometric action on $\tilde{N}$ does not have any exceptional orbits. Since a homogeneous foliation does not have singular orbits, it follows that the Weyl group W_Σ is trivial, cf. [11, Lemma 5.1]. Therefore, one may apply the Splitting Theorem 5.2 in [11] and it follows that $\tilde{N}$ is isometric to $\tilde{M} \times \tilde{\Sigma}$. $\qquad\square$

It is an interesting open question whether there are polar actions with non-flat section and singular orbits on irreducible symmetric spaces of non-compact type.

We briefly summarize the contents of this paper. In Section 2 we prove a criterion for polarity of actions with a fixed point. In Section 3 we use this criterion to show that the duality of symmetric spaces leads to a one-to-one correspondence between polar actions with a fixed point on a pair of dual symmetric spaces. We classify polar actions with a fixed point in rank one symmetric spaces in Section 4. Theorem 1 is proven in Section 5.

2. CRITERIA FOR POLARITY

We say that two isometric Lie group actions on two Riemannian manifolds M_1 and M_2, respectively, are *orbit equivalent*, if there is an isometry $M_1 \to M_2$ which maps connected components of orbits onto connected components of orbits. Two isometric actions of a Lie group H on two Riemannian manifolds M_1 and M_2 are called *conjugate* if there is an isometry $F \colon M_1 \to M_2$ such that $F(h \cdot p) = h \cdot F(p)$ for all $h \in H$ and $p \in M_1$. Obviously, the polarity of an action only depends on the orbit equivalence class of the action.

We will now prove an infinitesimal criterion which allows to decide if the orbits of an isometric Lie group action intersect a totally geodesic submanifold orthogonally. This is a simple but fundamental observation from which one may deduce for instance the criteria for polarity of actions on symmetric spaces which had been obtained in various special cases by different authors, cf. [9, Theorem 2.1 and Corollary 2.12], [8, Proposition], [11, Proposition 4.1], [2, Theorem 4.1].

Lemma 5. *Let M be a Riemannian manifold and let Σ be a connected totally geodesic submanifold of M. Let $p \in \Sigma$ and let X be a Killing vector field. Then $X(q) \in N_q\Sigma$ holds for all $q \in \Sigma$ if and only if $X(p) \in N_p\Sigma$ and $\nabla_v X \in N_p\Sigma$ for all $v \in T_p\Sigma$.*

Proof. Let $\gamma \colon \mathbb{R} \to \Sigma$ be a geodesic such that $\gamma(0) = p$. Then $J(t) := X(\gamma(t))$ is a Jacobi field along γ. Let $n = \dim M$. Let $e_1(t), \ldots, e_n(t)$ be parallel orthonormal fields along γ such that $e_1(0), \ldots, e_k(0) \in T_p\Sigma$ for $k = \dim \Sigma$. Since Σ is totally geodesic, we have that the vectors $e_1(t), \ldots, e_k(t)$ are contained in $T_{\gamma(t)}\Sigma$ for all t. Let $J(t) = \sum_i f_i(t)e_i(t)$ for some functions f_i. The Jacobi equation $\frac{D^2 J}{dt^2} = R(\gamma'(t), J(t))\gamma'(t)$ is then equivalent to the system

$$f_j''(t) + \sum_i a_{ij}(t)f_i(t) = 0, \quad j = 1, \ldots, n,$$

where the functions a_{ij} are given by $a_{ij}(t) = \langle R(\gamma'(t), e_i(t))\gamma'(t), e_j(t)\rangle$, see [7]. It follows from the symmetries of the Riemann tensor that the matrix $a_{ij}(t)$ is symmetric for all t. Moreover, since $\gamma'(t), e_1(t), \ldots, e_k(t)$ are tangent to the totally geodesic submanifold Σ, we have $a_{ij}(t) = a_{ji}(t) = 0$ if $i \leq k$ and $j > k$. It follows that $f_1(t) = f_1'(t) = \ldots = f_k(t) = f_k'(t) = 0$ for all t (which is equivalent to $J(t) \in N_{\gamma(t)}\Sigma$ for all t) if and only if we have $f_1(0) = f_1'(0) = \ldots = f_k(0) = f_k'(0) = 0$ (which is equivalent to $J(0) \in N_p\Sigma$ and $J'(0) = \frac{\nabla J}{dt}(0) = \nabla_{\gamma'(0)}X \in N_p\Sigma$). This shows the statement of the lemma. $\qquad \square$

From the above lemma, we obtain a general criterion for polarity of isometric actions.

