Complex Riemannian foliations of Kähler manifolds

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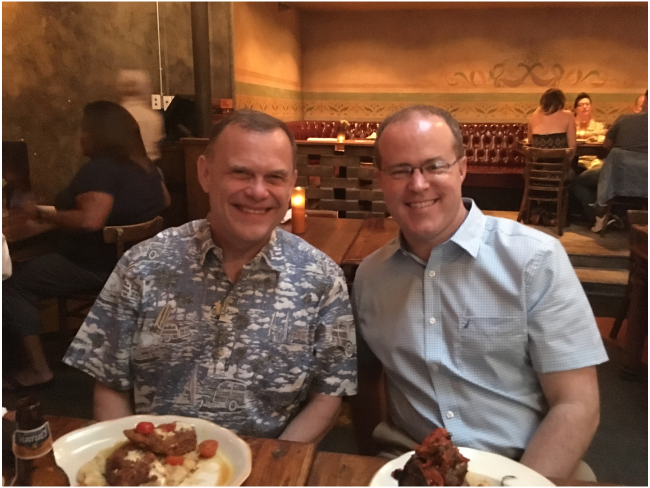
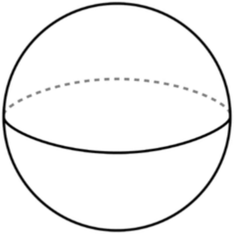
Happy birthday!



Riemannian foliations

- Given Data: (M^n, g) Riemannian manifold of dimension n, connected.
  $\mathcal{F}$ is a *Riemannian* foliation: leaves are equidistant.
- Occurs: isometric group actions, Riemannian submersions, construction of distinguished metrics.
- **Global question** For a given (M, g), classify the Riemannian foliations whose leaves satisfy a natural geometric condition.
- Examples:
  1. For a space form, classify isoparametric foliations: regular leaves are CMC hypersurfaces.
  2. Taut foliations: Riemannian foliations of M^3 by minimal surfaces (Sullivan, Thurston, Gabai).
  3. For a symmetric space, classify the isometric group actions of a given cohomogeneity (Kollross, Berndt and (many) coauthors).

- **Local question**: Classify submanifolds of M whose principal curvatures satisfy a natural geometric condition.
- Examples:
  1. Hypersurfaces of space forms with constant principal curvatures (Cartan, FKM, Cecil, Chi...),
  2. Minimal surfaces in $\mathbb{S}^3$.
  3. Totally geodesic submanifolds of symmetric spaces (Cartan, Wolf, Chen-Nagano, Klein...),
-  Local $\Longleftrightarrow$ Global

Temporary Digression

- **Conundrum** (Spivak/Berger): "Everybody knows" that generic Riemannian manifolds do not admit any non-trivial totally geodesic submanifolds:
- yet nobody knows a single example of such a metric.

- Theorem
  (M.-Wilhelm MMJ 2019.) Suppose $dim_{\mathbb{R}}(M) \geq 4$*. For any finite* $q \geq 2$*, the set of Riemannian metrics on M with no nontrivial immersed totally geodesic submanifolds contains a set that is open and dense in the* C^q*- topology.*

- Theorem
  Suppose $dim_{\mathbb{R}}(M) \geq 8$*. The set of Kähler metrics on M with no-nontrivial immersed* **complex** *totally geodesic submanifolds contains a set that is open and dense in the Kähler cone.*

Complex Riemannian foliations of Kähler manifolds

- Take now a Kähler metric (g, J), and study when the leaves of $\mathcal{F}$ are complex.
- Occurs naturally:
  1. Twistor space of quaternionic Kähler metrics with positive scalar curvature.
  2. nearly Kähler metrics: (M, g^{nk}, J^{nk}) such that $(\nabla^{nk}_X J^{nk})X = 0$ for all $X \in \Gamma(TM)$.
  3. Given any complex, totally geodesic $\mathcal{F}$ Eells–Sampson construction $\implies$

$$g^{nk}\Big|_{\mathcal{V}} = \frac{1}{2} g\Big|_{\mathcal{V}}, \quad g^{nk}\Big|_{\mathcal{H}} = g\Big|_{\mathcal{H}}, \quad g^{nk}(\mathcal{H}, \mathcal{V}) = 0$$

$$J^{nk} = -J_{|\mathcal{V}} + J_{|\mathcal{H}}.$$

- Theorem

  (Nagy 2002 JGA.) If M is closed, $\mathcal{F}$ is either totally geodesic, or polar.

  - Fixing (M, g) to be a Hermitian symmetric space, this tell us (in the compact case) the problem is similar to classifying (complex) totally geodesic submanifolds.
  - Idea of proof: every holomorphic one-form is closed on a compact Kähler manifold. Adapt this to bundle-valued holomorphic forms on M, namely spaces of $\mathcal{V}$ and $\mathcal{H}$-valued holomorphic one-forms.

General Structure Theorem

- Let $\mathcal{V}$ be the distribution associated to $\mathcal{F}$

- Theorem
  (M.–Nagy TAMS 2019.) Either $\mathcal{F}$ is totally geodesic, or there is a subdistribution $\mathcal{V}_0 \subset \mathcal{V}$ which is polar.
  - Proof: Consider the Bott connection $\overline{\nabla}$ of $\mathcal{F}$. Set $\mathcal{V}_1 = A_{\mathcal{H}}\mathcal{H}$ and split $\mathcal{V} = \mathcal{V}_1 \oplus \mathcal{V}_0$, and prove (i) $\mathcal{V}_0$ is integrable and (ii) $\mathcal{V}_1 \oplus \mathcal{H}$ is integrable and totally geodesic.

