Realizations of contact metric (κ, μ)-spaces as homogeneous real hypersurfaces in noncompact two-plane Grassmanians

Takahiro Hashinaga
National Institute Technolofgy of Kitakyushu collage,
Santiago de Compositera university
(as overseas research fellow ∼28/03/2020)

Symmetry and shape
Celebrating the 60th birthday of Prof. J. Berndt

Aim of this talk

(κ, μ)-space was defined by Blair, Koufogiorgos and Papantoniou.

Definition

A contact metric manifold $(M, \eta, \xi, \varphi, g)$ with $\Phi = d\eta$ is called a (κ, μ)-space if the Riemannian curavture tensor R satisfies

$$R(X, Y)\xi = (\kappa I + \mu h)(\eta(Y)X - \eta(X)Y), \quad (\forall X, Y \in \mathfrak{X}(M))$$

where I denotes the identity transformation and $h := (1/2)\mathcal{L}_\xi \varphi$ is the Lie derivative of φ along ξ.

Main results:

Theorem (Cho-Kubo-Taketomi-Tamaru-H., Kubo-Taketomi-Tamaru-H., Cho, Inoguchi-Hashinaga)

$\forall \mu \in \mathbb{R}$, $(0, \mu)$-spaces can be realized as homogeneous real hypersurfaces in noncompact real two-plane Grassmanian $G_2^*(\mathbb{R}^{n+2})$ with some min. sec. curv. $-c(\mu)$.

Aim of this talk

- $G_2^*(\mathbb{R}^{n+2}) \cong Q^{n*}$(: noncompact dual of complex quadric)
- homogeneous contact real hypersurfaces with constant mean curvature in Q^{n*} have been classified by Berndt-Suh:

Theorem (Berndt-Suh, 2015)

M: a connected orientable real hypersurface with constant mean curvature in Q^{n*} $(n \geq 3)$.
M is a contact real hypersurface
$\Longleftrightarrow$ M is congruent to an open part of one of the following:

- a tube around the totally geodesic Q^{n-1*}
- a "certain horosphere"
- a tube around the totally real $\mathbb{R}H^n$

Aim of this talk:
Roughly speaking, above homogeneous contact real hypersurfaces in Q^{n*} satisfy (κ, μ) condition:

$$R(X,Y)\xi = (\kappa I + \mu h)(\eta(Y)X - \eta(X)Y), \quad (\forall X, Y \in \mathfrak{X}(M))$$

Aim of this talk

- $G_2^*(\mathbb{R}^{n+2}) \cong Q^{n*}$(: noncompact dual of complex quadric)
- homogeneous contact real hypersurfaces with constant mean curvature in Q^{n*} have been classified by Berndt-Suh:

Theorem (Berndt-Suh, 2015)

M: a connected orientable real hypersurface with constant mean curvature in Q^{n*} $(n \geq 3)$.
M is a contact real hypersurface
$\Longleftrightarrow$ M is congruent to an open part of one of the following:

- a tube around the totally geodesic Q^{n-1*}
- a "certain horosphere"
- a tube around the totally real $\mathbb{R}H^n$

Aim of this talk:
Roughly speaking, above homogeneous contact real hypersurfaces in Q^{n*} satisfy (κ, μ) condition:

$$R(X,Y)\xi = (\kappa I + \mu h)(\eta(Y)X - \eta(X)Y), \quad (\forall X, Y \in \mathfrak{X}(M))$$

- M: a $(2n-1)$ dimensional manifold,
- η: a 1-form on M,
- ξ: a vector field on M,
- φ:a $(1,1)$-tensor field on M
- g: a Riemannian metric on M

Definition

$(M, \eta, \xi, \varphi, g)$ is said to be an almost contact metric manifold If the following conditions hold:

$$\eta(\xi) = 1, \quad \varphi^2 X = -X + \eta(X)\xi \quad (\forall X \in \mathfrak{X}(M)).$$

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y) \quad (\forall X, Y \in \mathfrak{X}(M)).$$

- $\Phi(X, Y) := g(X, \varphi Y) \quad (X, Y \in \mathfrak{X}(M))$: 2-form on M.