Corollary 6. *Let M be a complete connected Riemannian manifold and let Σ be a connected totally geodesic embedded submanifold of M. Then the proper isometric action of a Lie group H on M is polar with section Σ if and only if there is a point $p \in \Sigma$ such that (i) $T_p\Sigma \subseteq N_p(H \cdot p)$, (ii) the slice representation of the isotropy subgroup H_p on $N_p(H \cdot p)$ is such that $T_p\Sigma$ meets all H_p-orbits and (iii) $\nabla_v X \in N_p\Sigma$ for all $v \in T_p\Sigma$ and all Killing vector fields X induced by the H-action on M.*

Proof. Let M be a Riemannian H-manifold and let Σ be a connected totally geodesic embedded submanifold such that (i) and (ii) hold for some point $p \in \Sigma$. We show that then Σ intersects all H-orbits. Let $q \in M$. Since the orbit $H \cdot p$ is closed [5] and M is complete and connected, there is a shortest geodesic γ joining q and $H \cdot p$, which intersects $H \cdot p$ orthogonally at some point $h \cdot p$. Then $h^{-1} \circ \gamma$ is a geodesic joining p and $h^{-1}(q)$. Since the slice representation of H_p on the normal space $N_p(H \cdot p)$ is such that $T_p\Sigma$ meets all orbits, there is an element $g \in H_p$ such that $g^{-1} \circ h^{-1} \circ \gamma$ is a geodesic joining the two points p and $g^{-1}(h^{-1}(q))$ whose image is contained in Σ. This shows that the H-orbit through q intersects Σ. The rest of the statement follows immediately from Lemma 5. $\square$

The key observation for this article is the following criterion for polarity of isometric actions with a fixed point, which enables us to use the duality of symmetric spaces.

Theorem 7. *Let M be a connected Riemannian manifold with isometry group $I(M)$ and let $p \in M$. A closed subgroup $H \subseteq I(M)_p$ acts polarly on M if and only if the slice representation of $H = H_p$ on T_pM is polar and for any of its sections $\mathfrak{s}$, the exponential image $\Sigma := \exp_p(\mathfrak{s})$ is an embedded totally geodesic submanifold of M. In this case, Σ is a section for the H-action on M.*

Proof. Assume H acts polarly on T_pM and for any of its sections $\mathfrak{s}$, the exponential image $\exp_p(\mathfrak{s})$ is an embedded totally geodesic submanifold of M. Let x be an element of the Lie algebra of H. Then for all $q \in M$, the Killing vector field X corresponding to x is given by $X(p) = \frac{d}{ds}\big|_{s=0} (h_s(p))$, where h_s denotes the isometry of M given by the group element $\exp(sx)$. Let $\exp_p\colon T_pM \to M$ denote the Riemannian exponential map of M at the point p and let $v \in \mathfrak{s}$. Then we have

$$\nabla_v X = \frac{\nabla}{\partial t}\frac{\partial}{\partial s} h_s(\exp_p(tv))\bigg|_{s=t=0} = \frac{\nabla}{\partial s}\frac{\partial}{\partial t} h_s(\exp_p(tv))\bigg|_{s=t=0} =$$
$$= \frac{\nabla}{\partial s}\left(\frac{\partial}{\partial t} h_s(\exp_p(tv))\big|_{t=0}\right)\bigg|_{s=0} = \frac{\nabla}{\partial s} (h_s)_{*p}(v)\big|_{s=0} \in N_p\Sigma,$$

since the slice representation of H on T_pM is polar with section $\mathfrak{s}$. Now it follows from Corollary 6 that the H-action is polar with section Σ. The converse is a well known fact. $\square$

To find all polar actions on a given space, it is often useful to know *a priori* that certain subgroups of a group acting isometrically do not act polarly. The following theorem gives such an information for irreducible representations which are not transitive on the sphere.

