HOMOGENEOUS HYPERSURFACES IN COMPLEX HYPERBOLIC SPACES

JÜRGEN BERNDT, JOSÉ CARLOS DÍAZ-RAMOS

ABSTRACT. We study the geometry of homogeneous hypersurfaces and their focal sets in complex hyperbolic spaces. In particular, we provide a characterization of the focal set in terms of its second fundamental form and determine the principal curvatures of the homogeneous hypersurfaces together with their multiplicities.

1. INTRODUCTION

An s-representation is the isotropy representation of a semisimple Riemannian symmetric space. A result by Hsiang and Lawson [10] implies that a hypersurface in the Riemannian sphere S^m is homogeneous if and only if it is a principal orbit of the s-representation of an $(m + 1)$-dimensional semisimple Riemannian symmetric space G/K of rank two. The classification of homogeneous hypersurfaces in S^m can therefore be easily deduced from Cartan's classification of Riemannian symmetric spaces.

If G/K is Hermitian symmetric, then m is odd, say $m = 2n+1$, and the s-representation induces an action on the corresponding complex projective space $\mathbb{C}P^n$ via the Hopf map $S^{2n+1} \to \mathbb{C}P^n$. Takagi [12] showed in 1973 that a real hypersurface in $\mathbb{C}P^n$ is homogeneous if and only if it is a principal orbit of an action that is induced in this way from the s-representation of a semisimple Hermitian symmetric space of rank two. Thus the classification of homogeneous hypersurfaces in $\mathbb{C}P^n$ can easily be deduced from the classification of Hermitian symmetric spaces.

A remarkable consequence of Takagi's result is that each homogeneous hypersurface in $\mathbb{C}P^n$ is a Hopf hypersurface. A real hypersurface M of an almost Hermitian manifold $\bar{M}$ is a Hopf hypersurface if the one-dimensional foliation on M induced by the rank one distribution $J(\nu M)$ is totally geodesic, where νM is the normal bundle of M and J is the complex structure on $\bar{M}$. In 1989 the first author classified in [1] the homogeneous Hopf hypersurfaces in the complex hyperbolic space $\mathbb{C}H^n$. Any such Hopf hypersurface is either a horosphere in $\mathbb{C}H^n$, or a tube around a totally geodesic $\mathbb{R}H^n$ or $\mathbb{C}H^k$ for some $k \in \{0, \ldots, n-1\}$.

For some time it was believed that, as is the case for $\mathbb{C}P^n$, every homogeneous hypersurface in $\mathbb{C}H^n$ was a Hopf hypersurface. It came as a kind of surprise when Lohnherr [11] constructed in 1998 a counterexample: the ruled real hypersurface W^{2n-1} in $\mathbb{C}H^n$ which is

determined by a horocycle in a totally geodesic $\mathbb{R}H^2 \subset \mathbb{C}H^n$ is a non-Hopf homogeneous real hypersurface. Recently the first author and Tamaru [7] obtained the classification of homogeneous hypersurfaces in $\mathbb{C}H^n$. A connected real hypersurface in $\mathbb{C}H^n$, $n \geq 2$, is homogeneous if and only if it is holomorphically congruent to

- (A) a tube around a totally geodesic $\mathbb{C}H^k$ for some $k \in \{0, \ldots, n-1\}$, or
- (B) a tube around a totally geodesic $\mathbb{R}H^n$, or
- (H) a horosphere in $\mathbb{C}H^n$, or
- (S) the ruled real hypersurface W^{2n-1} or one of its equidistant hypersurfaces, or
- (W) a tube around the minimal ruled submanifold W_φ^{2n-k} for some $\varphi \in (0, \pi/2]$ and $k \in \{2, \ldots, n-1\}$, where k is even if $\varphi \neq \pi/2$.

The construction of the minimal ruled submanifolds W_φ^{2n-k} will be described later in this article. We just mention here that the normal bundle of W_φ^{2n-k} has rank k and constant Kähler angle φ.

The hypersurfaces of type (A), (B) and (H) are Hopf hypersurfaces and their geometry is well understood. The first author gave in [2] a Lie theoretic construction of the homogeneous hypersurfaces of type (S) and investigated their geometry. The aim of this paper is to investigate the geometry of the other homogeneous hypersurfaces and their focal sets W_φ^{2n-k}.

The motivation for our investigations originates from the question: Is every real hypersurface with constant principal curvatures in a complex hyperbolic space $\mathbb{C}H^n$ an open part of a homogeneous hypersurface? Élie Cartan [9] gave an affirmative answer for the corresponding question in real hyperbolic space. Some time ago the first author demonstrated in [1] that for $\mathbb{C}H^n$ the answer is yes within the class of Hopf hypersurfaces. For arbitrary real hypersurfaces, we obtained recently an affirmative answer in [6] in case of $\mathbb{C}H^2$, and for $n \geq 3$ we obtained an affirmative answer in [5] provided that the number g of distinct principal curvatures satisfies $g \leq 3$. It is a well-established fact that the classification problem of hypersurfaces with constant principal curvatures is intimately related to the understanding of the geometric structure of their focal sets. Therefore the next step is to understand more thoroughly the geometry of the homogeneous hypersurfaces in $\mathbb{C}H^n$ and their focal sets. The two main results of this paper are as follows. (1) We determine explicitly the principal curvatures and their multiplicities for all homogeneous hypersurfaces in $\mathbb{C}H^n$. A consequence is that $g \in \{2, 3, 4, 5\}$ for any such hypersurface. (2) We give a characterization of the non-totally geodesic focal sets of homogeneous hypersurfaces in $\mathbb{C}H^n$ in terms of their second fundamental form.

We now describe the contents of this paper. In Section 2 we summarize some basic material about the complex hyperbolic space. In Section 3 we characterize the minimal ruled submanifolds W_φ^{2n-k} in terms of the second fundamental form. More precisely, we prove

Theorem (Rigidity of the submanifold W_φ^{2n-k})**.** *Let M be a $(2n-k)$-dimensional connected submanifold in $\mathbb{C}H^n$, $n \geq 2$, with normal bundle $\nu M \subset T\mathbb{C}H^n$ of constant Kähler angle $\varphi \in (0, \pi/2]$. Assume that there exists a unit vector field Z tangent to the maximal complex distribution on M such that the second fundamental form II of M is given by the trivial*

symmetric bilinear extension of

$$2II(Z, P\xi) = \sin^2(\varphi)\,\xi$$

for all $\xi \in \nu M$, where $P\xi$ is the tangential component of $J\xi$. Then M is holomorphically congruent to an open part of the ruled minimal submanifold W_φ^{2n-k}. Conversely, the second fundamental form of W_φ^{2n-k} is of this form.

This shows that the minimal ruled submanifold W_φ^{2n-k} has three distinct constant principal curvatures $\sin(\varphi)/2$, $-\sin(\varphi)/2$ and 0 with multiplicities 1, 1 and $2n - k - 2$ with respect to each unit normal vector. In Section 4 we determine the principal curvatures of the tubes around W_φ^{2n-k} together with their multiplicities. A table containing the principal curvatures and their multiplicities of all homogeneous hypersurfaces in $\mathbb{C}H^n$ is given at the end of the paper.

The investigation of the geometry of the tubes around W_φ^{2n-k} for $0 < \varphi < \pi/2$ can be reduced to the one of the tubes around W_φ^4 in $\mathbb{C}H^3$. In the final part of the paper we derive some geometric information about these particular tubes in terms of some autoparallel distributions of their tangent bundles.

2. Preliminaries

In this section we summarize some basic facts about the complex hyperbolic space. For details we refer to [8].

Let $\mathbb{C}H^n$, $n \geq 2$, be the n-dimensional complex hyperbolic space equipped with the Fubini-Study metric $\langle \cdot, \cdot \rangle$ of constant holomorphic sectional curvature -1. The Riemannian curvature tensor $\bar{R}$ on $\mathbb{C}H^n$ is given by

$$4\bar{R}(X, Y)Z = \langle X, Z \rangle Y - \langle Y, Z \rangle X + \langle JX, Z \rangle JY - \langle JY, Z \rangle JX + 2\langle JX, Y \rangle JZ,$$

where J is the complex structure on $\mathbb{C}H^n$.

We denote by $\mathbb{C}H^n(\infty)$ the ideal boundary of $\mathbb{C}H^n$. Each element x of $\mathbb{C}H^n(\infty)$ is an equivalence class of asymptotic geodesics in $\mathbb{C}H^n$. We equip $\mathbb{C}H^n \cup \mathbb{C}H^n(\infty)$ with the cone topology. Then $\mathbb{C}H^n \cup \mathbb{C}H^n(\infty)$ is homeomorphic to a closed ball in the Euclidean space $\mathbb{R}^{2n}$. For each $o \in \mathbb{C}H^n$ and each $x \in \mathbb{C}H^n(\infty)$ there exists a unique geodesic $\gamma_{ox} : \mathbb{R} \to \mathbb{C}H^n$ such that $\|\dot{\gamma}_{ox}\| = 1$, $\gamma_{ox}(0) = o$ and $\lim_{t \to \infty} \gamma_{ox}(t) = x$.

The connected component of the isometry group of $\mathbb{C}H^n$ is the special unitary group $G = SU(1, n)$. We fix a point $o \in \mathbb{C}H^n$ and denote by K the isotropy subgroup of G at o. Then K is isomorphic to $S(U(1) \times U(n)) \subset SU(1, n)$ and (G, K) is a symmetric pair. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the corresponding Cartan decomposition of the Lie algebra $\mathfrak{g}$ of G. As usual we identify $T_o\mathbb{C}H^n$ with $\mathfrak{p}$.

We now fix a point $x \in \mathbb{C}H^n(\infty)$ and denote by $\mathfrak{a}$ the one-dimensional linear subspace of $\mathfrak{p}$ spanned by $\dot{\gamma}_{ox}(0) \in T_o\mathbb{C}H^n \cong \mathfrak{p}$. As the rank of $\mathbb{C}H^n$ is one, $\mathfrak{a}$ is a maximal abelian subspace of $\mathfrak{p}$. Let $\mathfrak{g} = \mathfrak{g}_{-2\alpha} + \mathfrak{g}_{-\alpha} + \mathfrak{g}_0 + \mathfrak{g}_\alpha + \mathfrak{g}_{2\alpha}$ be the root space decomposition of $\mathfrak{g}$

induced by $\mathfrak{a}$. Then $\mathfrak{n} = \mathfrak{g}_\alpha + \mathfrak{g}_{2\alpha}$ is a 2-step nilpotent subalgebra of $\mathfrak{g}$ which is isomorphic to the $(2n-1)$-dimensional Heisenberg algebra. The center of $\mathfrak{n}$ is the one-dimensional subalgebra $\mathfrak{g}_{2\alpha}$. Moreover, $\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$ is an Iwasawa decomposition of $\mathfrak{g}$. We denote by A and N the connected closed subgroup of G with Lie algebra $\mathfrak{a}$ and $\mathfrak{n}$, respectively. The orbit $A \cdot o$ of A through o is just the path $\gamma_{ox}(\mathbb{R})$ of the geodesic γ_{ox} in $\mathbb{C}H^n$, and the orbits of N are the horospheres in $\mathbb{C}H^n$ centered at x. The solvable subgroup $AN \subset KAN = G$ acts simply transitively on $\mathbb{C}H^n$. Thus we can identify $\mathfrak{a}+\mathfrak{n}$ with $T_o\mathbb{C}H^n$. The Riemannian metric on $\mathbb{C}H^n$ induces an inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{a} + \mathfrak{n}$, and we may identify $\mathbb{C}H^n$ with the solvable Lie group AN equipped with the left-invariant Riemannian metric which is induced from $\langle \cdot, \cdot \rangle$.

The complex structure J on $\mathbb{C}H^n$ induces a complex structure on the vector space $\mathfrak{a}+\mathfrak{n} \cong T_o\mathbb{C}H^n$ which we will also denote by J. We define $B := \dot{\gamma}_{ox}(0) \in \mathfrak{a}$ and $Z := JB \in \mathfrak{g}_{2\alpha}$. Note that $\mathfrak{g}_\alpha$ is J-invariant. The Lie algebra structure on $\mathfrak{a} + \mathfrak{n}$ is given by the trivial skew-symmetric bilinear extension to $(\mathfrak{a} + \mathfrak{n}) \times (\mathfrak{a} + \mathfrak{n})$ of the relations

$$[B,Z] = Z \ , \ 2[B,U] = U \ , \ [U,V] = \langle JU, V \rangle Z \qquad (U, V \in \mathfrak{g}_\alpha). \tag{1}$$

This shows that $\mathfrak{a}+\mathfrak{n}$ is a semidirect sum of the two Lie algebras $\mathfrak{a}$ and $\mathfrak{n}$. Let $\mathrm{Exp}_\mathfrak{n} : \mathfrak{n} \to N$ be the Lie exponential map. The group structure on the semidirect product AN is given by

$$\begin{aligned} &\Big(a, \mathrm{Exp}_\mathfrak{n}(U + xZ)\Big) \cdot \Big(b, \mathrm{Exp}_\mathfrak{n}(V + yZ)\Big) \\ &= \Big(a + b, \mathrm{Exp}_\mathfrak{n}\Big(U + e^{a/2}V + \Big(x + e^a y + \frac{1}{2}e^{a/2}\langle JU, V \rangle\Big)Z\Big)\Big) \end{aligned} \tag{2}$$

for all $a, b, x, y \in \mathbb{R}$ and $U, V \in \mathfrak{g}_\alpha$. Here we identify the one-dimensional Lie group A in the canonical way with $\mathbb{R}$ such that $1 \in \mathbb{R}$ corresponds to $\mathrm{Exp}_\mathfrak{a}(B)$ with the Lie exponential map $\mathrm{Exp}_\mathfrak{a} : \mathfrak{a} \to A$. Finally, the Lie exponential map $\mathrm{Exp}_{\mathfrak{a}+\mathfrak{n}} : \mathfrak{a} + \mathfrak{n} \to AN$ is given by

$$\mathrm{Exp}_{\mathfrak{a}+\mathfrak{n}}\big(aB + U + xZ\big) = \Big(a, \mathrm{Exp}_\mathfrak{n}\Big(\rho(a/2)U + \rho(a)xZ\Big)\Big) \tag{3}$$

for all $a, x \in \mathbb{R}$ and $U \in \mathfrak{g}_\alpha$, where the analytic function $\rho : \mathbb{R} \to \mathbb{R}$ is defined by

$$\rho(s) = \begin{cases} \dfrac{e^s - 1}{s} & , \ \text{if } s \neq 0, \\ 1 & , \ \text{if } s = 0. \end{cases}$$

