Jacobi relations on naturally reductive spaces

Tillmann Jentsch (joint work with Gregor Weingart)

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Santiago de Compostela, 28.10.2019

T. Jentsch, G. Weingart:
Jacobi relations on naturally reductive spaces,
arXiv:1909.04764.

A distinguished class of naturally reductive homogeneous spaces

Let V be a Euclidean vector space and $\sigma : V \times V \times V \to \mathbb{R}$ be an alternating 3-tensor. For each $x \in V$ we denote by σ_x the skew-symmetric endomorphism defined by $\langle \sigma_x y, z \rangle := \sigma(x, y, z)$.

Definition

σ is a vector cross product in the sense of A. Gray if $|\sigma_x(y)|^2 = \|x \wedge y\|^2$ for all $x, y \in V$.

There are only the following two examples:

- In three dimensions there exists the well-known vector product which measures the directed area of two vectors.
- The octonionic multiplication on $\mathbb{R}^8$ yields a vector cross product on $\mathbb{R}^7$. Its stabilizer defines the exceptional Riemannian holonomy group G_2 as a subgroup of the orthogonal group $\mathrm{O}(7)$.

A. Gray:
Vector cross products on Manifolds,
T. Am. Math. Soc. **141**, 465 – 504 (1969).

In order to understand the following definition, note that σ is a vector cross product if and only if σ_x is a Hermitian structure (i.e. $\sigma_x^2 = -\mathrm{Id}$) on the orthogonal complement $x^\perp$ for every unit vector $x \in V$. In particular σ_{x_1} and σ_{x_2} are conjugate in $\mathrm{O}(V)$ for all unit vectors x_1 and x_2.

Definition (M. Barberis, A. Moroianu, U. Semmelmann)

Let V be a Euclidean vector space. An alternating 3-tensor τ is called a generalized vector cross product if $\tau \neq 0$ and τ_{x_1} is conjugate to τ_{x_2} in $\mathrm{O}(V)$ for all unit vectors x_1 and x_2.

A different characterization of a generalized vector cross product is the following: Let τ be a 3-form and $x_0 \in V$ with $\|x_0\| = 1$. Let $\lambda_1 > \lambda_2 > \cdots >$ denote the different eigenvalues of the square $\tau_{x_0}^2$ on T_pM. In general these define continuous functions $\lambda_i(x)$ in a small neighbourhood of x_0. Then τ is a generalized vector cross product if and only if the λ_i are constant on the unit sphere $\mathrm{S}^1(V)$.

Theorem (M. Barberis, A. Moroianu, U. Semmelmann)

Let V be a Euclidean vector space of dimension n.

1. *If $n = 2m + 1$ is odd, then every generalized vector cross product on V is, up to constant rescaling, a standard vector cross product. Hence $n = 3$ or $n = 7$.*

M. Barberis, A. Moroianu, U. Semmelmann
Generalized vector cross products and Killing forms on negatively curved manifolds,
Geom Dedicata **DOI:10.1007/s10711-019-00467-9** (2019).

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2. *If $n = 2m + 2$ and there exists a generalized vector cross product τ on V, then $n = 6$ and $\tau = \sigma|_{V \times V \times V}$, where σ is the vector cross product on $\mathbb{R}^7 = V \oplus \mathbb{R}$.*

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3. *In dimension $n = 4m$ there exists no generalized vector cross product.*

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Generalized vector cross products and Killing forms on negatively curved manifolds,
Geom Dedicata **DOI:10.1007/s10711-019-00467-9** (2019).

Recall that a naturally reductive (homogeneous) space is a triple $(M, g, \bar{\nabla})$ where (M, g) is a Riemannian manifold and $\bar{\nabla}$ is an Ambrose-Singer connection whose torsion tensor τ is a 3-form. Since τ is parallel with respect to the Ambrose-Singer connection, its algebraic type is pointwise the same. Hence we will say that τ is a generalized vector cross product if τ_p has this property at some $p \in M$.

Theorem (-,G. Weingart)

The torsion tensor of a naturally reductive space M is a generalized vector cross product if and only if:

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1. $\dim(M) = 3$;
2. $\dim(M) = 6$ *and M is a nearly Kähler 3-symmetric space;*
3. $\dim(M) = 7$ *and M is a normal homogeneous nearly parallel* G_2*-space.*

Theorem

A six-dimensional Riemannian space M is a nearly Kähler 3-symmetric space if and only if M is a standard normal space from the following list:

- *the round sphere* $\mathrm{S}^6 = \mathrm{G}_2/\mathrm{SU}(3)$ *realized as the purely imaginary octonions of unit length and with the nearly Kähler structure coming from the octonionic multiplication,*

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- *the complex projective space* $\mathbb{C}\mathrm{P}^3 = \mathrm{SO}(5)/\mathrm{SU}(2) \times \mathrm{U}(1)$,
- *the product space* $\mathrm{S}^3 \times \mathrm{S}^3 = \mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(2)/\mathrm{SU}(2)$ *with the 3-symmetric structure onconstructed by Ledger-Obata.*

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Normal homogeneous nearly parallel G_2*-spaces are standard normal spaces of positive sectional curvature. They are from the following list:*

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- *Berger's manifold* $V_1 = \mathrm{SO}(5)/\mathrm{SO}(3)$,
- *Wilking's manifold* $V_3 = \mathrm{SO}(3) \times \mathrm{SU}(3)/\mathrm{U}^{\bullet}(2)$.