Theorem 8. [13] *Let $G \subset \mathrm{SO}(n)$ be a closed connected subgroup which acts irreducibly on $\mathbb{R}^n$ and non-transitively on the sphere $\mathsf{S}^{n-1} \subset \mathbb{R}^n$. Let $H \subset G$ be a closed connected subgroup $\neq \{e\}$ that acts polarly on $\mathbb{R}^n$. Then the H-action and the G-action on $\mathbb{R}^n$ are orbit equivalent.*

This can be used to show the following fact on polar actions with a zero- or one-dimensional orbit on strongly isotropy irreducible Riemannian homogeneous spaces. In particular, it follows that polar actions with a fixed point on irreducible symmetric spaces of rank greater than one are hyperpolar and in fact orbit equivalent to the isotropy action. This had been shown already in [4, Theorem 2.2].

Corollary 9. [11, Corollary 6.2] *Let X be a strongly isotropy irreducible Riemannian homogeneous space. Assume a connected compact Lie group H acts polarly and non-trivially on X and the H-action has a one-dimensional orbit $H \cdot p$ or a fixed point $p \in X$. Then the space X is locally symmetric. Furthermore, X is a rank-one symmetric space or the action of H is orbit equivalent to the action of the connected component of the isotropy group of X at p.*

3. Duality

The statement of Theorem 1 was previously known in the case of irreducible symmetric spaces of higher rank [4] and for compact rank one symmetric spaces it follows from [14]. Thus it essentially remains to extend the classification to the noncompact rank-one symmetric spaces. Theorem 7 enables us to do this by using duality between symmetric spaces of the compact and the non-compact type.

Let (G, K) be a symmetric pair and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be an $\mathrm{Ad}(K)$-invariant decomposition. Then one may define a Lie algebra $\mathfrak{g}^*$ be setting $\mathfrak{g}^* = \mathfrak{k} \oplus i\mathfrak{p}$, where $i = \sqrt{-1}$. If a symmetric pair (G^*, K) is such that $\mathfrak{g}^*$ is isomorphic to the Lie algebra of G^*, we say that (G^*, K) is *dual* to (G, K).

Theorem 10. *Let (G, K) be a symmetric pair such that G/K is an irreducible space of the compact or noncompact type and let (G^*, K) be a dual symmetric pair. Let $H \subseteq K$ be a closed subgroup. Then the action of H on G/K is polar if and only if the action of H on G^*/K is polar.*

Proof. Assume first that (G, K) is a symmetric pair and G/K is of rank one. Let (G^*, K) be a dual of (G, K). Let H be a closed subgroup of K acting polarly on G/K with a fixed point. By Theorem 7, H acts polarly on $T_{eK}(G/K)$. Let $\Sigma \subset T_{eK}(G/K)$ be a section of this action. Then Σ is a Lie triple system in $\mathfrak{p} = T_{eK}(G/K)$ and hence $i\Sigma$ is a Lie triple system in $i\mathfrak{p} = T_{eK}(G^*/K)$. In a rank one symmetric space, every totally geodesic submanifold is embedded, hence it follows from Theorem 7 that the action of H on G^*/K is polar if $\mathrm{rank}\, G^*/K = \mathrm{rank}\, G/K = 1$. For spaces of higher rank, the assertion follows immediately from Corollary 9. $\qquad\square$

Corollary 11. *Let Σ be the section of a polar action with fixed point on an irreducible symmetric space of noncompact type. Then Σ is isometric to a product $\mathbb{R}^{n_0} \times \mathsf{H}^{n_1} \times \ldots \times \mathsf{H}^{n_k}$.*

Proof. This follows via Theorem 10 from an analogous statement for spaces of compact type [11, Theorem 5.4]. □

We remark that Corollary 11 does not hold anymore if one drops the assumption that the polar action has a fixed point, see [2, Proposition 4.2].

4. Actions on rank one symmetric spaces

Let us recall some results of [14] about polar actions on compact rank one symmetric spaces. We say that an isometric action of a connected compact Lie group $H \subseteq \mathrm{U}(n+1)$ on $\mathbb{CP}^n = \mathrm{U}(n+1)/(\mathrm{U}(n) \times \mathrm{U}(1))$ is *induced by the isotropy representation of a Hermitian symmetric space U/K* if there is a semisimple Hermitian symmetric space U/K such that (U, K) is a symmetric pair and the natural action of H on $\mathbb{C}^{n+1}$ is equivalent to the isotropy representation of U/K.