Sharpness of Theorem

Let $f : N \to \mathbb{U}$ be holomorphic, N admit a complex, totally geodesic Riemannian foliation $TN = \mathcal{V}_1 \oplus \mathcal{H}$, $\mathcal{V}_0$ denote the fibres of the projection $\mathbb{U} \times N \to N$. Fix g_N, J_N on N, and J_0 on $T\mathbb{U}$.

$$\Phi = \begin{pmatrix} \text{Re } f & \text{Im } f \\ \text{Im } f & -\text{Re } f \end{pmatrix}.$$

Construct the metric

$$g = g_0\Big((1+\Phi)^{-1}(1-\Phi)\cdot, \cdot\Big) + g_N$$
$$J = (1-\Phi)^{-1} J_0 (1+\Phi) + J_N$$

Splitting, $\mathcal{V} \oplus \mathcal{H}$ of $T(\mathbb{U} \times N)$, with $\mathcal{V}$ equal to $\mathcal{V}_0 \oplus \mathcal{V}_1$, then $\mathbb{U} \times N$ admits a complex Riemannian foliation $(\mathcal{V}_0 + \mathcal{V}_1) \oplus \mathcal{H}$ which is neither totally geodesic nor polar.

Examples in Hermitian symmetric space

- Twistor space of $\mathbb{H}P^n$:

$$\mathbb{R}^{4n+4} = \mathbb{C}^{2n+2} = \mathbb{H}^{n+1}$$

$$\begin{array}{ccccc}
S^1 & \longrightarrow & \mathbb{S}^{4n+3} & \longrightarrow & \mathbb{C}P^{2n+1} \\
\downarrow & \nearrow & & \searrow & \downarrow \pi \\
SU(2) & & & & \mathbb{H}P^n
\end{array}$$

- Twistor space of $\mathbb{S}^{2n}$:

$$H_n = SO_{2n+1}/U_n = SO_{2n+2}/U_{n+1}$$

and is given by the fibration

$$H_{n-1} \to H_n \to SO_{2n+1}/SO_{2n} = \mathbb{S}^{2n}$$

Classification

Theorem

(M.–Nagy, TAMS 2019). Let M be an open subset of an irreducible Hermitian symmetric space N and $\mathcal{F}$ a complex Riemannian foliation on M.

(i) *If N has non-negative sectional curvature, then $\mathcal{F}$ is either the twistor fibration of $\mathbb{H}P^n$ restricted to $M \subset \mathbb{C}P^{2n+1}$, or the twistor fibration of $\mathbb{S}^{2n}$ restricted to $M \subset SO_{2n+1}/U_n$.*

(ii) *If N has non-positive sectional curvature, then $\mathcal{F}$ is polar.*

Proof (compact case)

- Structure theorem $\implies$ $\mathcal{F}$ is totally geodesic. Consider the Bott connection $\overline{\nabla}$.
- The *canonical* Hermitian connection for g^{nk}:

$$\overline{\nabla} g^{nk} = \overline{\nabla} J^{nk} = T_{\overline{\nabla}}^{(1,1)} = 0.$$

- $$\overline{\nabla} = \nabla^{nk} + \frac{1}{2}\left(\nabla^{nk} J^{nk}\right) J^{nk}.$$
- $\nabla R = 0 \implies \overline{\nabla}\, \overline{R} = 0$ and $\overline{\nabla}\, \overline{T} = 0$: i.e. $\overline{\nabla}$ is an Ambrose–Singer connection.

- Study the associated infinitesimal model generated by

$$\mathfrak{h}_{nk} = \mathfrak{hol}(\overline{\nabla}).$$

- $\implies$ Two descriptions of M as a locally homogeneous space $\mathbb{V} = T_pM = \mathfrak{p} = \mathfrak{p}_{nk}$, where

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$$

  and

$$\mathfrak{g}_{nk} = \mathfrak{h}_{nk} \oplus \mathfrak{p}_{nk}$$

- $\mathfrak{h}_{nk} \subset \mathfrak{h}$ which implies $\mathfrak{g}_{nk} \subset \mathfrak{g}$.
- $$\mathfrak{g} = \mathfrak{h} + \sigma(\mathfrak{g}_{nk})$$

  where $\sigma : \mathfrak{g}_{nk} \to \mathfrak{g}$, $\sigma(\mathfrak{g}_{nk}) \neq \mathfrak{g}$, $\mathfrak{h}_{nk} = \mathfrak{h} \cap \sigma(\mathfrak{g}_{nk})$.

$\mathfrak{g}$	$\mathfrak{h}$	$\mathfrak{g}_{nk}$	$\mathfrak{h}_{nk}$
$\mathfrak{su}(2n)$	$s(\mathfrak{u}(1) \oplus \mathfrak{u}(2n-1))$	$\mathfrak{sp}(n)$	$\mathfrak{sp}(n-1) \oplus \mathfrak{u}(1)$
$\mathfrak{so}(2n+2)$	$\mathfrak{u}(n+1)$	$\mathfrak{so}(2n+1)$	$\mathfrak{u}(n)$
$\mathfrak{so}(7)$	$\mathfrak{so}(2) \oplus \mathfrak{so}(5)$	$\mathfrak{g}_2$	$\mathfrak{u}(2)$

Future Directions

- Construct complex polar foliations on a wide variety of homogeneous manifolds
- $\rho : L \to \mathbb{V}$ an irreducible representation: can produce foliations on $L \rtimes_\rho \mathbb{V}$