T. Friedrich, I. Kath, A. Moroianu, U. Semmelmann:
On nearly parallel G_2-structures,
J. Geom. Phys. **23**, 259 – 286 (1997).

R. Storm:
The classification of 7- and 8-dimensional naturally reductive spaces,
Canad. J. Math. **DOI:10.4153/S0008414X19000300** (2019).

B. Wilking, W. Ziller:
Revisiting homogeneous spaces with positive curvature,
J. reine angew. Math. **738**, 313 – 328 (2018).

Jacobi relations on Riemannian manifolds

Motivation: It is known that Riemannian symmetric spaces are characterized by the condition $\nabla R = 0$. What other equations satisfies the curvature tensor of some (homogeneous) Riemannian manifold?

Fact: $\nabla^k R = 0$ for $k \geq 2$ on a (complete) Riemannian manifold implies that $\nabla R = 0$. In order to address non-symmetric Riemannian spaces we hence need a better idea.

K. Nomizu, H. Ozeki:
A theorem on curvature tensor fields
Proc. Natl. Acad. Sci. U.S.A. **48** no. 2, 206 – 207.

Let M be a Riemannian manifold with Levi Civita connection ∇ and Riemannian curvature tensor R. For every geodesic γ let $\mathcal{R}_\gamma : x \mapsto R(x, \dot\gamma, \dot\gamma)$ denote the Jacobi operator and $\mathcal{R}^i_\gamma = \frac{\nabla^i}{\mathrm{d}t^i}\mathcal{R}_\gamma$ the i-fold iterated covariant derivative.

Definition

A linear Jacobi relation of (even) order k is a linear dependence relation

$$\mathcal{R}^{k+1}_\gamma = a_1 \mathcal{R}^{k-1}_\gamma + a_2 \mathcal{R}^{k-3}_\gamma + \cdots + a_{k+1}\mathcal{R}_\gamma \qquad (1)$$

for all unit-speed geodesics with real numbers a_i that can be choosen independently of γ.

Every Riemannian symmetric space satisfies a linear Jacobi relation of order zero and vice versa. More interesting examples of linear Jacobi relations are provided by the naturally reductive spaces whose torsion tensor is a generalized vector cross product.

Theorem (-,G. Weingart)

1. *For a three-dimensional naturally reductive space with the Berger metric* $g_{\kappa,\tau}$ *we have*

$$\mathcal{R}^3_\gamma = -\tau^2 \mathcal{R}^1_\gamma. \tag{2}$$

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1. *For a three-dimensional naturally reductive space with the Berger metric* $g_{\kappa,\tau}$ *we have*

$$\mathcal{R}_\gamma^3 = -\tau^2 \mathcal{R}_\gamma^1. \tag{2}$$

2. *For a six-dimensional nearly Kähler 3-symmetric space with scalar curvature* $\mathrm{scal} = 30$ *we have*

$$\mathcal{R}_\gamma^5 = -\frac{5}{4}\mathcal{R}_\gamma^3 - \frac{1}{4}\mathcal{R}_\gamma^1. \tag{3}$$

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$$\mathcal{R}^5_\gamma = -\frac{5}{4}\mathcal{R}^3_\gamma - \frac{1}{4}\mathcal{R}^1_\gamma. \tag{3}$$

3. *For a naturally reductive nearly-parallel G_2-manifold with $\mathrm{scal} = \frac{21}{8}$ we have*

$$\mathcal{R}^3_\gamma = -\frac{1}{36}\mathcal{R}^1_\gamma. \tag{4}$$

Proof of the previous Theorem

Let $(M, g, \bar{\nabla})$ be a naturally reductive space whose torsion 3-form τ is a generalized vector cross product. For every unit-speed geodesic $\gamma\colon \mathbb{R} \to M$ let $\mathrm{Sym}^2(\dot{\gamma}^\perp)$ denote the vector bundle of symmetric 2-tensors on the orthogonal complement of $\dot{\gamma}$. Furthermore we consider the skew-symmetric endomorphism field defined by $\tau_\gamma := \tau(\dot{\gamma}, \cdot\,, \cdot)$ and set

$$\begin{aligned} &\mathcal{T}_\gamma\colon \mathrm{Sym}^2(\dot{\gamma}^\perp) \to \mathrm{Sym}^2(\dot{\gamma}^\perp), \beta \mapsto \mathcal{T}_\gamma\beta \\ &\mathcal{T}_\gamma\beta(u, v) := \tfrac{1}{2}\big(\beta(\tau_\gamma\, u, v) + \beta(u, \tau_\gamma\, v)\big). \end{aligned} \tag{5}$$

for all $(u, v) \in \dot{\gamma}^\perp \times_{\mathbb{R}} \dot{\gamma}^\perp$. Thus $-2\mathcal{T}_\gamma$ is the standard action of τ_γ on symmetric 2-tensors.