The Lie exponential map $\mathrm{Exp}_{\mathfrak{a}+\mathfrak{n}}$ is a diffeomeorphism. If $V \in \mathfrak{g}_\alpha$ is a unit vector, then the geodesic γ in $\mathbb{C}H^n$ with $\gamma(0) = o$ and $\dot{\gamma}(0) = V$ is given by

$$\gamma(t) = (\ln \mathrm{sech}^2(t/2), \mathrm{Exp}_\mathfrak{n}(2\tanh(t/2)V)), \tag{4}$$

and its tangent vector field $\dot{\gamma}$ is given by

$$\dot{\gamma}(t) = -\tanh(t/2)B + \mathrm{sech}(t/2)V. \tag{5}$$

We denote by $\bar{\nabla}$ the Levi-Civita covariant derivative of $\mathbb{C}H^n$. The standard method for calculating the Levi-Civita covariant derivative of a Lie group equipped with a left-invariant

Riemannian metric yields

$$
\begin{aligned}
\bar{\nabla}_{aB+U+xZ}\big(bB + V + yZ\big) \ = \ & \Big(\frac{1}{2}\langle U, V\rangle + x\,y\Big)B - \frac{1}{2}\big(bU + yJU + xJV\big) \\
& + \Big(\frac{1}{2}\,\langle JU, V\rangle - b\,x\Big)Z,
\end{aligned}
\tag{6}
$$

where $a, b, x, y \in \mathbb{R}$ and $U, V \in \mathfrak{g}_\alpha$ and all elements in $\mathfrak{a} + \mathfrak{n}$ are considered as left-invariant vector fields on $AN \cong \mathbb{C}H^n$.

3. The ruled submanifolds W^{2n-k} and W_φ^{2n-k}

The submanifolds W^{2n-k} and W_φ^{2n-k} were first constructed by the first author and Brück in [3].

Let $\mathfrak{v}$ be a linear subspace of $\mathfrak{g}_\alpha$. For each $0 \neq v \in \mathfrak{v}$ the Kähler angle of $\mathfrak{v}$ with respect to v is the angle $\varphi(v) \in [0, \pi/2]$ between $\mathfrak{v}$ and the real span of Jv. Thus $\varphi(v) \in [0, \pi/2]$ is determined by requiring that $\cos(\varphi(v))\|v\|$ is the length of the orthogonal projection of Jv onto $\mathfrak{v}$. We say that $\mathfrak{v}$ has *constant Kähler angle φ* if $\varphi(v) = \varphi$ for all nonzero vectors $v \in \mathfrak{v}$. The subspaces of $\mathfrak{g}_\alpha$ with constant Kähler angle $\varphi = 0$ are precisely the complex subspaces of $\mathfrak{g}_\alpha$, and the subspaces of $\mathfrak{g}_\alpha$ with constant Kähler angle $\varphi = \pi/2$ are precisely the real subspaces of $\mathfrak{g}_\alpha$.

Let $\mathfrak{w}$ be a linear subspace of $\mathfrak{g}_\alpha$ such that the orthogonal complement $\mathfrak{w}^\perp = \mathfrak{g}_\alpha \ominus \mathfrak{w}$ of $\mathfrak{w}$ in $\mathfrak{g}_\alpha$ has constant Kähler angle $\varphi \in [0, \pi/2]$. Then $\mathfrak{s} = \mathfrak{a} + \mathfrak{w} + \mathfrak{g}_{2\alpha}$ is a subalgebra of $\mathfrak{a} + \mathfrak{n}$. Denote by S the connected closed subgroup of AN with Lie algebra $\mathfrak{s}$ and by $N_K^0(S)$ the identity component of the normalizer of S in K. Then $N_K^0(S)S \subset KAN$ acts on $\mathbb{C}H^n$ with cohomogeneity one. The orbit $W_\varphi^{2n-k} = N_K^0(S)S \cdot o = S \cdot o$ of this action containing the point o is a $(2n-k)$-dimensional submanifold of $\mathbb{C}H^n$, where $k = \dim \mathfrak{w}^\perp$.

If $\varphi = 0$, that is, $\mathfrak{w}^\perp$ is a complex subspace of $\mathfrak{g}_\alpha$, then W_0^{2n-k} is a totally geodesic complex hyperbolic subspace $\mathbb{C}H^{n-k'}$, where $k = 2k'$.

If $\varphi = \pi/2$, then $\mathfrak{w}^\perp$ is a k-dimensional real subspace of $\mathfrak{g}_\alpha$. If $k = 1$, then $W_{\pi/2}^{2n-1}$ is the ruled real hypersurface W^{2n-1} determined by a horocycle in a totally geodesic $\mathbb{R}H^2 \subset \mathbb{C}H^n$. The orbits of $N_K^0(S)S$ form a homogeneous codimension one foliation on $\mathbb{C}H^n$ whose geometry has been investigated by the first author in [2]. If $k > 1$, then $W_{\pi/2}^{2n-k}$ is a $(2n-k)$-dimensional homogeneous submanifold of $\mathbb{C}H^n$ with totally real normal bundle of rank k. We will sometimes use the notation $W^{2n-k} := W_{\pi/2}^{2n-k}$.

If $0 < \varphi < \pi/2$, then k is even and W_φ^{2n-k} is a $(2n-k)$-dimensional homogeneous submanifold of $\mathbb{C}H^n$ whose normal bundle has constant Kähler angle φ and rank k.

As $\mathbb{C}H^n$ is a two-point homogeneous space, the construction of the submanifolds W^{2n-k} and W_φ^{2n-k} does not depend on the choice of the two points $o \in \mathbb{C}H^n$ and $x \in \mathbb{C}H^n(\infty)$, or equivalently, on the choice of the Iwasawa decomposition of G. Our next aim is to investigate the geometry of the submanifolds W_φ^{2n-k}, $0 < \varphi \leq \pi/2$.

Let $\mathbb{C}\mathfrak{w}^\perp$ be the complex subspace of $\mathfrak{g}_\alpha$ spanned by $\mathfrak{w}^\perp$ and $\mathfrak{d} = \mathbb{C}\mathfrak{w}^\perp \ominus \mathfrak{w}^\perp$ be the orthogonal complement of $\mathfrak{w}^\perp$ in $\mathbb{C}\mathfrak{w}^\perp$. As $\varphi > 0$, we have $k = \dim_\mathbb{C} \mathbb{C}\mathfrak{w}^\perp$ and hence $k = \dim \mathfrak{w}^\perp = \dim \mathfrak{d}$. For each $\xi \in \mathfrak{w}^\perp$ we decompose $J\xi \in \mathbb{C}\mathfrak{w}^\perp = \mathfrak{d} + \mathfrak{w}^\perp$ into $J\xi =$

$P\xi + F\xi$ with $P\xi \in \mathfrak{d}$ and $F\xi \in \mathfrak{w}^\perp$. Since $\mathfrak{w}^\perp$ has constant Kähler angle φ, we have $\langle F\xi, F\xi \rangle = \cos^2(\varphi)\langle \xi, \xi \rangle$ and hence $\langle P\xi, P\xi \rangle = \sin^2(\varphi)\langle \xi, \xi \rangle$. As $\varphi > 0$, the homomorphism $P : \mathfrak{w}^\perp \to \mathfrak{d}$ is injective, and as $\dim \mathfrak{w}^\perp = \dim \mathfrak{d}$ we see that $P : \mathfrak{w}^\perp \to \mathfrak{d}$ is an isomorphism. From $-\xi = JJ\xi = JP\xi + JF\xi = JP\xi + PF\xi + F^2\xi$ we see that the $\mathfrak{d}$-component $(JP\xi)_\mathfrak{d}$ of $JP\xi$ is equal to $-PF\xi$, and hence $\langle (JP\xi)_\mathfrak{d}, (JP\xi)_\mathfrak{d} \rangle = \langle PF\xi, PF\xi \rangle = \sin^2(\varphi)\langle F\xi, F\xi \rangle = \sin^2(\varphi)\cos^2(\varphi)\langle \xi, \xi \rangle = \cos^2(\varphi)\langle P\xi, P\xi \rangle$. Since $P : \mathfrak{w}^\perp \to \mathfrak{d}$ is an isomorphism, this implies that $\mathfrak{d}$ has constant Kähler angle φ as well.

We denote by $\mathfrak{c}$ the maximal complex subspace of $\mathfrak{s}$. Note that $\mathfrak{a} + \mathfrak{g}_{2\alpha} \subset \mathfrak{c}$, $\dim_\mathbb{C} \mathfrak{c} = n - k$ and $\mathfrak{s} = \mathfrak{c} + \mathfrak{d}$. Then we have the orthogonal decomposition

$$\mathfrak{a} + \mathfrak{n} = \mathfrak{c} + \mathfrak{d} + \mathfrak{w}^\perp.$$

We denote by $\mathfrak{A}$, $\mathfrak{C}$, $\mathfrak{D}$ and $\mathfrak{W}^\perp$ the left-invariant distributions on $\mathbb{C}H^n$ along W_φ^{2n-k} which are induced by $\mathfrak{a}$, $\mathfrak{c}$, $\mathfrak{d}$ and $\mathfrak{w}^\perp$, respectively. By construction, we have $\mathfrak{C} + \mathfrak{D} = TW_\varphi^{2n-k}$ and $\mathfrak{W}^\perp = \nu W_\varphi^{2n-k}$.

Proposition 3.1. *The submanifold W_φ^{2n-k}, $0 < \varphi \leq \pi/2$, of $\mathbb{C}H^n$ has the following properties:*

  (i) *The maximal holomorphic subbundle $\mathfrak{C}$ of TW_φ^{2n-k} is autoparallel and the leaves of the induced foliation on W_φ^{2n-k} are totally geodesic $\mathbb{C}H^{n-k} \subset \mathbb{C}H^n$. Hence W_φ^{2n-k} is a ruled submanifold of $\mathbb{C}H^n$.*
  (ii) *The following statements are equivalent:*
  (a) *the distribution $\mathfrak{D}$ on W_φ^{2n-k} is integrable;*
  (b) *the distribution $\mathfrak{A} + \mathfrak{D}$ on W_φ^{2n-k} is integrable;*
  (c) *the normal bundle $\mathfrak{W}^\perp$ is flat with respect to the normal connection;*
  (d) *$\varphi = \pi/2$.*
  In this case the leaves of the foliation on $W_{\pi/2}^{2n-k}$ induced by $\mathfrak{A} + \mathfrak{D}$ are totally geodesic $\mathbb{R}H^{k+1} \subset \mathbb{C}H^n$ and the leaves of the foliation on $W_{\pi/2}^{2n-k}$ induced by $\mathfrak{D}$ are horospheres with center x in these totally geodesic $\mathbb{R}H^{k+1} \subset \mathbb{C}H^n$.
  (iii) *For each $0 \neq \xi \in \mathfrak{w}^\perp$ the left-invariant distribution $\mathfrak{A} + \mathbb{R}P\xi$ on W_φ^{2n-k} is autoparallel and the leaves of the induced foliation on W_φ^{2n-k} are totally geodesic $\mathbb{R}H^2 \subset \mathbb{C}H^n$.*
  (iv) *For each $0 \neq \xi \in \mathfrak{w}^\perp$ the left-invariant distribution $\mathbb{R}P\xi$ on W_φ^{2n-k} is integrable and the leaves of the induced foliation on W_φ^{2n-k} are horocycles with center x in the totally geodesic $\mathbb{R}H^2 \subset \mathbb{C}H^n$ given by the distribution $\mathfrak{A} + \mathbb{R}P\xi$.*

Proof. (i) follows immediately from (6) and the fact that the only complex totally geodesic submanifolds of $\mathbb{C}H^n$ are complex hyperbolic spaces.

From (1) we see that $2[aB + U, bB + V] = aV - bU + 2\langle JU, V\rangle Z$ for all $a, b \in \mathbb{R}$ and $U, V \in \mathfrak{D}$. This shows that $\mathfrak{A} + \mathfrak{D}$ is integrable if and only if $\mathfrak{D}$ is integrable if and only if $\mathfrak{D}$ is real, that is, $\varphi = \pi/2$. On the other hand, (6) implies $2\nabla^\perp_{aB+U+xZ}\xi = -xF\xi$ for all $a, x \in \mathbb{R}$, $U \in \mathfrak{D}$ and $\xi \in \mathfrak{W}^\perp$. Hence, the normal bundle $\mathfrak{W}^\perp$ is flat if and only if $F = 0$, or equivalently, $\varphi = \pi/2$. In this case (6) yields

$$2\bar{\nabla}_{aB+U}(bB + V) = \langle U, V\rangle B - bU \in \mathfrak{A} + \mathfrak{D}$$

for all $a, b \in \mathbb{R}$ and $U, V \in \mathfrak{D}$. This shows that $\mathfrak{A} + \mathfrak{D}$ is autoparallel and its leaves are totally geodesic real submanifolds of $\mathbb{C}H^n$. The only real totally geodesic submanifolds of $\mathbb{C}H^n$ are real hyperbolic spaces. Finally, for all $U, V \in \mathfrak{D}$ we have $2\bar{\nabla}_U V = \langle U, V \rangle B$ and $2\bar{\nabla}_U B = -U$, which implies that the leaves of $\mathfrak{D}$ are spherical hypersurfaces of the corresponding real hyperbolic subspaces. Since the sectional curvature of a totally geodesic real hyperbolic subspace is $-1/4$, and the mean curvature vector field of any leaf of $\mathfrak{D}$ is $(1/2)B$, it follows that the leaves of $\mathfrak{D}$ are horospheres centered at x in the real hyperbolic subspaces. This finishes the proof of (ii).