Definition

$(M, \eta, \xi, \varphi, g)$ is called a contact metric manifold if $\Phi = d\eta$ holds.

- M: a $(2n-1)$ dimensional manifold,
- η: a 1-form on M,
- ξ: a vector field on M,
- φ:a $(1,1)$-tensor field on M
- g: a Riemannian metric on M

Definition

(M,η,ξ,φ,g) is said to be an almost contact metric manifold If the following conditions hold:

$$\eta(\xi)=1, \quad \varphi^2 X = -X + \eta(X)\xi \quad (\forall X \in \mathfrak{X}(M)).$$

$$g(\varphi X,\varphi Y) = g(X,Y) - \eta(X)\eta(Y) \quad (\forall X, Y \in \mathfrak{X}(M)).$$

- $\Phi(X,Y) := g(X,\varphi Y) \quad (X,Y \in \mathfrak{X}(M))$: 2-form on M.

Definition

(M,η,ξ,φ,g) is called a contact metric manifold if $\Phi = d\eta$ holds.

Definition

A contact metric manifold $(M, \eta, \xi, \varphi, g)$ with $\Phi = d\eta$ is called a **(κ, μ)-space** if the Riemannian curavture tensor R satisfies

$$R(X,Y)\xi = (\kappa I + \mu h)(\eta(Y)X - \eta(X)Y), \quad (\forall X, Y \in \mathfrak{X}(M))$$

where I denotes the identity transformation and $h := (1/2)\mathcal{L}_\xi \varphi$ is the Lie derivative of φ along ξ.

Remark

- (κ, μ)-spaces satisfy the inequality $\kappa \leq 1$.
- If $\kappa = 1$, then $h = 0$, (κ, μ)-spaces are Sasakian.
- If $\kappa \neq 1$, then (κ, μ)-spaces are non-Sasakian.
  Ex.: the unit tangent sphere bundles of a Riemannian manifold with constant sectional curvature $c \neq 1$ are non-Sasakian $(c(2-c), -2c)$-spaces.

Definition

A contact metric manifold $(M, \eta, \xi, \varphi, g)$ with $\Phi = d\eta$ is called a (κ, μ)-space if the Riemannian curavture tensor R satisfies

$$R(X,Y)\xi = (\kappa I + \mu h)(\eta(Y)X - \eta(X)Y), \quad (\forall X, Y \in \mathfrak{X}(M))$$

where I denotes the identity transformation and $h := (1/2)\mathcal{L}_\xi \varphi$ is the Lie derivative of φ along ξ.

Remark

- (κ, μ)-spaces satisfy the inequality $\kappa \leq 1$.
- If $\kappa = 1$, then $h = 0$, (κ, μ)-spaces are Sasakian.
- If $\kappa \neq 1$, then (κ, μ)-spaces are non-Sasakian.
  Ex.: the unit tangent sphere bundles of a Riemannian manifold with constant sectional curvature $c \neq 1$ are non-Sasakian $(c(2-c), -2c)$-spaces.

(κ, μ)-spaces have fruitful geometric properties:

- the class of (κ, μ)-spaces is invariant under D-homothetic transformations.
- (κ, μ)-spaces have strongly pseudo-convex CR-structure.

For non-Sasakian (κ, μ)-spaces, Boeckx proved that

- every non-Sasakian (κ, μ)-space is a locally homogeneous.
- Local geometry of non-Sasakian (κ, μ)-space is completely determined by the dimension and the numbers (κ, μ).
- For non-Sasakian (κ, μ)-space M (note that M satisfies $\Phi = 1 \cdot d\eta$), he defined invarinat

$$I_M = \frac{1 - (\mu/2)}{\sqrt{1 - \kappa}} \quad (1)$$

- Two non-Sasakian (κ, μ)-spaces M_1 and M_2 are locally isometric as contact metric manifolds up to a D-homothetic transformation if and only if their Boeckx invariant agree $I_{M_1} = I_{M_2}$.
- Up to D-homothetic transformation, the local model of non-Sasakian (κ, μ)-spaces have been decided.

(κ, μ)-spaces have fruitful geometric properties:

- the class of (κ, μ)-spaces is invariant under D-homothetic transformations.
- (κ, μ)-spaces have strongly pseudo-convex CR-structure.