Similarly, let $H_i \subseteq \mathrm{Sp}(n_i)$ be connected closed subgroups for $i = 1, \ldots, k$; we say that the isometric action of $H_1 \times \ldots \times H_k \subseteq \mathrm{Sp}(n_1) \times \ldots \times \mathrm{Sp}(n_k) \subseteq \mathrm{Sp}(n+1)$ on $\mathbb{HP}^n$, where $n + 1 = n_1 + \ldots + n_k$, is *induced by the isotropy representation of a product of k quaternion-Kähler symmetric spaces* if there are quaternion-Kähler symmetric spaces $U_1/K_1, \ldots, U_k/K_k$ such that the action of $H_i \times \mathrm{Sp}(1)$ on $\mathbb{H}^{n_i}$ (given by restriction of the usual $\mathrm{Sp}(n_i) \times \mathrm{Sp}(1)$-representation) is equivalent (on the Lie algebra level) to the isotropy representation of U_i/K_i.

It was proved in [14] that the actions on $\mathbb{CP}^n$ as described above are polar and exhaust the orbit equivalence classes of polar actions on $\mathbb{CP}^n$. Furthermore, the actions on $\mathbb{HP}^n$ where at most one of the quaternion-Kähler symmetric spaces U_i/K_i is of rank ≥ 2, are polar and exhaust the orbit equivalence classes of polar actions on $\mathbb{HP}^n$.

We remark that an action on $\mathbb{CP}^n$, resp. $\mathbb{HP}^n$, induced by the isotropy representation of a Hermitian symmetric space, resp. a product of k quaternion-Kähler symmetric spaces, has fixed point if and only if the symmetric space decomposes as a Riemannian product containing a $\mathbb{CP}^1$, resp. an $\mathbb{HP}^1$, factor.

The next proposition shows that the orbit equivalence classes of polar actions with a fixed point on $\mathbb{KP}^n$ resp. $\mathbb{KH}^n$ are in one-to-one correspondence with the orbit equivalence classes of polar actions on $\mathbb{KP}^{n-1}$ for $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$.

Proposition 12. *Let $n \geq 2$.*

(i) *Let H be closed subgroup of $\mathrm{SO}(n)$. Then the group H acts polarly on $\mathsf{S}^n = \mathrm{SO}(n+1)/\mathrm{SO}(n)$ or $\mathsf{H}^n = \mathrm{SO}(n,1)_0/\mathrm{SO}(n)$, respectively, if and only if H acts polarly on S^{n-1} by restriction of the n-dimensional standard representation of $\mathrm{SO}(n)$.*

(ii) *Let H be a closed subgroup of $\mathrm{U}(n)$. If the action of H on $\mathbb{CP}^n = \mathrm{SU}(n+1)/\mathrm{U}(n)$ or $\mathbb{CH}^n = \mathrm{SU}(n,1)/\mathrm{U}(n)$, respectively, is polar, then the H-action on $\mathbb{CP}^{n-1} = \mathrm{U}(n)/(\mathrm{U}(n-1) \times \mathrm{U}(1))$ is polar. Conversely, if the action of H on $\mathbb{CP}^{n-1}$ is polar and nontrivial, then the action of $\mathrm{U}(1) \cdot H \subseteq \mathrm{U}(n)$ on $\mathbb{CP}^n$ or $\mathbb{CH}^n$, respectively, is polar with a fixed point.*

(iii) *Let H be a closed subgroup of $\mathrm{Sp}(n) \times \mathrm{Sp}(1)$. If the action of H on $\mathbb{HP}^n = \mathrm{Sp}(n+1)/(\mathrm{Sp}(n) \times \mathrm{Sp}(1))$ or $\mathbb{HH}^n = \mathrm{Sp}(n,1)/(\mathrm{Sp}(n) \times \mathrm{Sp}(1))$, respectively, is polar, then*

the $\pi(H)$-action on $\mathbb{H}\mathsf{P}^{n-1} = \mathrm{Sp}(n)/(\mathrm{Sp}(n-1) \times \mathrm{Sp}(1))$, where $\pi \colon \mathrm{Sp}(n) \times \mathrm{Sp}(1) \to \mathrm{Sp}(n)$ is the natural projection onto the first factor, is polar. Conversely, let H be a closed subgroup of $\mathrm{Sp}(n)$ acting polarly and nontrivially on $\mathbb{H}\mathsf{P}^{n-1}$, then $H \times \mathrm{Sp}(1)$ acts polarly with a fixed point on $\mathbb{H}\mathsf{P}^n$ or $\mathbb{H}\mathsf{H}^n$, respectively.