For any $aB + xP\xi, bB + yP\xi \in \mathfrak{A} + \mathbb{R}P\xi$ we get from (6) and using $\langle P\xi, P\xi \rangle = \sin^2(\varphi)\langle \xi, \xi \rangle$ that $2\bar{\nabla}_{aB+xP\xi}(bB + yP\xi) = xy\sin^2(\varphi)B - bxP\xi \in \mathfrak{A} + \mathbb{R}P\xi$. From this we easily get the assertion (iii).

Finally, define $U_\xi = P\xi/\sin(\varphi)$. Then (6) implies $2\bar{\nabla}_{U_\xi}U_\xi = B$ and $4\bar{\nabla}_{U_\xi}\bar{\nabla}_{U_\xi}U_\xi = -U_\xi$. Since the real hyperbolic planes in (iii) have constant sectional curvature $-1/4$, this shows that the integral curves of U_ξ are horocycles with center x in the corresponding real hyperbolic planes. This proves (iv). $\qquad \square$

The previous proposition implies a nice geometric construction of the ruled submanifolds $W_\varphi^{2n-k} \subset \mathbb{C}H^n$.

Corollary 3.2. *Let $k \in \{1, \ldots, n-1\}$, and fix a totally geodesic $\mathbb{C}H^{n-k} \subset \mathbb{C}H^n$ and points $o \in \mathbb{C}H^{n-k}$ and $x \in \mathbb{C}H^{n-k}(\infty)$. Let KAN be the Iwasawa decomposition of $SU(1, n)$ with respect to o and x, and let H' be the subgroup of AN which acts simply transitively on $\mathbb{C}H^{n-k}$. Next, let W be a subspace of $\nu_o\mathbb{C}H^{n-k}$ with constant Kähler angle $\varphi \in (0, \pi/2]$ such that $\mathbb{C}W = \nu_o\mathbb{C}H^{n-k}$. Left translation of W by H' to all points in $\mathbb{C}H^{n-k}$ determines a subbundle $\mathfrak{V}$ of the normal bundle $\nu\mathbb{C}H^{n-k}$. At each point $p \in \mathbb{C}H^{n-k}$ attach the horocycles determined by x and the linear lines in $\mathfrak{V}_p$. The resulting subset M of $\mathbb{C}H^n$ is holomorphically congruent to the ruled submanifold W_φ^{2n-k}.*

Proof. Let W_φ^{2n-k} be the ruled minimal submanifold of $\mathbb{C}H^n$ constructed from the Iwasawa decomposition KAN associated with o and x and from the choice of $\mathfrak{w}^\perp = \nu_o\mathbb{C}H^{n-k} \ominus W$. We use the above notations. We will show that $M = W_\varphi^{2n-k}$. Let $p \in W_\varphi^{2n-k}$. Then there exists an isometry $s \in S$ with $p = s(o)$. There is a unique vector X in the Lie algebra $\mathfrak{s}$ of S such that $s = \operatorname{Exp}_{\mathfrak{a}+\mathfrak{n}}(X)$. We can write $X = aB + U + V + zZ$ with some $U \in \mathfrak{c} \ominus (\mathfrak{a} + \mathfrak{g}_{2\alpha})$, $V \in \mathfrak{d}$, and $a, z \in \mathbb{R}$. Note that $[V, U] = 0$ because they are complex orthogonal. We now define $g = \operatorname{Exp}_{\mathfrak{a}+\mathfrak{n}}(\rho(a/2)V)$ and $h = \operatorname{Exp}_{\mathfrak{a}+\mathfrak{n}}(aB + U + zZ)$. Note that $h \in H'$. Using (2) and (3) we get

$$
\begin{aligned}
gh &= \operatorname{Exp}_{\mathfrak{a}+\mathfrak{n}}\left(\rho\left(\frac{a}{2}\right)V\right)\operatorname{Exp}_{\mathfrak{a}+\mathfrak{n}}(aB + U + zZ) \\
&= \left(0, \operatorname{Exp}_{\mathfrak{n}}\left(\rho\left(\frac{a}{2}\right)V\right)\right) \cdot \left(a, \operatorname{Exp}_{\mathfrak{n}}\left(\rho\left(\frac{a}{2}\right)U + \rho(a)z\,Z\right)\right) \\
&= \left(a, \operatorname{Exp}_{\mathfrak{n}}\left(\rho\left(\frac{a}{2}\right)(U + V) + \rho(a)z\,Z\right)\right) \\
&= \operatorname{Exp}_{\mathfrak{a}+\mathfrak{n}}(aA + U + V + zZ) = \operatorname{Exp}_{\mathfrak{a}+\mathfrak{n}}(X) = s.
\end{aligned}
$$

By construction, $h(o) \in \mathbb{C}H^{n-k}$, and $s(o) = g(h(o))$ is on the horocycle with center x through $h(o)$ tangent to $\mathbb{R}V$. From this we conclude that $W_\varphi^{2n-k} \subset M$. From Proposition 3.1 we already know that $M \subset W_\varphi^{2n-k}$. Altogether this implies $M = W_\varphi^{2n-k}$.   □

We now describe the geometry of W_φ^{2n-k} in terms of the second fundamental form.

Proposition 3.3. *The second fundamental form II of W_φ^{2n-k} is given by*

$$II\big(aB + U + P\xi + xZ, bB + V + P\eta + yZ\big) = \frac{\sin^2(\varphi)}{2}\big(y\xi + x\eta\big)$$

for all $\xi, \eta \in \mathfrak{w}^\perp$, $U, V \in \mathfrak{c} \ominus (\mathfrak{a} + \mathfrak{g}_{2\alpha})$ and $a, b, x, y \in \mathbb{R}$. Thus II is given by the trivial symmetric bilinear extension of $2II(Z, P\xi) = \sin^2(\varphi)\xi$ for all $\xi \in \mathfrak{w}^\perp$.

Proof. We denote by $(\cdot)^\perp$ the orthogonal projection onto νW_φ^{2n-k}. The Gauß formula and (6) imply

$$
\begin{aligned}
II\big(aB + U + P\xi + xZ, bB + V + P\eta + yZ\big) &= \big(\bar{\nabla}_{aB+U+P\xi+xZ}(bB + V + P\eta + yZ)\big)^\perp \\
&= -\Big(\frac{y}{2}JP\xi + \frac{x}{2}JP\eta\Big)^\perp \\
&= \frac{\sin^2(\varphi)}{2}\big(y\xi + x\eta\big),
\end{aligned}
$$

since $(JP\xi)^\perp = -\sin^2(\varphi)\xi$, which follows from the fact that $\mathfrak{w}^\perp$ has constant Kähler angle φ.   □

As an immediate consequence we get

Corollary 3.4. *W_φ^{2n-k} is a minimal ruled submanifold of $\mathbb{C}H^n$.*

For $k > 1$ the previous corollary follows also from the general fact that each singular orbit of a cohomogeneity one action is a minimal submanifold. We will now show that the above equation for the second fundamental form in fact characterizes the minimal ruled submanifolds W_φ^{2n-k} in $\mathbb{C}H^n$.

Theorem 3.5 (Rigidity of the submanifold W_φ^{2n-k}). *Let M be a $(2n - k)$-dimensional connected submanifold in $\mathbb{C}H^n$, $n \geq 2$, with normal bundle $\nu M \subset T\mathbb{C}H^n$ of constant Kähler angle $\varphi \in (0, \pi/2]$. Assume that there exists a unit vector field Z tangent to the maximal complex distribution on M such that the second fundamental form II of M is given by the trivial symmetric bilinear extension of*

$$2II(Z, P\xi) = \sin^2(\varphi)\,\xi \tag{7}$$

for all $\xi \in \nu M$, where $P\xi$ is the tangential component of $J\xi$. Then M is holomorphically congruent to an open part of the ruled minimal submanifold W_φ^{2n-k}. Conversely, the second fundamental form of W_φ^{2n-k} is of this form.

Proof. The last statement is a consequence of Proposition 3.3. For the other part we use Corollary 3.2.

We decompose the tangent bundle TM of M orthogonally into $TM = \mathfrak{C} + \mathfrak{D}$, where $\mathfrak{C}$ is the maximal complex subbundle of TM. For each $\xi \in \Gamma(\nu M)$ we decompose $J\xi$ orthogonally into $J\xi = P\xi + F\xi$ with $P\xi \in \Gamma(\mathfrak{D})$ and $F\xi \in \Gamma(\nu M)$. As above one can show that $\mathfrak{D}$ has constant Kähler angle φ as well, and the bundle homomorphisms $P : \nu M \to \mathfrak{D}$ and $F : \nu M \to \nu M$ are homomorphisms satisfying $\langle P\xi, P\xi \rangle = \sin^2(\varphi)\langle \xi, \xi \rangle$ and $\langle F\xi, F\xi \rangle = \cos^2(\varphi)\langle \xi, \xi \rangle$. Note that if $\varphi = \pi/2$ then F is trivial and $P = J|_{\nu M}$; otherwise, P and F are isomorphisms.

For all $U, V \in \Gamma(\mathfrak{C})$ and $\xi \in \Gamma(\nu M)$ we have, using (7) and $\bar{\nabla}J = 0$,

$$\langle \bar{\nabla}_U V, \xi \rangle = \langle II(U, V), \xi \rangle = 0$$

and

$$\langle \bar{\nabla}_U V, J\xi \rangle = -\langle J\bar{\nabla}_U V, \xi \rangle = -\langle II(U, JV), \xi \rangle = 0.$$

This shows that $\mathfrak{C}$ is an autoparallel subbundle of TM and each integral manifold is a totally geodesic submanifold of $\mathbb{C}H^n$. As $\mathfrak{C}$ is a complex subbundle of complex rank $n - k$, each of these integral manifolds must be an open part of a totally geodesic $\mathbb{C}H^{n-k} \subset \mathbb{C}H^n$.

Let $o \in M$ and $\mathcal{F}_o$ be the leaf of $\mathfrak{C}$ through o, which is an open part of a totally geodesic $\mathbb{C}H^{n-k} \subset \mathbb{C}H^n$. Let $\gamma : I \to \mathcal{F}_o$ be a curve with $\gamma(0) = o$. We prove that the normal spaces of M along γ are uniquely determined by the differential equation

$$2\bar{\nabla}_{\dot{\gamma}}X + \langle \dot{\gamma}, Z \rangle JX = 0 \tag{8}$$

along $\gamma^*\nu\mathcal{F}_o$. Let $X \in \Gamma(TM)$ and $\xi \in \Gamma(\nu M)$. Using (7) we get

$$-\langle \bar{\nabla}_{\dot{\gamma}}\xi, X \rangle = \langle II(\dot{\gamma}, X), \xi \rangle = \langle \dot{\gamma}, Z \rangle \langle X, P\xi \rangle \langle II(Z, P\xi), \xi \rangle / \langle P\xi, P\xi \rangle = \frac{1}{2}\langle \dot{\gamma}, Z \rangle \langle P\xi, X \rangle,$$

which implies

$$\bar{\nabla}_{\dot{\gamma}}\xi = -\frac{1}{2}\langle \dot{\gamma}, Z \rangle P\xi + \nabla^{\perp}_{\dot{\gamma}}\xi, \tag{9}$$

where $\nabla^{\perp}$ is the normal connection of M. Now, let X be a vector field along γ with $X_0 \in \nu_o M$ and satisfying (8). We may write $X = U + J\eta + \xi$ with $U \in \Gamma(\gamma^*\mathfrak{C})$, $\xi, \eta \in \Gamma(\gamma^*\nu M)$ and $U_0 = \eta_0 = 0$. Then, using (9) and $\bar{\nabla}J = 0$, we get

$$
\begin{aligned}
0 &= 2\bar{\nabla}_{\dot{\gamma}}X + \langle \dot{\gamma}, Z \rangle JX \\
&= 2\bar{\nabla}_{\dot{\gamma}}U + 2J\bar{\nabla}_{\dot{\gamma}}\eta + 2\bar{\nabla}_{\dot{\gamma}}\xi + \langle \dot{\gamma}, Z \rangle JU + \langle \dot{\gamma}, Z \rangle J^2\eta + \langle \dot{\gamma}, Z \rangle J\xi \\
&= 2\bar{\nabla}_{\dot{\gamma}}U + \langle \dot{\gamma}, Z \rangle JU + P\left(2\nabla^{\perp}_{\dot{\gamma}}\eta + \langle \dot{\gamma}, Z \rangle F\eta\right) \\
&\quad + 2\nabla^{\perp}_{\dot{\gamma}}\xi + \langle \dot{\gamma}, Z \rangle F\xi + F\left(2\nabla^{\perp}_{\dot{\gamma}}\eta + \langle \dot{\gamma}, Z \rangle F\eta\right).
\end{aligned}
$$

We have that $2\bar{\nabla}_{\dot{\gamma}}U + \langle \dot{\gamma}, Z \rangle JU$ is tangent to $\mathfrak{C}$ because $\mathfrak{C}$ is a complex autoparallel distribution. Hence, it follows that $2\bar{\nabla}_{\dot{\gamma}}U + \langle \dot{\gamma}, Z \rangle JU = 0$. Since $U_0 = 0$, the uniqueness of solutions to ordinary differential equations implies $U_t = 0$ for all t and thus X is normal to $\mathcal{F}_o$ along γ. Similarly, the component tangent to $P\nu M$ yields $2\nabla^{\perp}_{\dot{\gamma}}\eta + \langle \dot{\gamma}, Z \rangle F\eta = 0$ and since $\eta_0 = 0$ we have $\eta_t = 0$ for all t. Hence, $X_t \in \nu_{\gamma(t)}M$ for any t, which proves our previous assertion.