For non-Sasakian (κ, μ)-spaces, Boeckx proved that

- every non-Sasakian (κ, μ)-space is a locally homogeneous.
- Local geometry of non-Sasakian (κ, μ)-space is completely determined by the dimension and the numbers (κ, μ).
- For non-Sasakian (κ, μ)-space M (note that M satisfies $\Phi = 1 \cdot d\eta$), he defined invarinat

$$I_M = \frac{1 - (\mu/2)}{\sqrt{1 - \kappa}} \tag{1}$$

- Two non-Sasakian (κ, μ)-spaces M_1 and M_2 are locally isometric as contact metric manifolds up to a D-homothetic transformation if and only if their Boeckx invariant agree $I_{M_1} = I_{M_2}$.
- Up to D-homothetic transformation, the local model of non-Sasakian (κ, μ)-spaces have been decided.

Local models of non-Sasakian (κ, μ)-spaces

Local model of (κ, μ)-spaces	Boeckx Invariant
unit tangent speher bundle of Riemannian manifolds with $c \neq 1$	$-1 < I = \frac{1+c}{\lvert 1-c \rvert}$
$G_{\alpha,\beta}$, tangent hyperquadric bundle of Lorentzian manifolds with $c \neq -1$	$I = \frac{c-1}{\lvert c+1 \rvert} \leq -1$

- Boeckx Invariants of the unit tangent sphere bundle $\mathrm{T}_1 M$ of a Riemannian manifold (M, g) with constant sectioanl curvature $c \neq 1$ are given by $I = \frac{1+c}{\lvert 1-c \rvert} > -1$.
- For the case of $I \leq -1$, Boeckx constructed examples of non-Sasakian (κ, μ)-spaces for any odd dimension and any value $I \leq -1$. as non-unimodular Lie groups $G_{\alpha,\beta}$ $(\beta > \alpha \geq 0)$ with certain left-invariant contact metric structure
- Loiudice and Lotta constructs non-Sasakian (κ, μ)-spaces with $I \leq -1$ as tangent hyperquadric bundle $\mathrm{T}_{-1} M$ of a Lorentzian manifold (M, g) with constant sectioanl curvature $c \neq -1$

Local models of non-Sasakian (κ, μ)-spaces

Local model of (κ, μ)-spaces	Boeckx Invariant
unit tangent speher bundle of Riemannian manifolds with $c \neq 1$	$-1 < I = \frac{1+c}{\lvert 1-c \rvert}$
$G_{\alpha,\beta}$, tangent hyperquadric bundle of Lorentzian manifolds with $c \neq -1$	$I = \frac{c-1}{\lvert c+1 \rvert} \leq -1$

- Boeckx Invariants of the unit tangent sphere bundle $\mathrm{T}_1 M$ of a Riemannian manifold (M, g) with constant sectioanl curvature $c \neq 1$ are given by $I = \frac{1+c}{\lvert 1-c \rvert} > -1$.
- For the case of $I \leq -1$, Boeckx constructed examples of non-Sasakian (κ, μ)-spaces for any odd dimension and any value $I \leq -1$. as non-unimodular Lie groups $G_{\alpha,\beta}$ $(\beta > \alpha \geq 0)$ with certain left-invariant contact metric structure
- Loiudice and Lotta constructs non-Sasakian (κ, μ)-spaces with $I \leq -1$ as tangent hyperquadric bundle $\mathrm{T}_{-1} M$ of a Lorentzian manifold (M, g) with constant sectioanl curvature $c \neq -1$

Local models of non-Sasakian (κ, μ)-spaces

Local model of (κ, μ)-spaces	Boeckx Invariant
unit tangent speher bundle of Riemannian manifolds with $c \neq 1$	$-1 < I = \frac{1+c}{\lvert 1-c \rvert}$
$G_{\alpha,\beta}$, tangent hyperquadric bundle of Lorentzian manifolds with $c \neq -1$	$I = \frac{c-1}{\lvert c+1 \rvert} \leq -1$