(iv) Let H be a closed connected subgroup of $\mathrm{Spin}(9)$. Then the action of H on $\mathbb{O}\mathsf{P}^2 = \mathrm{F}_4/\mathrm{Spin}(9)$ or $\mathbb{O}\mathsf{H}^2 = \mathrm{F}_4^*/\mathrm{Spin}(9)$, respectively, is polar if and only if H is conjugate to one of the following: $\mathrm{Spin}(9)$, $\mathrm{Spin}(8)$, $\mathrm{Spin}(7) \cdot \mathrm{SO}(2)$, $\mathrm{Spin}(6) \cdot \mathrm{Spin}(3)$.

Proof. In view of Theorem 10, it suffices to prove the proposition in the compact case. Part (i) follows immediately from the fact that polar actions on the sphere S^n are precisely the restrictions of polar representations on $\mathbb{R}^{n+1}$.

We will now prove part (ii). Let $H \subseteq \mathrm{U}(n)$ be a closed subgroup acting polarly (and with a fixed point) on $\mathbb{C}\mathsf{P}^n$, homogeneously presented as $\mathrm{SU}(n+1)/\mathrm{U}(n)$, where $\mathrm{U}(n)$ is embedded into $\mathrm{SU}(n+1)$ as the subgroup $\mathrm{S}(\mathrm{U}(n) \times \mathrm{U}(1))$. The action of this group leaves invariant a totally geodesic $\mathbb{C}\mathsf{P}^{n-1}$, which we may identify with $\mathrm{U}(n)/(\mathrm{U}(n-1) \times \mathrm{U}(1))$. By [11, Lemma 4.2], the restriction of the H-action to this submanifold is polar.

Now assume $H \subseteq \mathrm{U}(n)$ is a closed subgroup acting polarly and nontrivially on $\mathbb{C}\mathsf{P}^{n-1} = \mathrm{U}(n)/(\mathrm{U}(n-1) \times \mathrm{U}(1))$. By the results of [14], the H-action on $\mathbb{C}\mathsf{P}^{n-1}$ is orbit equivalent to an action induced by the isotropy representation of a Hermitian symmetric space and we may assume that the action of H on $\mathbb{C}^n$ is equivalent to the isotropy representation of a Hermitian symmetric space X. Then the action of $\mathrm{U}(1) \cdot H \subseteq \mathrm{U}(n)$ on $\mathbb{C}\mathsf{P}^n = \mathrm{SU}(n+1)/\mathrm{U}(n)$ is orbit equivalent to the action induced by the Hermitian symmetric space $X \times \mathbb{C}\mathsf{P}^1$, showing that the action of $\mathrm{U}(1) \cdot H$ on $\mathbb{C}\mathsf{P}^n$ is polar [14] with a fixed point.

To prove part (iii), assume $H \subset \mathrm{Sp}(n) \times \mathrm{Sp}(1)$ is a closed subgroup acting polarly (and with a fixed point) on $\mathbb{H}\mathsf{P}^n$. The action of this group leaves invariant a totally geodesic $\mathbb{H}\mathsf{P}^{n-1}$. By [11, Lemma 4.2], the restriction of the H-action to this submanifold is polar.

Now assume $H \subseteq \mathrm{Sp}(n)$ is a closed subgroup acting polarly and nontrivially on $\mathbb{H}\mathsf{P}^{n-1}$. By the results of [14], the H-action on $\mathbb{H}\mathsf{P}^{n-1}$ is orbit equivalent to an action induced by the isotropy representation of a product of k quaternion-Kähler symmetric spaces $U_1/K_1 \times \ldots \times U_k/K_k$ where at most one factor is of higher rank. Then the action induced by the isotropy representation of $U_1/K_1 \times \ldots \times U_k/K_k \times \mathbb{H}\mathsf{P}^1$ on $\mathbb{H}\mathsf{P}^n$ is orbit equivalent to the action of $H \times \mathrm{Sp}(1)$, which is hence polar (and has a fixed point).