We define $B := -JZ$. The tangent vector B_o determines a point $x \in \mathbb{C}H^n(\infty)$, and thus, o and x give rise to an Iwasawa decomposition of the Lie algebra of the isometry group of $\mathbb{C}H^n$. Let us consider the ruled submanifold W_φ^{2n-k} determined by the above Iwasawa decomposition and the normal space $\mathfrak{w}^\perp = \nu_o M$ of constant Kähler angle φ. The leaf $\mathcal{F}_o$ is an open part of the totally geodesic $\mathbb{C}H^{n-k}$ tangent to the maximal complex distribution of W_φ^{2n-k} at o. We have just proved that (8) determines the normal bundle of a submanifold satisfying all the hypotheses of Theorem 3.5, which implies $\nu_p M = \nu_p W_\varphi^{2n-k}$ for all $p \in \mathcal{F}_o$, that is, $\nu_p M$ is obtained by left translation of $\nu_o M$ for all $p \in \mathcal{F}_o$. According to Corollary 3.2 it just remains to prove that at any point $p \in \mathcal{F}_o$ the horocycles determined by x and the linear lines in $P\nu_p M$ are contained in a neighborhood of p in M.

We now prove that the vector field B is a geodesic vector field and all its integral curves are geodesics in $\mathbb{C}H^n$ converging to the point $x \in \mathbb{C}H^n(\infty)$. Since B belongs to the maximal complex distribution we have $\bar{\nabla}_B B \in \Gamma(\mathfrak{C})$. Since B is a unit vector $\langle \bar{\nabla}_B B, B \rangle = 0$. Let $X \in \Gamma(\mathfrak{C} \ominus \mathbb{R}B)$ and $\eta \in \Gamma(\nu M)$ be a local unit normal vector field of M. Using the explicit expression for $\bar{R}$, the Codazzi equation, (7) and $\bar{\nabla}J = 0$ we get

$$
\begin{aligned}
0 &= 2\bar{R}_{BP\eta JX\eta} = 2\langle (\nabla_B^\perp II)(P\eta, JX) - (\nabla_{P\eta}^\perp II)(B, JX), \eta \rangle \\
&= -2\langle II(P\eta, \nabla_B JX), \eta \rangle = -2\langle \nabla_B JX, Z \rangle \langle II(P\eta, Z), \eta \rangle \\
&= -\sin^2(\varphi)\langle \bar{\nabla}_B JX, Z \rangle = \sin^2(\varphi)\langle \bar{\nabla}_B B, X \rangle.
\end{aligned}
$$

This implies $\langle \bar{\nabla}_B B, X \rangle = 0$ and therefore $\bar{\nabla}_B B = 0$, which means that the integral curves of B are geodesics in $\mathbb{C}H^n$.

Now let $X \in \Gamma(TM \ominus \mathbb{R}B)$ and γ an integral curve of X. We consider the geodesic variation $F(s,t) = \exp_{\gamma(s)}(tB_{\gamma(s)})$ of $\alpha(t) = F(0,t)$, the geodesic in $\mathbb{C}H^n$ with initial condition $\dot{\alpha}(0) = B_o$. We prove that $d(\alpha(t), F(s,t))$ tends to 0 as t goes to infinity, where d is the Riemannian distance function of $\mathbb{C}H^n$.

The transversal vector field of the geodesic variation F, $\zeta(t) = (\partial F/\partial s)(0,t)$, is a Jacobi field along α (thus, $4\zeta'' - \zeta - 3\langle \zeta, Z \rangle Z = 0$) with initial conditions $\zeta(0) = X_{\gamma(0)}$, $\zeta'(0) = \bar{\nabla}_{X_{\gamma(0)}} B$. Hence, we need to calculate $\bar{\nabla}_X B$.

Let $\eta \in \Gamma(\nu M)$ be a local unit vector field. Using (7) we have $\langle \bar{\nabla}_X B, \eta \rangle = \langle II(X, B), \eta \rangle = 0$. We also have $\langle \bar{\nabla}_X B, B \rangle = 0$. Now, using (7) we get

$$
\begin{aligned}
2\langle \bar{\nabla}_X B, P\eta \rangle &= -2\langle \bar{\nabla}_X JZ, J\eta - F\eta \rangle = -2\langle II(X, Z), \eta \rangle - 2\langle II(X, B), F\eta \rangle \\
&= -2\langle X, P\eta \rangle \langle II(P\eta, Z), \eta \rangle / \sin^2(\varphi) = -\langle X, P\eta \rangle.
\end{aligned} \tag{10}
$$

Next, let $Y \in \Gamma(\mathfrak{C} \ominus \mathbb{R}B)$ and assume that $X \in \Gamma(\mathfrak{C} \ominus \mathbb{R}B)$. Given $\xi \in \Gamma(\nu M)$ we have $\langle \nabla_{P\eta} JY, P\xi \rangle = \langle II(P\eta, Y), \xi \rangle - \langle II(P\eta, JY), F\xi \rangle = \frac{1}{2}\langle Y, Z \rangle \langle P\eta, P\xi \rangle$. This, the expression for $\bar{R}$, the Codazzi equation, (7) and $\bar{\nabla}J = 0$ imply

$$
\begin{aligned}
-\sin^2(\varphi)\langle X, Y \rangle &= 4\bar{R}_{XP\eta JY\eta} = 4\langle (\nabla_X^\perp II)(P\eta, JY) - (\nabla_{P\eta}^\perp II)(X, JY), \eta \rangle \\
&= -4\langle II(P\eta, \nabla_X JY), \eta \rangle + 4\langle II(X, \nabla_{P\eta} JY), \eta \rangle \\
&= -4\langle \nabla_X JY, Z \rangle \langle II(P\eta, Z), \eta \rangle + 4\langle X, Z \rangle \langle II(Z, \nabla_{P\eta} JY), \eta \rangle \\
&= 2\sin^2(\varphi)\langle \bar{\nabla}_X B, Y \rangle + \sin^2(\varphi)\langle X, Z \rangle \langle Z, Y \rangle.
\end{aligned}
$$

Hence, if $X \in \Gamma(\mathfrak{C} \ominus \mathbb{R}B)$ we have, as $\bar{\nabla}_X B \in \Gamma(\mathfrak{C})$, that $2\bar{\nabla}_X B = -X - \langle X, Z \rangle Z$.

Now assume that $X \in \Gamma(P\nu M)$ and write $X = P\xi$ with $\xi \in \Gamma(\nu M)$. We have $\langle \nabla_{JY} P\xi, Z \rangle = -\langle \bar{\nabla}_{JY} Z, J\xi - F\xi \rangle = -\langle II(JY, B), \xi \rangle + \langle II(JY, Z), F\xi \rangle = 0$. This, together with the explicit expression for $\bar{R}$, the Codazzi equation, (7) and $\bar{\nabla} J = 0$ implies

$$
\begin{aligned}
0 &= 2\bar{R}_{P\xi JY P\xi\xi} = 2\langle (\nabla^{\perp}_{P\xi} II)(JY, P\xi) - (\nabla^{\perp}_{JY} II)(P\xi, P\xi), \xi \rangle \\
&= -2\langle II(\nabla_{P\xi} JY, P\xi), \xi \rangle + 4\langle II(\nabla_{JY} P\xi, P\xi), \xi \rangle \\
&= -2\langle \nabla_{P\xi} JY, Z \rangle \langle II(Z, P\xi), \xi \rangle + 4\langle \nabla_{JY} P\xi, Z \rangle \langle II(Z, P\xi), \xi \rangle \\
&= -\sin^2(\varphi)\langle \bar{\nabla}_{P\xi} JY, Z \rangle = \sin^2(\varphi)\langle \bar{\nabla}_{P\xi} B, Y \rangle.
\end{aligned}
$$

Thus we get $\langle \bar{\nabla}_{P\xi} B, Y \rangle = 0$, and as a consequence, we get $2\bar{\nabla}_{P\xi} B = -P\xi$ using (10).

All in all, this implies

$$
\bar{\nabla}_X B = -\frac{1}{2} X - \frac{1}{2}\langle X, Z \rangle Z \quad \text{for all } X \in \Gamma(TM \ominus \mathbb{R}B). \tag{11}
$$

Therefore, if $X \in T_{\alpha(0)} M \ominus \mathbb{R}B_{\alpha(0)}$ is a unit vector and $\mathcal{B}_X$ denotes $\bar{\nabla}$-parallel translation of X along α, we get

$$
\zeta(t) = e^{-t/2}\mathcal{B}_X(t) + (e^{-t} - e^{-t/2})\langle X, Z_{\alpha(0)} \rangle Z_{\alpha(t)}.
$$

Note that $Z_{\alpha(t)}$ is a parallel vector field along α since $Z'_{\alpha(t)} = \bar{\nabla}_{B_{\alpha(t)}} Z = J\bar{\nabla}_{B_{\alpha(t)}} B = 0$. We easily see that $\lim_{t \to \infty} \|\zeta(t)\| = 0$, which implies $\lim_{t \to \infty} d(\alpha(t), F(s, t)) = 0$ as $d(\alpha(t), F(s, t)) \leq s\|\zeta(t)\|$. Altogether this shows that the integral curves of B are asymptotic geodesics corresponding to the point $x \in \mathbb{C}H^n(\infty)$.

Now let $p \in \mathcal{F}_o$ and $\xi_p \in \nu_p M$ be a unit vector. The theorem now follows if we prove that the horocycle determined by $P\xi_p/\sin(\varphi)$ and the point $x \in \mathbb{C}H^n(\infty)$ is locally contained in M. To achieve this we will construct a unit local vector field $\xi \in \Gamma(\nu M)$ such that the previous horocycle is an integral curve of $P\xi/\sin(\varphi)$.

Let $\gamma : I \to M$ be a curve in M satisfying the differential equation

$$
\nabla_{\dot{\gamma}} \dot{\gamma} = \frac{1}{2}\langle \dot{\gamma}, \dot{\gamma} \rangle B, \qquad \dot{\gamma}(0) = P\xi_p/\sin(\varphi). \tag{12}
$$

We first prove that γ is parametrized by arc length and that it remains tangent to $P\nu M$.

Write $\dot{\gamma} = aB + xZ + X + P\eta$ for some differentiable functions $a, x : I \to \mathbb{R}$, and vector fields $X \in \Gamma(\gamma^*(\mathfrak{C} \ominus (\mathbb{R}B + \mathbb{R}Z)))$ and $\eta \in \Gamma(\gamma^*\nu M)$. Since $Z = JB$, the definition of γ and (11) show

$$
\frac{dx}{dt} = \frac{d}{dt}\langle \dot{\gamma}, Z \rangle = \langle \nabla_{\dot{\gamma}} \dot{\gamma}, Z \rangle + \langle \nabla_{\dot{\gamma}} Z, \dot{\gamma} \rangle = \langle xB - \frac{1}{2}JX - \frac{1}{2}JP\eta, \dot{\gamma} \rangle = ax.
$$

Since $x(0) = 0$, the uniqueness of solutions to ordinary differential equations implies that $x(t) = 0$ for all t.

Let $Y \in \Gamma(\mathbb{R}B + P\nu M)$ and $\zeta \in \Gamma(\nu M)$. Then, (7) yields $\langle \bar{\nabla}_Y X, \zeta \rangle = \langle II(Y, X), \zeta \rangle = 0$ and $\langle \bar{\nabla}_Y X, J\zeta \rangle = -\langle II(Y, JX), \zeta \rangle = 0$. Moreover, since $\bar{\nabla}_Y B \in \Gamma(P\nu M)$ by (11) we have $\langle \bar{\nabla}_Y X, B \rangle = -\langle \bar{\nabla}_Y B, X \rangle = 0$. Also, $2\langle \bar{\nabla}_X X, B \rangle = -2\langle \bar{\nabla}_X B, X \rangle = \langle X, X \rangle$ and

$\langle \bar{\nabla}_X X, P\eta \rangle = -\langle II(X, JX), \eta \rangle - \langle II(X, X), F\eta \rangle = 0$. Hence,

$$\begin{aligned}
\frac{d}{dt}\langle X, X \rangle &= \frac{d}{dt}\langle \dot{\gamma}, X \rangle = \langle \nabla_{\dot{\gamma}}\dot{\gamma}, X \rangle + \langle \nabla_{\dot{\gamma}}X, \dot{\gamma} \rangle \\
&= a\langle \bar{\nabla}_{\dot{\gamma}}X, B \rangle + \langle \bar{\nabla}_{\dot{\gamma}}X, X \rangle + \langle \bar{\nabla}_{\dot{\gamma}}X, P\eta \rangle = \langle \bar{\nabla}_{\dot{\gamma}}X, X \rangle + a\langle \bar{\nabla}_X X, B \rangle \\
&= \langle \nabla_{\dot{\gamma}}X, X \rangle + \frac{a}{2}\langle X, X \rangle = \frac{1}{2}\frac{d}{dt}\langle X, X \rangle + \frac{a}{2}\langle X, X \rangle.
\end{aligned}$$

This yields $(d/dt)\langle X, X \rangle = a\langle X, X \rangle$ and since $\langle X(0), X(0) \rangle = 0$ we obtain $\langle X(t), X(t) \rangle = 0$ for all t and thus $X = 0$.