- Boeckx Invariants of the unit tangent sphere bundle $\mathrm{T}_1 M$ of a Riemannian manifold (M, g) with constant sectioanl curvature $c \neq 1$ are given by $I = \frac{1+c}{\lvert 1-c \rvert} > -1$.
- For the case of $I \leq -1$, Boeckx constructed examples of non-Sasakian (κ, μ)-spaces for any odd dimension and any value $I \leq -1$. as non-unimodular Lie groups $G_{\alpha,\beta}$ $(\beta > \alpha \geq 0)$ with certain left-invariant contact metric structure
- Loiudice and Lotta constructs non-Sasakian (κ, μ)-spaces with $I \leq -1$ as tangent hyperquadric bundle $\mathrm{T}_{-1} M$ of a Lorentzian manifold (M, g) with constant sectioanl curvature $c \neq -1$

Boeckx defined a real $(2n+1)$-dimensional Lie algebra $\mathfrak{g}_{\alpha,\beta}$ with basis $\{\xi, X_1, X_2, \ldots, X_n, Y_1, Y_2, \ldots, Y_n\}$ whose bracket products are given by

$$
\begin{aligned}
&[\xi, X_1] = -(1/2)\alpha\beta X_2 - (1/2)\alpha^2 Y_1, \quad [\xi, X_2] = (1/2)\alpha\beta X_1 - (1/2)\alpha^2 Y_2,\\
&[\xi, X_i] = -(1/2)\alpha^2 Y_i (i \neq 1,2), \quad [\xi, Y_1] = (1/2)\beta^2 X_1 - (1/2)\alpha\beta Y_2,\\
&[\xi, Y_2] = (1/2)\beta^2 X_2 + (1/2)\alpha\beta Y_1, \quad [\xi, Y_i] = (1/2)\beta^2 X_i (i \neq 1,2),\\
&[X_1, X_i] = \alpha X_i (i \neq 1), \quad [X_i, X_j] = 0 (i,j \neq 1),\\
&[Y_2, Y_i] = \beta Y_i (i \neq 2), \quad [Y_i, Y_j] = 0 (i,j \neq 2),\\
&[X_1, Y_1] = -\beta X_2 + 2\xi, \quad [X_1, Y_i] = 0 (i \neq 1),\\
&[X_2, Y_1] = \beta X_1 - \alpha Y_2, \quad [X_2, Y_2] = \alpha Y_1 + 2\xi,\\
&[X_2, Y_i] = \beta X_i (i \neq 1,2), \quad [X_i, Y_1] = -\alpha Y_i (i \neq 1,2),\\
&[X_i, Y_2] = 0 (i \neq 1,2), \quad [X_i, Y_j] = \delta_{ij}(-\beta X_2 + \alpha Y_1 + 2\xi)(i,j \neq 1,2)
\end{aligned}
$$

$G_{\alpha,\beta}$: the simply-connected Lie group with Lie algebra $\mathfrak{g}_{\alpha,\beta}$.

- g: the Riemannain metric so that the above basis is orthonormal,
- the characteristic vector field is given by ξ,
- the 1-form η is the metric dual of ξ, that is, $\eta(X) = g(X, \xi)$,
- the $(1,1)$-tensor field φ is defined by
  $\varphi(\xi) = 0, \quad \varphi(X_i) = Y_i, \quad \varphi(Y_i) = -X_i.$

Boeckx defined a real $(2n+1)$-dimensional Lie algebra $\mathfrak{g}_{\alpha,\beta}$ with basis $\{\xi, X_1, X_2, \ldots, X_n, Y_1, Y_2, \ldots, Y_n\}$ whose bracket products are given by