Let $p_{\min}(\lambda)$ denote the minimal polynomial of $\mathcal{T}_\gamma$. Since τ_γ is parallel along γ with respect to the Ambrose-Singer connection, the coefficients of $p_{\min}(\lambda)$ are constant along γ. Since τ is a generalized vector cross product, these coefficients do also not depend on γ. Moreover the theorem of Cayley-Hamilton implies that $p_{\min}(\mathcal{T}_\gamma) = 0$, in particular $p_{\min}(\mathcal{T}_\gamma)\mathcal{R}_\gamma = 0$. We claim that this yields already a linear Jacobi relation: In fact, the Riemannian curvature tensor R is parallel with respect to the Ambrose-Singer connection $\bar{\nabla} = \nabla + \frac{1}{2}\tau$. Thus we have

$$\mathcal{T}_\gamma \mathcal{R}_\gamma^k = \mathcal{R}_\gamma^{k+1}. \tag{6}$$

Therefore the polynomial algebraic identity $p_{\min}(\mathcal{T}_\gamma)\mathcal{R}_\gamma = 0$ yields a Jacobi relation where the numbers a_i are the same as the coefficients of $p_{\min}$. This will also become clear from the examples.

- It suffices to study two cases anyway. Suppose that $\tau = c\,\sigma$ where c is a constant and $\sigma \in \Omega^3 M$ is pointwise a classical vector cross product. Then the eigenvalues of $\frac{\tau_\gamma}{2}$ on $\dot{\gamma}^\perp$ are $\{\pm \frac{c}{2} i\}$ for every unit-geodesic γ. Thus the eigenvalues of $\mathcal{T}_\gamma$ on $\mathrm{Sym}^2(\dot{\gamma}^\perp)$ are $\{0, \pm ci\}$ and the minimal polynomial is given by

$$p(\lambda) := \lambda(\lambda^2 + c^2). \tag{7}$$

For $c^2 = \tau^2$ or $c^2 = \frac{1}{36}$ this yields (2) and (4).

$\square$

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  For $c^2 = \tau^2$ or $c^2 = \frac{1}{36}$ this yields (2) and (4).
- In a similar way, if $\dim(M) = 6$, then the eigenvalues of $\frac{\tau_\gamma}{2}$ on $\dot\gamma^\perp$ are $\{0, \pm\frac{1}{2} i\}$. Thus the eigenvalues of $\mathcal{T}_\gamma$ on $\mathrm{Sym}^2(\dot\gamma^\perp)$ are $\{0, \pm\frac{1}{2} i, \pm i\}$ and the minimal polynomial is

$$p(\lambda) := \lambda(\lambda^2 + \frac{1}{4})(\lambda^2 + 1) = \lambda^5 + \frac{5}{4}\lambda^3 + \frac{1}{4}\lambda. \qquad (8)$$

  This establishes (3).

$\square$

T. Arias-Marco, A. Arvanitoyeorgos, A. Naveira:
Constancy of Jacobi osculating rank of g.o. spaces of compact and non-compact type,
A. M. RACSAM **105**, 207 – 221 (2011).

T. Arias-Marco, A. Naveira:
Constant Jacobi osculating rank of a g.o. space.
A method to obtain explicitly the Jacobi operator.
Publ. Math. Debrecen bf 74 no. 1-2, 135 – 157 (2009).

T. Arias-Marco:
Constant Jacobi osculating rank of
$\mathrm{U}(3)/(\mathrm{U}(1) \times \mathrm{U}(1) \times \mathrm{U}(1))$,
Arch. Math. (Brno) **45** no. 4, 241 – 254 (2009).

J. C. Gonzàlez-Dàvila:
Jacobi osculating rank and isotropic geodesics on naturally reductive 3-manifolds,
Diff. Geom. Appl. **27**, 482 – 495 (2009).

J. C. Gonzàlez-Dàvila, A. M. Naveira:
Jacobi fields and osculating rank of the Jacobi operator in some special classes of homogeneous Riemannian spaces,
Note Mat. **28** Suppl. 1, 191 – 208 (2008).

E. Macías-Virgós, A. M. Naveira, A. Tarrío:
The constant osculating rank of the Wilking manifold V_3,
C. R. Math. Acad. Sci. Paris **346** no. 1-2, 67-70 (2008).

A. M. Naveira, A. Tarrío:
A method for the resolution of the Jacobi equation $\mathbf{Y}'' + \mathbf{R}\mathbf{Y} = \mathbf{0}$ on the manifold $\mathrm{Sp}(\mathbf{2})/\mathrm{SU}(\mathbf{2})$,
Monatsh. Math. **154** no. 3, 231 – 246 (2008).