Using the definition of γ we obtain

$$\frac{d}{dt}\langle \dot{\gamma}, \dot{\gamma} \rangle = 2\langle \nabla_{\dot{\gamma}}\dot{\gamma}, \dot{\gamma} \rangle = a\langle \dot{\gamma}, \dot{\gamma} \rangle.$$

The definition of γ, the fact that B is geodesic and (11) yield

$$\frac{da}{dt} = \frac{d}{dt}\langle \dot{\gamma}, B \rangle = \langle \nabla_{\dot{\gamma}}\dot{\gamma}, B \rangle + \langle \nabla_{\dot{\gamma}}B, \dot{\gamma} \rangle = \frac{1}{2}\langle \dot{\gamma}, \dot{\gamma} \rangle - \frac{1}{2}\langle P\eta, \dot{\gamma} \rangle = \frac{1}{2}\langle \dot{\gamma}, \dot{\gamma} \rangle - \frac{1}{2}\langle P\eta, P\eta \rangle.$$

Now we calculate $(d/dt)\langle P\eta, P\eta \rangle$. Let $\xi, \zeta \in \Gamma(\nu M)$ and $Y \in \Gamma(\mathfrak{C})$. Since $\mathfrak{C}$ is autoparallel, we have $\langle \bar{\nabla}_B P\xi, Y \rangle = 0$. Using (7) we obtain $\langle \bar{\nabla}_B P\xi, \zeta \rangle = \langle II(B, P\xi), \zeta \rangle = 0$. On the other hand we have $JP\xi = -\xi - PF\xi - F^2\xi = -PF\xi - \sin^2(\varphi)\xi$, which gives, using (7),

$$\begin{aligned}
\langle \bar{\nabla}_B P\xi, P\zeta \rangle &= \langle \bar{\nabla}_B P\xi, J\zeta - F\zeta \rangle = -\langle \bar{\nabla}_B JP\xi, \zeta \rangle - \langle II(B, P\xi), F\zeta \rangle \\
&= \langle II(B, PF\xi), \zeta \rangle + \sin^2(\varphi)\langle \bar{\nabla}_B^\perp \xi, \zeta \rangle = \langle P\nabla_B^\perp \xi, P\zeta \rangle.
\end{aligned}$$

This readily implies,

$$\bar{\nabla}_B P\xi = P\nabla_B^\perp \xi \quad \text{for all } \xi \in \Gamma(\nu M). \tag{13}$$

Using again (7) we get

$$\begin{aligned}
2\langle \bar{\nabla}_{P\xi} P\xi, Y \rangle &= -2\langle \bar{\nabla}_{P\xi} Y, J\xi - F\xi \rangle = 2\langle JY, Z \rangle\langle II(P\xi, Z), \xi \rangle + 2\langle Y, Z \rangle\langle II(P\xi, Z), F\xi \rangle \\
&= -\sin^2(\varphi)\langle JZ, Y \rangle\langle \xi, \xi \rangle + \sin^2(\varphi)\langle Y, Z \rangle\langle \xi, F\xi \rangle = \langle P\xi, P\xi \rangle\langle B, Y \rangle.
\end{aligned}$$

Clearly, equation (7) implies $\langle \bar{\nabla}_{P\xi} P\xi, \zeta \rangle = \langle II(P\xi, P\xi), \zeta \rangle = 0$. Using (7) and the fact that $JP\xi = -PF\xi - \sin^2(\varphi)\xi$ we obtain

$$\begin{aligned}
\langle \bar{\nabla}_{P\xi} P\xi, P\zeta \rangle &= \langle \bar{\nabla}_{P\xi} P\xi, J\zeta - F\zeta \rangle = -\langle \bar{\nabla}_{P\xi} JP\xi, \zeta \rangle - \langle II(P\xi, P\xi), F\zeta \rangle \\
&= \langle II(P\xi, PF\xi), \zeta \rangle + \sin^2(\varphi)\langle \bar{\nabla}_{P\xi}\xi, \zeta \rangle = \langle P\nabla_{P\xi}^\perp \xi, P\zeta \rangle.
\end{aligned}$$

Altogether this implies,

$$\bar{\nabla}_{P\xi} P\xi = \frac{1}{2}\langle P\xi, P\xi \rangle B + P\nabla_{P\xi}^\perp \xi \quad \text{for all } \xi \in \Gamma(\nu M). \tag{14}$$

Finally, equations (13) and (14) yield

$$\frac{d}{dt}\langle P\eta, P\eta\rangle = \frac{d}{dt}\langle \dot\gamma, P\eta\rangle = \langle \nabla_{\dot\gamma}\dot\gamma, P\eta\rangle + \langle \nabla_{\dot\gamma}P\eta, \dot\gamma\rangle$$

$$= \frac{a}{2}\langle P\eta, P\eta\rangle + a\langle P\nabla_B^{\perp}\eta, P\eta\rangle + \langle P\nabla_{P\eta}^{\perp}\eta, P\eta\rangle$$

$$= \frac{a}{2}\langle P\eta, P\eta\rangle + \langle P\nabla_{\dot\gamma}^{\perp}\eta, P\eta\rangle = \frac{a}{2}\langle P\eta, P\eta\rangle + \frac{1}{2}\frac{d}{dt}\langle P\eta, P\eta\rangle.$$

and hence

$$\frac{d}{dt}\langle P\eta, P\eta\rangle = a\langle P\eta, P\eta\rangle.$$

Putting $b = \langle \dot\gamma, \dot\gamma\rangle$ and $c = \langle P\eta, P\eta\rangle$ we then have the initial value problem:

$$a' = \frac{1}{2}(b - c), \quad b' = ab, \quad c' = ac, \quad a(0) = 0, \quad b(0) = c(0) = 1.$$

Again, the uniqueness of solutions to differential equations yields $a(t) = 0$, $b(t) = c(t) = 1$ for all t. Therefore, $\langle \dot\gamma(t), \dot\gamma(t)\rangle = 1$ and $\dot\gamma(t) \in P\nu M$ for all t as desired.

Let us assume then that $\gamma : I \to M$ is a curve satisfying equation (12). There exists a unit normal vector field η of M in a neighborhood of p such that $\dot\gamma(t) = P\eta_{\gamma(t)}/\sin(\varphi)$ for all sufficiently small t. Since B is nonsingular and γ is normal to B, there exists a hypersurface N in M containing γ and transversal to B in a neighborhood of p. The restriction of η to N is a smooth unit normal vector field along N. We define ξ as the unit normal vector field on a neighborhood of p satisfying $\xi = \eta$ on N and such that ξ is obtained by $\nabla^{\perp}$-parallel translation along the integral curves of B. The smooth dependance on initial conditions of ordinary differential equations implies that ξ is smooth. Also, note that $\nabla_B^{\perp}\xi = 0$ and that ξ is a local unit vector field extending $\xi_p \in \nu_p M$.

The definition of ξ and equations (11) and (13) yield $[B, P\xi] = \bar\nabla_B P\xi - \bar\nabla_{P\xi}B = \frac{1}{2}P\xi$, and hence the distribution generated by B and $P\xi$ is integrable. Let U denote the integral submanifold through p. We prove that U is an open part of a totally geodesic $\mathbb{R}H^2 \subset \mathbb{C}H^n$.

Since B is geodesic we have $\bar\nabla_B B = 0$. Equation (11) implies $2\bar\nabla_{P\xi}B = -P\xi$, and by definition of ξ we have using (13) that $\bar\nabla_B P\xi = P\bar\nabla_B^{\perp}\xi = 0$. Now we prove that $2\bar\nabla_{P\xi}P\xi = \langle P\xi, P\xi\rangle B$.

Let $\eta \in \nu M$ and denote by $\mathcal{S}_\eta$ the shape operator of M with respect to the normal vector η. Equation (7) implies that $\mathcal{S}_\eta B = 0$ for all η, and thus, for any $\eta, \zeta \in \nu M$ the Ricci equation of M reads

$$\langle R_{BP\xi}^{\perp}\eta, \zeta\rangle = \langle \bar R(B, P\xi)\eta, \zeta\rangle + \langle [\mathcal{S}_\eta, \mathcal{S}_\zeta]B, P\xi\rangle = 0,$$

where $R^{\perp}$ denotes the curvature tensor of the normal connection $\nabla^{\perp}$. The previous equation, $2[B, P\xi] = P\xi$, and the definition of ξ imply

$$0 = R_{BP\xi}^{\perp}\xi = \nabla_B^{\perp}\nabla_{P\xi}^{\perp}\xi - \nabla_{P\xi}^{\perp}\nabla_B^{\perp}\xi - \nabla_{[B,P\xi]}^{\perp}\xi = \nabla_B^{\perp}\nabla_{P\xi}^{\perp}\xi - \frac{1}{2}\nabla_{P\xi}^{\perp}\xi,$$

that is,

$$2\nabla_B^{\perp}\nabla_{P\xi}^{\perp}\xi = \nabla_{P\xi}^{\perp}\xi. \tag{15}$$

By definition of ξ, we have along γ that $2\bar{\nabla}_{P\xi}P\xi = 2\sin^2(\varphi)\bar{\nabla}_{\dot{\gamma}}\dot{\gamma} = 2\sin^2(\varphi)\nabla_{\dot{\gamma}}\dot{\gamma} = \langle P\xi, P\xi\rangle B$. On the other hand, (14) yields $2\bar{\nabla}_{P\xi}P\xi = \langle P\xi, P\xi\rangle B + 2P\nabla^{\perp}_{P\xi}\xi$, and hence $\nabla^{\perp}_{P\xi}\xi = 0$ along γ. Now, let us take α an integral curve of B through $\alpha(0) = \gamma(s)$. We have just seen that $\nabla^{\perp}_{P\xi}\xi_{|\alpha(0)} = \nabla^{\perp}_{P\xi}\xi_{|\gamma(s)} = 0$. Moreover, using (15) and the fact that $\mathcal{S}_\eta B = 0$ for any $\eta \in \nu M$, we obtain

$$2\bar{\nabla}_{\dot{\alpha}}\nabla^{\perp}_{P\xi}\xi_{|t} = 2\nabla^{\perp}_B\nabla^{\perp}_{P\xi}\xi_{|\alpha(t)} - 2\mathcal{S}_{\nabla^{\perp}_{P\xi}\xi}B_{|\alpha(t)} = \nabla^{\perp}_{P\xi}\xi_{|\alpha(t)}.$$

Therefore, by the uniqueness of solutions to differential equations we get $\nabla^{\perp}_{P\xi}\xi_{|\alpha(t)} = 0$ for all t, and as a consequence $2\bar{\nabla}_{P\xi}P\xi = \langle P\xi, P\xi\rangle B$ along the integral submanifold U. Hence, U is an open part of a totally geodesic $\mathbb{R}H^2 \subset \mathbb{C}H^n$.

We define $\bar{P}\xi = P\xi/\|P\xi\| = P\xi/\sin(\varphi)$ along U. From (14) we obtain $2\bar{\nabla}_{\bar{P}\xi}\bar{P}\xi = B$ since ξ is unit normal. Using this and (11) we get

$$\bar{\nabla}_{\bar{P}\xi}\bar{\nabla}_{\bar{P}\xi}\bar{P}\xi + \langle\bar{\nabla}_{\bar{P}\xi}\bar{P}\xi, \bar{\nabla}_{\bar{P}\xi}\bar{P}\xi\rangle\bar{P}\xi = \frac{1}{2}\bar{\nabla}_{\bar{P}\xi}B + \frac{1}{4}\langle B, B\rangle\bar{P}\xi = 0.$$

From this we see that the integral curves of $\bar{P}\xi$ are horocycles with center x at infinity contained in an open part of a totally geodesic real hyperbolic plane contained in $\mathbb{C}H^n$. Corollary 3.2, the rigidity of totally geodesic submanifolds of Riemannian manifolds (see e.g. [4], p. 230), and of horocycles in real hyperbolic planes (see e.g. [4], pp. 24-26), then imply the assertion.   □

Remark 3.6. The proof shows that the differential equation (8) characterizes left translation of the normal spaces by $S_{\mathfrak{c}}$.

4. The tubes around W^{2n-k} and W^{2n-k}_φ

To accomplish the task of investigating the geometry of orbits of the cohomogeneity one actions on $\mathbb{C}H^n$ we will deal with two different possibilities depending on the constant Kähler angle $\varphi \in (0, \pi/2)$ or $\varphi = \pi/2$ of $\mathfrak{w}$. For this we first we recall a few properties of the solvable foliation already studied in [2].

4.1. **The solvable foliation.** The solvable foliation is the foliation on $\mathbb{C}H^n$ arising from $k = 1$. In this case $\varphi = \pi/2$ and the orbit $S \cdot o = W^{2n-1}$ is a minimal homogeneous ruled real hypersurface. Its principal curvatures are $1/2$, $-1/2$ and 0 with multiplicities 1, 1 and $2n - 3$. The following theorem shows that this eigenvalue structure is characteristic of this orbit.

Theorem 4.1 (Rigidity of the submanifold W^{2n-1}). *Let M be a connected real hypersurface in $\mathbb{C}H^n$, $n \geq 2$, with three distinct principal curvatures $1/2$, $-1/2$ and 0 and multiplicities 1, 1 and $2n - 3$, respectively. Then M is holomorphically congruent to an open part of the minimal homogeneous ruled real hypersurface W^{2n-1}.*

This result was proved in [5] for $n \geq 3$. The analogous statement for $n = 2$ is more involved and follows from the classification of real hypersurfaces with constant principal curvatures in the complex hyperbolic plane [6]. Any other orbit of the action of S is an equidistant hypersurface to this minimal one. Any two such orbits are congruent to each

other if and only if their distance to $S \cdot o = W^{2n-1}$ is the same. None of them is ruled by a totally geodesic $\mathbb{C}H^{n-1}$. Let $M(r)$ denote an orbit of S at a distance $r > 0$ from $S \cdot o$. The shape operator of $M(r)$ has exactly three eigenvalues

$$\lambda_{1/2} = \frac{3}{4} \tanh\left(\frac{r}{2}\right) \pm \frac{1}{2}\sqrt{1 - \frac{3}{4}\tanh^2\left(\frac{r}{2}\right)}, \quad \lambda_3 = \frac{1}{2}\tanh\left(\frac{r}{2}\right).$$

with corresponding multiplicities $m_1 = 1$, $m_2 = 1$ and $m_3 = 2n - 3$. The Hopf vector field $J\xi$ has nontrivial projection onto the principal curvature spaces of λ_1 and λ_2.