Part (iv) follows immediately from [14]. $\qquad\square$

5. Proof of Theorem 1

Proposition 13. *Let $(G_0, K_0), \ldots, (G_\ell, K_\ell)$ be symmetric pairs such that G_0/K_0 is of Euclidean type and $G_1/K_1, \ldots, G_\ell/K_\ell$ are irreducible symmetric spaces. Define $G = G_0 \times G_1 \times \ldots \times G_\ell$ and $K = K_0 \times K_1 \times \ldots \times K_\ell$. Let $H \subseteq K$ be a closed subgroup. Then there is a natural action of H on the spaces G_i/K_i given by $\pi_i \circ \iota$, where $\iota \colon H \to K$ is the inclusion and $\pi_i \colon K \to K_i$ is the projection. If the action of H on G/K is polar then the action of H on each of the spaces G_i/K_i is polar. Moreover, the H-action on G/K is orbit equivalent to the product action of $\Pi_{i=1}^{\ell} H$ on $G_1/K_1 \times \ldots \times G_\ell/K_\ell$.*

Proof. Assume the H-action on G/K is polar. Then by Theorem 7, the slice representation of H on $T_{eK}G/K$ is polar as well and for any of its sections $\mathfrak{s}$, the exponential image $\exp_{eK}(\mathfrak{s})$ is an embedded totally geodesic submanifold of G/K. Since $H \subseteq K$, the group H leaves each $T_{eK}G_i/K_i$ invariant. Then [6, Theorem 4] implies that the H-action on each invariant summand $T_{eK}G_i/K_i$ is also polar and the section $\mathfrak{s}$ is of the form $\mathfrak{s}_0 \oplus \ldots \oplus \mathfrak{s}_\ell$, where $\mathfrak{s}_i$ is a section of the H-action on $T_{eK}G_i/K_i$. From this we see that $\exp_{eK_i}(\mathfrak{s}_i)$ is an embedded totally geodesic submanifold of G_i/K_i for each $i = 0, \ldots, \ell$. By using Theorem 7 once more, we get that the H-action on each factor G_i/K_i is polar with section $\exp_{eK_i}(\mathfrak{s}_i)$. The last statement follows since the orbits of the H-action on G/K are certainly contained in those of the $\Pi_{i=1}^\ell H$-action and both actions are of the same cohomogeneity. $\square$

Proof of Theorem 1. Let $M = G/K$ be an effective presentation and $G = \mathrm{Isom}(M)$. We may assume that H is a closed subgroup of K. Let $\tilde{G}$ be the universal cover of G with covering map $\pi \colon \tilde{G} \to G$. Define $\tilde{K} := \pi^{-1}(K)$ and identify $M = \tilde{G}/\tilde{K}$. Then $\pi|_{\tilde{K}} \colon \tilde{K} \to K$ is a covering map whose kernel agrees with the effectivity kernel of the $\tilde{K}$-action on $M = \tilde{G}/\tilde{K}$. Let $h \in H$. Since H is connected, there is a path $h(t)$ defined on $[0, 1]$ and connecting $h = h(1)$ with the identity element $e = h(0)$. Let $\tilde{h}(t)$ be the continuous lift of $h(t)$ such that $\pi(\tilde{h}(t)) = h(t)$ and $\tilde{h}(0) = e$. Then $\tilde{h} := \tilde{h}(1)$ lies in the connected component $\tilde{K}_0$. Define an action of H on $\tilde{M}$ by requiring that an element $h \in H$ acts on $\tilde{M} = \tilde{G}/\tilde{K}_0$ by $\tilde{h}$ as constructed above. Since the action of the isometry $\tilde{h}$ is determined by its action on $T_{e\tilde{K}}\tilde{M}$ and the representation of $\tilde{K}_0$ on $T_{e\tilde{K}}\tilde{M}$ is equivalent to $\rho \circ \pi \colon \tilde{K}_0 \to \mathrm{O}(T_{eK}M)$, where ρ is the isotropy representation of K on M, the action of H on $\tilde{M}$ is well-defined, i.e. does not depend on the path $h(t)$. It is now clear that we have defined an isometric Lie group action of H on $\tilde{M}$.

It follows from Theorem 6 that the action of H on $\tilde{M}$ is polar if and only if the action of H on M is. Assume the H-action on $\tilde{M}$ is polar. It follows from Proposition 13 that the induced actions on the factors M_i are all polar, hence either trivial, as described in Proposition 12 or Corollary 9, or given by a polar linear representation if M_i is Euclidean. $\square$

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