The subspace $\mathfrak{a} + \mathfrak{w}^\perp + J\mathfrak{w}^\perp + \mathfrak{g}_{2\alpha}$ of $\mathfrak{a} + \mathfrak{n}$ is a subalgebra of $\mathfrak{a} + \mathfrak{n}$, and the orbit through o of the corresponding connected closed subgroup of AN is a totally geodesic $\mathbb{C}H^2$. The action of the connected closed subgroup of S with Lie algebra $\mathfrak{a} + J\mathfrak{w}^\perp + \mathfrak{g}_{2\alpha}$ induces the solvable foliation on this totally geodesic $\mathbb{C}H^2$. The relevant geometric information of the solvable foliation on $\mathbb{C}H^n$ is contained in the "slice" $\mathbb{C}H^2$. We describe in more detail the geometry of the leaves of the solvable foliation on $\mathbb{C}H^2$ in what follows.

Let γ be the geodesic in $\mathbb{C}H^2$ determined by $\gamma(0) = o$ and $\dot\gamma(0) = \xi$, where $\xi \in \mathfrak{w}^\perp$ is a unit vector. Let $r \in \mathbb{R}$ and denote by $M(r)$ the leaf of the solvable foliation containing $\gamma(r)$. According to (5) the tangent vector field $\dot\gamma$ of the geodesic γ can be written with respect to left-invariant vector fields as

$$\dot\gamma(t) = -\tanh(t/2)B + \mathrm{sech}(t/2)\xi.$$

Then $\{Z, J\xi, \mathrm{sech}(r/2)B + \tanh(r/2)\xi\}$ is an orthonormal basis of $T_{\gamma(r)}M(r)$. The distribution on $M(r)$ generated by Z and $J\xi$ is integrable by (1). Moreover, using (6) we get

$$\bar\nabla_Z Z = B\,, \ 2\bar\nabla_{J\xi}J\xi = B\,, \ 2\bar\nabla_Z J\xi = \xi\,, \ 2\bar\nabla_{J\xi}Z = \xi. \tag{16}$$

Thus the shape operator of the leaf of this distribution containing $\gamma(r)$ with respect to the unit normal vector $\mathrm{sech}(r/2)B_{\gamma(r)} + \tanh(r/2)\xi_{\gamma(r)} \in T_{\gamma(r)}M(r)$ is given by the matrix

$$\frac{1}{2}\begin{pmatrix} 2\,\mathrm{sech}\left(\frac{r}{2}\right) & \tanh\left(\frac{r}{2}\right) \\ \tanh\left(\frac{r}{2}\right) & \mathrm{sech}\left(\frac{r}{2}\right) \end{pmatrix}$$

with respect to the basis $\{Z_{\gamma(r)}, J\xi_{\gamma(r)}\}$. Using the Gauss equation and (16) we get that the Gaussian curvature of the leaf through $\gamma(r)$ is equal to zero. For topological reasons it is clear that the leaf is a Euclidean plane $\mathbb{R}^2$.

On the other hand, using Lemma (6) we get

$$\bar\nabla_{\mathrm{sech}\left(\frac{r}{2}\right)B+\tanh\left(\frac{r}{2}\right)\xi}\left(\mathrm{sech}\left(\frac{r}{2}\right)B + \tanh\left(\frac{r}{2}\right)\xi\right) = -\frac{1}{2}\tanh\left(\frac{r}{2}\right)\dot\gamma(r).$$

Hence, every integral curve of $\mathrm{sech}(r/2)B + \tanh(r/2)\xi$ is a geodesic in $M(r)$.

All in all, this means

Theorem 4.2. *The leaves of the solvable foliation on $\mathbb{C}H^2$ are diffeomorphic to $\mathbb{R}^3$ and are foliated orthogonally by a one-dimensional totally geodesic foliation and a two-dimensional foliation whose leaves are Euclidean planes.*

4.2. **Constant Kähler angle $\varphi = \pi/2$.** In this case $\mathfrak{w}^\perp$ has constant Kähler angle $\varphi = \pi/2$, that is, $\mathfrak{w}^\perp$ is real. This means that the normal bundle νW^{2n-k} of W^{2n-k} is totally real. We recall that the second fundamental form of W^{2n-k} is given by the trivial symmetric bilinear extension of $II(Z, J\xi) = (1/2)\xi$ for all $\xi \in \nu W^{2n-k}$. Thus the eigenvalues of the shape operator of W^{2n-k} with respect to any unit vector $\xi \in \nu W^{2n-k}$ are $1/2$, $-1/2$ and 0 with multiplicities 1, 1 and $2n - 2 - k$ respectively. The corresponding principal curvature spaces are $\mathbb{R}(Z + J\xi)$, $\mathbb{R}(Z - J\xi)$ and $TW^{2n-k} \ominus (\mathbb{R}Z + \mathbb{R}J\xi)$ respectively.

The above information allows us to calculate the shape operator of the principal orbits using Jacobi field theory. Every principal orbit of this action is a tube around W^{2n-k}. We denote by $M(r)$ the tube at distance $r > 0$ and fix $o \in W^{2n-k}$ and a unit vector $\xi \in \nu_o W^{2n-k}$. Let γ_ξ be the geodesic in $\mathbb{C}H^n$ given by the initial conditions $\gamma_\xi(0) = o$ and $\dot{\gamma}_\xi(0) = \xi$. We recall that the Jacobi equation in the complex hyperbolic space of constant holomorphic sectional curvature -1 along γ_ξ reads $4\zeta_X'' - \zeta_X - 3\langle J\dot{\gamma}_\xi, \zeta_X\rangle J\dot{\gamma}_\xi = 0$.

For any $X \in T_o\mathbb{C}H^n$ we denote by $\mathcal{B}_X$ the parallel displacement of the vector X along γ_ξ. If $X \in T_o W^{2n-k}$ we denote by ζ_X the Jacobi field along γ_ξ defined by the initial conditions $\zeta_X(0) = X$ and $\zeta_X'(0) = -\mathcal{S}_\xi(X)$. If X is a principal curvature vector, that is, $\mathcal{S}_\xi X = \lambda X$ for some $\lambda \in \mathbb{R}$, then the Jacobi equation can be solved explicitly to get

$$\zeta_X(t) = f_\lambda(t)\mathcal{B}_X(t) + \langle X, J\xi\rangle g_\lambda(t)J\dot{\gamma}_\xi(t)$$

with

$$f_\lambda(t) = \cosh\left(\frac{t}{2}\right) - 2\lambda\sinh\left(\frac{t}{2}\right), \ g_\lambda(t) = \left(\cosh\left(\frac{t}{2}\right) - 1\right)\left(1 + 2\cosh\left(\frac{t}{2}\right) - 2\lambda\sinh\left(\frac{t}{2}\right)\right).$$

If $X \in \nu_o W^{2n-k} \ominus \mathbb{R}\xi$ we define the Jacobi field ζ_X along γ_ξ by the initial conditions $\zeta_X(0) = 0$ and $\zeta_X'(0) = X$. In this case we have

$$\zeta_X(t) = p(t)\mathcal{B}_X(t) + \langle X, J\xi\rangle q(t)J\dot{\gamma}_\xi(t)$$

with

$$p(t) = 2\sinh\left(\frac{t}{2}\right), \ q(t) = 2\sinh\left(\frac{t}{2}\right)\left(\cosh\left(\frac{t}{2}\right) - 1\right).$$

Using the above formulas one easily gets

$$\zeta_X(t) = \begin{cases} \cosh\left(\dfrac{t}{2}\right)\mathcal{B}_Z(t) - \dfrac{1}{2}\sinh(t)\mathcal{B}_{J\xi}(t) & , & \text{if } X = Z, \\[2mm] -\sinh\left(\dfrac{t}{2}\right)\mathcal{B}_Z(t) + \cosh(t)\mathcal{B}_{J\xi}(t) & , & \text{if } X = J\xi, \\[2mm] \cosh\left(\dfrac{t}{2}\right)\mathcal{B}_X(t) & , & \text{if } X \in TW^{2n-k} \ominus (\mathbb{R}Z + \mathbb{R}J\xi), \\[2mm] 2\sinh\left(\dfrac{t}{2}\right)\mathcal{B}_X(t) & , & \text{if } X \in \nu W^{2n-k} \ominus \mathbb{R}\xi. \end{cases}$$

We define the endomorphism $D(r)$ of $T_{\gamma_\xi(r)}M(r) \ominus \mathbb{R}\dot{\gamma}_\xi(r)$ by $D(r)B_X(r) = \zeta_X(r)$ for all $X \in T_o\mathbb{C}H^n \ominus \mathbb{R}\xi$. Jacobi field theory shows that the shape operator of $M(r)$ at $\gamma_\xi(r)$

with respect to $-\dot{\gamma}(r)$ is given by $\mathcal{S}(r) = D'(r)D(r)^{-1}$. In our case $\mathcal{S}(r)$ is represented by the matrix

$$
\mathcal{S}(r) = \frac{1}{2}\left(
\begin{array}{cc|c|c}
\tanh^3\left(\frac{r}{2}\right) & -\operatorname{sech}^3\left(\frac{r}{2}\right) & & \\
-\operatorname{sech}^3\left(\frac{r}{2}\right) & 2\left(1+\frac{1}{2}\operatorname{sech}^2\left(\frac{r}{2}\right)\right)\tanh\left(\frac{r}{2}\right) & & \\
\hline
& & \tanh\left(\frac{r}{2}\right)\operatorname{Id}_{2n-2-k} & \\
\hline
& & & \coth\left(\frac{r}{2}\right)\operatorname{Id}_{k-1}
\end{array}
\right).
$$

with respect to the orthogonal sum decomposition

$$
T_{\gamma_\xi(r)}M(r) = \mathcal{B}_{\mathbb{R}Z+\mathbb{R}J\xi}(r) + \mathcal{B}_{T_oW^{2n-k}\ominus(\mathbb{R}Z+\mathbb{R}J\xi)}(r) + \mathcal{B}_{\nu_oW^{2n-k}\ominus\mathbb{R}\xi}(r),
$$

where $\mathcal{B}_V$ denotes the parallel translation of any vector subspace $V \subset T_o\mathbb{C}H^n$ along γ_ξ.

A straightforward calculation shows that $M(r)$ has four principal curvatures

$$
\begin{aligned}
\lambda_1 &= \frac{3}{4}\tanh\left(\frac{r}{2}\right) - \frac{1}{2}\sqrt{1-\frac{3}{4}\tanh^2\left(\frac{r}{2}\right)}, & \lambda_2 &= \frac{3}{4}\tanh\left(\frac{r}{2}\right) + \frac{1}{2}\sqrt{1-\frac{3}{4}\tanh^2\left(\frac{r}{2}\right)}, \\
\lambda_3 &= \frac{1}{2}\tanh\left(\frac{r}{2}\right), & \lambda_4 &= \frac{1}{2}\coth\left(\frac{r}{2}\right)
\end{aligned}
$$

with corresponding multiplicities $m_1 = m_2 = 1$, $m_3 = 2n - 2 - k$ and $m_4 = k - 1$. The Hopf vector field on M has nontrivial orthogonal projection onto the principal curvature spaces of λ_1 and λ_2. A special situation occurs when $r = \ln(2+\sqrt{3})$. In this case we have $\lambda_2 = \lambda_4$ and the principal curvatures are $\lambda_1 = 0$, $\lambda_2 = \lambda_4 = \sqrt{3}/2$ and $\lambda_3 = \sqrt{3}/6$ with multiplicities 1, k and $2n - k - 2$ respectively.

The previous calculations show that the interesting part of the shape operator of both the singular orbit W^{2n-k} and the principal orbit $M(r)$ concerns the vectors Z and $J\xi$. More precisely, let $\xi \in \nu_o W^{2n-k}$ be a unit vector. Consider the subalgebra $\tilde{\mathfrak{g}} = \mathfrak{a}+\mathbb{R}\xi+\mathbb{R}J\xi+\mathfrak{g}_{2\alpha}$ of $\mathfrak{a} + \mathfrak{g}_\alpha + \mathfrak{g}_{2\alpha}$, and let $\tilde{G}$ be the connected closed subgroup of AN with Lie algebra $\tilde{\mathfrak{g}}$. The orbit $\tilde{G}\cdot o$ is a totally geodesic $\mathbb{C}H^2$ in $\mathbb{C}H^n$. This $\mathbb{C}H^2$ defines a "slice" of the action of $N_K^0(S)S$ through o. Next, $\tilde{\mathfrak{h}} = \mathfrak{s}\cap\tilde{\mathfrak{g}}$ is a subalgebra of $\tilde{\mathfrak{g}}$ of codimension one. Let $\tilde{H}$ be the connected closed subgroup of $\tilde{G}$ with Lie algebra $\tilde{\mathfrak{h}}$. Then $\tilde{H}$ acts on $\mathbb{C}H^2 = \tilde{G}\cdot o$ with cohomogeneity one and gives exactly the solvable foliation of $\mathbb{C}H^2$ described in Section 4.1. The orbits of the action of $\tilde{H}$ on $\mathbb{C}H^2$ are the equidistant hypersurfaces to the orbit $\tilde{H}\cdot o$. On the other hand, the intersection of the orbits of the cohomogeneity one action of $N_K^0(S)S$ on $\mathbb{C}H^n$ with the slice $\mathbb{C}H^2$ also gives tubes around $\tilde{H}\cdot o$ because $\mathbb{C}H^2 = \tilde{G}\cdot o$ is totally geodesic in $\mathbb{C}H^n$. Thus, the geometry of the orbits of the action of G on $\mathbb{C}H^n$ in the slice $\mathbb{C}H^2$ is exactly the geometry of the orbits of the action of $\tilde{H}$ on $\mathbb{C}H^2$. This study was accomplished in the previous subsection.

4.3. **Constant Kähler angle $\varphi \in (0, \pi/2)$.** Again, we assume the notation above, and consider the singular orbit W_φ^{2n-k} of the cohomogeneity one action determined by the Lie group $N_K^0(S)S$, where S is the connected, simply connected Lie group whose Lie algebra is $\mathfrak{s} = \mathfrak{a} + \mathfrak{w} + \mathfrak{g}_{2\alpha}$, where $\mathfrak{w}^\perp$ has constant Kähler angle $\varphi \in (0, \pi/2)$. In this case we have that k is an even number. The second fundamental form of W_φ^{2n-k} is given by the

trivial bilinear extension of $II(Z, P\xi) = (\sin^2(\varphi)/2)\xi$, for each unit $\xi \in \nu W_\varphi^{2n-k}$. Thus, the eigenvalues of the shape operator with respect to ξ are $\sin(\varphi)/2$, $-\sin(\varphi)/2$ and 0, with multiplicities 1, 1 and $2n - k - 2$.

It is convenient to introduce the notation

$$\bar{P}\xi = P\xi/\|P\xi\| = P\xi/\sin(\varphi) \qquad \text{and} \qquad \bar{F}\xi = F\xi/\|F\xi\| = F\xi/\cos(\varphi)$$

for each unit vector $\xi \in \nu W_\varphi^{2n-k}$. Then, the eigenspaces of $\sin(\varphi)/2$, $-\sin(\varphi)/2$ and 0 of the shape operator of W_φ^{2n-k} with respect to ξ are $\mathbb{R}(Z + \bar{P}\xi)$, $\mathbb{R}(-Z + \bar{P}\xi)$ and $TW_\varphi^{2(n-k)} \ominus (\mathbb{R}Z + \mathbb{R}\bar{P}\xi)$, respectively.

The shape operator of the principal orbits can be calculated using Jacobi field theory as in the previous section. We delete the calculations, which are straightforward (although long) and directly give the matrix representation of the shape operator $\mathcal{S}(r)$ in direction $-\dot{\gamma}_\xi(r)$ of the orbit at a distance $r > 0$ from W_φ^{2n-k}

$$\mathcal{S}(r) = \left(\begin{array}{c|c|c} s(r) & & \\ \hline & \frac{1}{2}\tanh\left(\frac{r}{2}\right)\,\mathrm{Id}_{2n-k-2} & \\ \hline & & \frac{1}{2}\coth\left(\frac{r}{2}\right)\,\mathrm{Id}_{k-2} \end{array} \right),$$

with respect to the direct sum decomposition

$$T_{\gamma_\xi(r)}M(r) = \mathcal{B}_{\mathbb{R}Z+\mathbb{R}\bar{P}\xi+\mathbb{R}\bar{F}\xi}(r) + \mathcal{B}_{TW_\varphi^{2n-k}\ominus(\mathbb{R}Z+\mathbb{R}\bar{P}\xi)}(r) + \mathcal{B}_{\nu W_\varphi^{2n-k}\ominus(\mathbb{R}\xi+\mathbb{R}\bar{F}\xi)}(r).$$

Here, $s(r)$ is a symmetric 3×3 real matrix whose explicit entries we do not provide (they can be obtained after some elementary but long calculations). The characteristic polynomial of $s(r)$ is

$$p_{r,\varphi}(x) = -x^3 + \frac{1}{2}\left\{\mathrm{csch}\left(\frac{r}{2}\right)\mathrm{sech}\left(\frac{r}{2}\right) + 4\tanh\left(\frac{r}{2}\right)\right\} x^2 - \frac{1}{4}\left\{2\,\mathrm{sech}^2\left(\frac{r}{2}\right) + 5\tanh^2\left(\frac{r}{2}\right)\right\} x$$
$$- \frac{1}{8}\mathrm{csch}\left(\frac{r}{2}\right)\mathrm{sech}^3\left(\frac{r}{2}\right)\left\{\sin^2(\varphi) - \sinh^2\left(\frac{r}{2}\right) - 2\sinh^4\left(\frac{r}{2}\right)\right\}.$$

If we introduce the variable $6x = \coth(r/2)\, z - \mathrm{csch}(r/2)\,\mathrm{sech}(r/2) - 4\tanh(r/2)$, then the polynomial equation $p_{r,\varphi}(x) = 0$ transforms into $z^3 - 3z + \beta_{r,\varphi} = 0$, where $\beta_{r,\varphi} = 27\sin^2(\varphi)\tanh^2(r/2)\,\mathrm{sech}^4(r/2) - 2$. The discriminant of this cubic equation is $\Delta_{r,\varphi} = 27(\beta_{r,\varphi}^2 - 4)$. It is easy to prove that $\Delta_{r,\varphi} < 0$ for all $r > 0$, which means that the above cubic equation has exactly three distinct real roots for any r. They can be calculated explicitly as follows. Let $u_{r,\varphi}^i$, $i \in \{1, 2, 3\}$, denote each cubic root of the unit complex number $(\beta_{r,\varphi} + \sqrt{\beta_{r,\varphi}^2 - 4})/2$. Then, $-u_{r,\varphi}^i - 1/u_{r,\varphi}^i$ is a solution to $z^3 - 3z + \beta_{r,\varphi} = 0$ and hence, the eigenvalues of $s(r)$ are given by

$$\lambda_i(r) = -\frac{1}{6}\left(\coth\left(\frac{r}{2}\right)\left(u_{r,\varphi}^i + \frac{1}{u_{r,\varphi}^i}\right) + \mathrm{csch}\left(\frac{r}{2}\right)\mathrm{sech}\left(\frac{r}{2}\right) + 4\tanh\left(\frac{r}{2}\right)\right), \ i \in \{1, 2, 3\}.$$

On the other hand, $p_{r,\varphi}((1/2)\tanh(r/2)) \neq 0$ and $p_{r,\varphi}((1/2)\coth(r/2)) \neq 0$. Thus, neither $(1/2)\tanh(r/2)$ nor $(1/2)\coth(r/2)$ are eigenvalues of $s(r)$. This implies that $M(r)$ has five

distinct constant principal curvatures when $k > 2$ and four distinct principal curvatures when $k = 2$.

Hereafter, we follow the procedure of the previous section and focus our study on the non-trivial part of the shape operator of $M(r)$. Let $\mathfrak{v}_0 \subset \mathfrak{g}_\alpha$ be a two-dimensional vector subspace with constant Kähler angle φ. Then, $\tilde{\mathfrak{g}} = \mathfrak{a} + \mathbb{C}\mathfrak{v}_0 + \mathfrak{g}_{2\alpha}$ is a Lie subalgebra of $\mathfrak{a} + \mathfrak{n}$. Let $\tilde{G} = \operatorname{Exp}(\tilde{\mathfrak{g}})$ be the connected, simply connected Lie subgroup of AN whose Lie algebra is $\tilde{\mathfrak{g}}$. Then, $\tilde{G} \cdot o$ is a totally geodesic $\mathbb{C}H^3$ in $\mathbb{C}H^n$ containing o. The vector subspace $\tilde{\mathfrak{h}} = \mathfrak{a} + \mathfrak{v}_0 + \mathfrak{g}_{2\alpha}$ is a Lie subalgebra of $\tilde{\mathfrak{g}}$ of codimension two. Denote by $\tilde{H} = \operatorname{Exp}(\tilde{\mathfrak{h}})$ the connected, simply connected Lie subgroup of $\tilde{G}$ whose Lie algebra is $\tilde{\mathfrak{h}}$. We know that the Lie group $N_K^0(\tilde{H})\tilde{H}$ acts on $\tilde{G} \cdot o$ with cohomogeneity one and its orbit through o is exactly $\tilde{H} \cdot o$. This cohomogeneity one action is the one we have been describing throughout this subsection. We are interested in this particular case because it is the simplest of all cases containing all of the interesting geometry of the tubes.

Let $M(r)$ denote the tube around $W_\varphi^4 \subset \mathbb{C}H^3$ at distance $r > 0$. Then, $M(r)$ is the principal orbit of the action $N_K^0(\tilde{H})\tilde{H}$ at a distance r from the singular orbit $\tilde{H} \cdot o = W_\varphi^4$. The normal exponential map $\exp^\perp : \nu W_\varphi^4 \to \mathbb{C}H^3$ of W_φ^4 is a diffeomorphism and hence for each $p \in M(r)$ there exists a unique unit vector $\xi(p) \in \nu W_\varphi^4$ such that $p = \exp(r\, \xi(p))$. Clearly, the map $p \mapsto \xi(p)$ is differentiable. Let $\gamma_{\xi(p)}(t) = \exp(t\, \xi(p))$ be the unique geodesic perpendicular to W_φ^4 that joins W_φ^4 and p. For any $X \in T_{\gamma_{\xi(p)}(0)}\mathbb{C}H^3$ we denote by $\mathcal{B}_X^p(r)$ the parallel displacement of X to the point p along the geodesic $\gamma_{\xi(p)}$. The smooth dependence on initial conditions of the solution to ordinary differential equations implies that $\mathcal{B}_X(r) : p \mapsto \mathcal{B}_X^p(r)$ is a smooth vector field on $\mathbb{C}H^3$. Moreover, if X is tangent to W_φ^4, then $\mathcal{B}_X^p(r)$ is tangent to $M(r)$. We have

Theorem 4.3. *The following two statements hold.*

    (i) *Let $\mathcal{D}$ be the rank one distribution on $M(r)$ defined by $\mathcal{B}_B^p(r)$, $p \in M(r)$, and denote by $\mathcal{D}^\perp$ the orthogonal complement of $\mathcal{D}$ in $TM(r)$. Then both $\mathcal{D}$ and $\mathcal{D}^\perp$ are integrable. Moreover, $\mathcal{D}$ is autoparallel, that is, each of its leaves is totally geodesic in $M(r)$. If $p \in M(r)$ and $\mathbb{R}H^2$ is the totally geodesic real hyperbolic plane in $\mathbb{C}H^3$ which is determined by $\xi(p)$ and B_o, where $o \in W_\varphi^4$ is the footpoint of $\xi(p)$, then the leaf of $\mathcal{D}$ through p is parametrized by the parallel curve through p in $\mathbb{R}H^2$ of the geodesic in $\mathbb{R}H^2$ through o and in direction B_o.*

    (ii) *Let $\mathcal{E}$ be the rank two distribution on $M(r)$ defined by $\mathbb{R}\mathcal{B}_B^p(r) + \mathbb{R}\mathcal{B}_{PF\xi(p)}^p(r)$. Then $\mathcal{E}$ is autoparallel and each integral manifold has constant sectional curvature $-(1/4)\operatorname{sech}(r/2)$. If $p \in M(r)$ and $\mathbb{R}H^3$ is the totally geodesic real hyperbolic 3-space in $\mathbb{C}H^3$ which is determined by $\xi(p)$, $PF\xi(p)$ and B_o, where $o \in W_\varphi^4$ is the footpoint of $\xi(p)$, then the leaf of $\mathcal{E}$ through p is the parallel surface through p in $\mathbb{R}H^3$ of the totally geodesic $\mathbb{R}H^2$ in $\mathbb{R}H^3$ through o determined by $PF\xi(p)$ and B_o.*

Proof. Let $p \in M(r)$ and $o \in W_\varphi^4$ the footpoint of $\xi(p)$. The vectors B_o and $\xi(p)$ determine a totally geodesic real hyperbolic plane $\mathbb{R}H^2 \subset \mathbb{C}H^3$ through o. Let $\tilde{p} \in M(r) \cap \mathbb{R}H^2$. Since the normal exponential map of $W_\varphi^4 \subset \mathbb{C}H^3$ is a diffeomorphism and $W_\varphi^4 \cap \mathbb{R}H^2$ is the

path of the geodesic determined by B_o we have that $\xi(\tilde{p}) \in T_{\tilde{o}} \mathbb{R}H^2$, where $\tilde{o} \in W_\varphi^4 \cap \mathbb{R}H^2$ is the footpoint of $\xi(\tilde{p})$. Since $\mathbb{R}H^2$ is totally geodesic, $\mathcal{B}_B^{\tilde{p}}(r)$ is tangent to $M(r) \cap \mathbb{R}H^2$. This proves that $M(r) \cap \mathbb{R}H^2$ is an integral manifold of $\mathcal{D}$ through p. Moreover, if $X \in \Gamma(T(M(r) \cap \mathbb{R}H^2))$ and $\nabla^{M(r)}$ denotes the Levi-Civita connection of $M(r)$, it is clear that $\nabla_X^{M(r)} X \in \Gamma(TM(r))$. On the other hand, since $\mathbb{R}H^2$ is totally geodesic, $\bar{\nabla}_X X \in \Gamma(T\mathbb{R}H^2)$ and hence $\nabla_X^{M(r)} X \in \Gamma(\mathcal{D})$, which proves that $\mathcal{D}$ is autoparallel and the first part of (i) follows.

Similarly, let $\mathbb{R}H^3 \subset \mathbb{C}H^3$ be the totally geodesic real hyperbolic space determined by B_o, $\xi(p)$ and $PF\xi(p)$. Then the integral submanifold of $\mathcal{E}$ through p is $M(r) \cap \mathbb{R}H^3$, and since $\mathbb{R}H^3$ is totally geodesic and intersects $M(r)$ perpendicularly, we see that $\mathcal{E}$ is autoparallel. The curvature of the integral submanifolds of $\mathcal{E}$ is $-(1/4) \operatorname{sech}(r/2)$ as they are equidistant to a totally geodesic $\mathbb{R}H^2 \subset \mathbb{R}H^3$ obtained as the intersection of W_φ^4 and $\mathbb{R}H^3$. This proves (ii).

Now we prove that $\mathcal{D}^\perp$ is integrable. We define the vector field $\tilde{\xi}$ along $\mathbb{C}H^3 \setminus W_\varphi^4$ by

$$\tilde{\xi}_{\exp(r\eta)} = L_{\exp(r\eta)*}\eta \quad \text{for all unit vectors } \eta \in \nu W_\varphi^4 \text{ and } r > 0.$$

Let $\eta \in \nu W_\varphi^4$ be a unit vector and denote also by η the unit vector field on $\mathbb{C}H^3$ obtained by left translation to all points of $\mathbb{C}H^3$. Let γ_η be the geodesic in $\mathbb{C}H^3$ such that $\dot{\gamma}_\eta(0) = \eta$. According to (5) we have $\dot{\gamma}_\eta(r) = -\tanh(r/2)B_{\gamma_\eta(r)} + \operatorname{sech}(r/2)\eta_{\gamma_\eta(r)}$ where B and η are considered as left-invariant vector fields. Using (6) we get

$$\begin{aligned}
\bar{\nabla}_{\dot{\gamma}_\eta}\left(\operatorname{sech}\left(\frac{r}{2}\right)B + \tanh\left(\frac{r}{2}\right)\eta\right) &= -\frac{1}{2}\operatorname{sech}\left(\frac{r}{2}\right)\tanh\left(\frac{r}{2}\right)B + \operatorname{sech}\left(\frac{r}{2}\right)\bar{\nabla}_{\dot{\gamma}_\eta}B \\
&\quad + \frac{1}{2}\operatorname{sech}^2\left(\frac{r}{2}\right)\eta + \tanh\left(\frac{r}{2}\right)\bar{\nabla}_{\dot{\gamma}_\eta}\eta = 0.
\end{aligned}$$

This proves that

$$\mathcal{B}_B^p(r) = \operatorname{sech}\left(\frac{r}{2}\right)B_p + \tanh\left(\frac{r}{2}\right)\tilde{\xi}_p. \tag{17}$$

Now, let $p \in M(r)$. Let us assume without loss of generality that $\gamma_{\xi(p)}(0) = o$ and write $\eta = \xi(p) \in \nu_o W_\varphi^4$. The formulas for $\dot{\gamma}_\eta(r)$ and $\mathcal{B}_B^p(r)$ show that $\mathcal{D}_p^\perp$ is spanned by Z_p, $P\eta_p$, $PF\eta_p$ and $F\eta_p$. Let $X, Y \in \Gamma(\mathcal{D}^\perp)$. Using (17) we get

$$\langle \bar{\nabla}_{X_p}Y, \mathcal{B}_B^p(r)\rangle = -\langle Y_p, \bar{\nabla}_{X_p}\mathcal{B}_B(r)\rangle = -\operatorname{sech}\left(\frac{r}{2}\right)\langle Y_p, \bar{\nabla}_{X_p}B\rangle - \tanh\left(\frac{r}{2}\right)\langle Y_p, \bar{\nabla}_{X_p}\tilde{\xi}\rangle.$$

The first term on the right-hand side of this equation may be calculated using (6) so we turn our attention to $\bar{\nabla}_{X_p}\tilde{\xi}$. Let χ_X be a curve such that $\dot{\chi}_X(0) = X_p$. The curve χ_X can be written as $\chi_X(t) = \exp_{g_X(t)\cdot o}\left(s_X(t)(j_X(t)\bar{F}\eta + h_X(t)\eta)\right)$ for certain smooth functions $g_X : I \to \tilde{H}$ and $s_X, j_X, h_X : I \to \mathbb{R}$ satisfying $s_X(0) = r$, $j_X(0) = 0$, $h_X(0) = 1$ and $j_X^2 + h_X^2 = 1$. Taking derivatives on the last equality we get $h_X'(0) = 0$. Since $\tilde{\xi}_{\chi_X(t)} = j_X(t)\bar{F}\eta_{\chi_X(t)} + h_X(t)\eta_{\chi_X(t)}$, (6) yields

$$\bar{\nabla}_{\dot{\chi}_X(0)}\tilde{\xi} = \bar{\nabla}_{\dot{\chi}_X(0)}\left(j_X\bar{F}\eta + h_X\eta\right) = j_X'(0)\bar{F}\eta_p + \bar{\nabla}_{\dot{\chi}_X(0)}\eta.$$

Again, the second term can be calculated using (6). All in all this means (interchanging the roles of X and Y) that

$$\langle [X,Y]_p, \mathcal{B}_B^p(r) \rangle = \langle \bar\nabla_{X_p} Y - \bar\nabla_{Y_p} X, \mathcal{B}_B^p(r) \rangle = -\tanh\left(\frac{r}{2}\right)\langle j_X'(0)Y_p - j_Y'(0)X_p, \bar F \eta_p \rangle.$$

Note that the vector fields $Z, P\tilde\xi, PF\tilde\xi, F\tilde\xi$ restricted to $M(r)$ form a global frame field of $\mathcal{D}^\perp$, and at the point p we have $\tilde\xi_p = \eta_p$. Therefore it is clear that the result follows if we prove $j_X'(0) = 0$ for $X_p \in \{Z_p, P\eta_p, PF\eta_p\}$.

The curve $\alpha(t) = \mathrm{Exp}_{\mathfrak{a}+\mathfrak{n}}(tX_o)$ is tangent to X_o for $t = 0$. Then, $\chi_X(t) = \gamma_\eta(r)\alpha(t)$ is tangent to X_p at $t = 0$. We define

$$U_Z = \mathrm{sech}^2\left(\frac{r}{2}\right)Z,\ U_{P\eta} = \mathrm{sech}\left(\frac{r}{2}\right)P\eta + \mathrm{sech}\left(\frac{r}{2}\right)\tanh\left(\frac{r}{2}\right)\sin^2(\varphi)Z,\ U_{PF\eta} = \mathrm{sech}\left(\frac{r}{2}\right)PF\eta.$$

Using (2), (3) and (4) we get

$$\begin{aligned}
\chi_X(t) &= \left(\ln\mathrm{sech}^2\left(\frac{r}{2}\right), \mathrm{Exp}_{\mathfrak{n}}\left(2\tanh\left(\frac{r}{2}\right)\eta\right)\right)\cdot\left(0, \mathrm{Exp}_{\mathfrak{n}}(tX)\right)\\
&= \left(\ln\mathrm{sech}^2\left(\frac{r}{2}\right), \mathrm{Exp}_{\mathfrak{n}}\left(2\tanh\left(\frac{r}{2}\right)\eta + tU_X\right)\right).
\end{aligned}$$

On the other hand, we have $\chi(t) := \chi_X(t) = \exp_{g(t)\cdot o}\big(s(t)(j(t)\bar F\eta + h(t)\eta)\big)$. We may write $g(t) = (b(t), \mathrm{Exp}_{\mathfrak{n}}(x(t)Z + V(t)))$ for certain functions $b, x : I \to \mathbb{R}$ and $V : I \to \mathfrak{v}_0$. Taking into account that $g(t)$ is an isometry, we get, using again (2) and (4),

$$\begin{aligned}
\chi(t) &= \exp_{g(t)\cdot o}\big(s(t)(j(t)\bar F\eta + h(t)\eta)\big) = g(t)\exp_o\big(s(t)(j(t)\bar F\eta + h(t)\eta)\big)\\
&= (b(t), \mathrm{Exp}_{\mathfrak{n}}(x(t)Z + V(t)))\cdot\left(\ln\mathrm{sech}^2\frac{s(t)}{2}, \mathrm{Exp}_{\mathfrak{n}}\left(2\tanh\frac{s(t)}{2}(j(t)\bar F\eta + h(t)\eta)\right)\right)\\
&= \Big(b(t) + \ln\mathrm{sech}^2\frac{s(t)}{2}, \mathrm{Exp}_{\mathfrak{n}}\Big(V(t) + 2e^{b(t)/2}\tanh\frac{s(t)}{2}(j(t)\bar F\eta + h(t)\eta)\\
&\qquad\qquad + \Big\{x(t) + e^{b(t)/2}\tanh\frac{s(t)}{2}\langle JV(t), j(t)\bar F\eta + h(t)\eta\rangle\Big\}Z\Big)\Big).
\end{aligned}$$

As $\mathrm{Exp}_{\mathfrak{n}} : \mathfrak{n} \to N$ is a diffeomorphism, we easily get $j(t)\tanh(s(t)/2) = 0$ for all t from the previous two equations by comparing the $\bar F\eta$-component. This eventually implies $j'(0) = 0$, and finishes the proof for the second part of (i). $\qquad\square$

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DEPARTMENT OF MATHEMATICS, UNIVERSITY COLLEGE, CORK, IRELAND
EMAIL: j.berndt@ucc.ie, jc.diazramos@ucc.ie

Homogeneous hypersurfaces in the complex hyperbolic space

Type	Group acting	Principal curvatures	Multiplicities	Comments
(A)	$S(U(1,k)U(n-k))$	$\frac{1}{2}\tanh\frac{r}{2}$	$2(n-k-1)$	Tubes around totally geodesic $\mathbb{C}H^k$,
		$\frac{1}{2}\coth\frac{r}{2}$	$2k$	$0 \le k \le n-1$.
		$\coth r$	1	Two principal curvatures if $k \in \{0, n-1\}$.
(B)	$SO^0(1,n)$	$\lambda_1 = \frac{1}{2}\tanh\frac{r}{2}$	$n-1$	Tubes around totally geodesic $\mathbb{R}H^n$.
		$\lambda_2 = \frac{1}{2}\coth\frac{r}{2}$	$n-1$	If $r = \ln(2+\sqrt{3})$ then $\lambda_2 = \lambda_3$.
		$\lambda_3 = \tanh r$	1	
(H)	N (nilpotent part of	$1/2$	$2(n-1)$	Horosphere foliation.
	Iwasawa decomposition)	1	1	
(S)	S (Lie algebra of S:	$\frac{3}{4}\tanh\frac{r}{2} + \frac{1}{2}\sqrt{1-\frac{3}{4}\tanh^2\frac{r}{2}}$	1	Solvable foliation.
	$\mathfrak{s} = \mathfrak{a} + \mathfrak{w} + \mathfrak{g}_{2\alpha}$ with	$\frac{3}{4}\tanh\frac{r}{2} - \frac{1}{2}\sqrt{1-\frac{3}{4}\tanh^2\frac{r}{2}}$	1	
	$\mathfrak{w}$ hyperplane in $\mathfrak{g}_\alpha$)	$\frac{1}{2}\tanh\frac{r}{2}$	$2n-3$	
$(\text{W})_{\pi/2}$	$N_K^0(S)S$ (Lie algebra of S:	$\lambda_1 = \frac{3}{4}\tanh\frac{r}{2} + \frac{1}{2}\sqrt{1-\frac{3}{4}\tanh^2\frac{r}{2}}$	1	Tubes around $W_{\pi/2}^{2n-k}$, $2 \le k \le n-1$.
	$\mathfrak{s} = \mathfrak{a} + \mathfrak{w} + \mathfrak{g}_{2\alpha}$ with	$\lambda_2 = \frac{3}{4}\tanh\frac{r}{2} - \frac{1}{2}\sqrt{1-\frac{3}{4}\tanh^2\frac{r}{2}}$	1	If $r = \ln(2+\sqrt{3})$ then $\lambda_2 = \lambda_4$.
	$\mathfrak{w}$ such that $\mathfrak{g}_\alpha \ominus \mathfrak{w}$ is real)	$\lambda_3 = \frac{1}{2}\tanh\frac{r}{2}$	$2n-k-2$	
		$\lambda_4 = \frac{1}{2}\coth\frac{r}{2}$	$k-1$	
$(\text{W})_\varphi$	$N_K^0(S)S$ (Lie algebra of S:	$-\frac{1}{6}\left(\coth\frac{r}{2}\left(u_{r,\varphi}^1 + \frac{1}{u_{r,\varphi}^1}\right) + \operatorname{csch}\frac{r}{2}\operatorname{sech}\frac{r}{2} + 4\tanh\frac{r}{2}\right)$	1	Tubes around W_φ^{2n-k}, $2 \le k \le n-1$,
	$\mathfrak{s} = \mathfrak{a} + \mathfrak{w} + \mathfrak{g}_{2\alpha}$ with	$-\frac{1}{6}\left(\coth\frac{r}{2}\left(u_{r,\varphi}^2 + \frac{1}{u_{r,\varphi}^2}\right) + \operatorname{csch}\frac{r}{2}\operatorname{sech}\frac{r}{2} + 4\tanh\frac{r}{2}\right)$	1	k even. The number $u_{r,\varphi}^i$ is the ith cubic
	$\mathfrak{w}$ such that $\mathfrak{g}_\alpha \ominus \mathfrak{w}$ has	$-\frac{1}{6}\left(\coth\frac{r}{2}\left(u_{r,\varphi}^3 + \frac{1}{u_{r,\varphi}^3}\right) + \operatorname{csch}\frac{r}{2}\operatorname{sech}\frac{r}{2} + 4\tanh\frac{r}{2}\right)$	1	root of $(\beta_{r,\varphi} + \sqrt{\beta_{r,\varphi}^2 - 4})/2$, where
	constant Kähler angle	$\frac{1}{2}\tanh\frac{r}{2}$	$2n-k-2$	$\beta_{r,\varphi} = 27\sin^2(\varphi)\tanh^2(r/2)\operatorname{sech}^4(r/2) - 2$.
	$0 < \varphi < \pi/2$)	$\frac{1}{2}\coth\frac{r}{2}$	$k-2$	Four principal curvatures if $k = 2$.