$$
\begin{aligned}
&[\xi, X_1] = -(1/2)\alpha\beta X_2 - (1/2)\alpha^2 Y_1, \qquad [\xi, X_2] = (1/2)\alpha\beta X_1 - (1/2)\alpha^2 Y_2,\\
&[\xi, X_i] = -(1/2)\alpha^2 Y_i (i \neq 1,2), \qquad [\xi, Y_1] = (1/2)\beta^2 X_1 - (1/2)\alpha\beta Y_2,\\
&[\xi, Y_2] = (1/2)\beta^2 X_2 + (1/2)\alpha\beta Y_1, \qquad [\xi, Y_i] = (1/2)\beta^2 X_i (i \neq 1,2),\\
&[X_1, X_i] = \alpha X_i (i \neq 1), \qquad [X_i, X_j] = 0 (i,j \neq 1),\\
&[Y_2, Y_i] = \beta Y_i (i \neq 2), \qquad [Y_i, Y_j] = 0 (i,j \neq 2),\\
&[X_1, Y_1] = -\beta X_2 + 2\xi, \qquad [X_1, Y_i] = 0 (i \neq 1),\\
&[X_2, Y_1] = \beta X_1 - \alpha Y_2, \qquad [X_2, Y_2] = \alpha Y_1 + 2\xi,\\
&[X_2, Y_i] = \beta X_i (i \neq 1,2), \qquad [X_i, Y_1] = -\alpha Y_i (i \neq 1,2),\\
&[X_i, Y_2] = 0 (i \neq 1,2), \qquad [X_i, Y_j] = \delta_{ij}(-\beta X_2 + \alpha Y_1 + 2\xi)(i,j \neq 1,2)
\end{aligned}
$$

$G_{\alpha,\beta}$: the simply-connected Lie group with Lie algebra $\mathfrak{g}_{\alpha,\beta}$.

- g: the Riemannain metric so that the above basis is orthonormal,
- the characteristic vector field is given by ξ,
- the 1-form η is the metric dual of ξ, that is, $\eta(X) = g(X, \xi)$,
- the $(1,1)$-tensor field φ is defined by $\varphi(\xi) = 0, \quad \varphi(X_i) = Y_i, \quad \varphi(Y_i) = -X_i$.

Realization problem of non-Sasakian (κ, μ)-spaces

Motivations:

I am interested in the realization problem of (κ, μ)-space

- In CR geometry: realization problem of strongly pseudo-convex CR-mfd as real hypersurfaces in complex manifolds
- In Submfd geometry: $\forall$ real hypersurfaces in Kähler manifolds are almost contact metric mfds.
  It would be natural question which is contact? or (κ, μ)-space?

Known results:

- Berndt proved that Sasakian space forms are realized as specific homogeneous hypersurfaces in non-flat complex space forms.
- For non-Sasakian cases, Cho-Inoguchi proved that

(κ, μ)	real hypersurface	Boeckx Inv.
$(-\frac{c}{4}, -\frac{\sqrt{c}}{2})$ $(c > 0)$	a tube of rad. $r = r(c)$ around Q^{n-1} in $\mathbb{C}P^n(c)$	$1 < I$
$(\frac{3c}{4}, -\frac{\sqrt{c}}{2})$ $(-4 < c < 0)$	a tube of rad. $r = r(c)$ around $\mathbb{R}H^n(\frac{c}{4})$ in $\mathbb{C}H^n(c)$	$0 < I < 1$

Realization problem of non-Sasakian (κ, μ)-spaces

Motivations:

I am interested in the realization problem of (κ, μ)-space

- In CR geometry: realization problem of strongly pseudo-convex CR-mfd as real hypersurfaces in complex manifolds
- In Submfd geometry: $\forall$ real hypersurfaces in Kähler manifolds are almost contact metric mfds.
  It would be natural question which is contact? or (κ, μ)-space?

Known results:

- Berndt proved that Sasakian space forms are realized as specific homogeneous hypersurfaces in non-flat complex space forms.
- For non-Sasakian cases, Cho-Inoguchi proved that

(κ, μ)	real hypersurface	Boeckx Inv.
$(-\frac{c}{4}, -\frac{\sqrt{c}}{2})$ $(c > 0)$	a tube of rad. $r = r(c)$ around Q^{n-1} in $\mathbb{C}P^n(c)$	$1 < I$
$(\frac{3c}{4}, -\frac{\sqrt{c}}{2})$ $(-4 < c < 0)$	a tube of rad. $r = r(c)$ around $\mathbb{R}H^n(\frac{c}{4})$ in $\mathbb{C}H^n(c)$	$0 < I < 1$

Main results

- $Q^{n*}(c)$: noncompact complex quadric with min. sec. curv. $-c$

Main results(Cho-Kubo-Taketomi-Tamaru-H., $\cdots$):
$\forall \mu \in \mathbb{R}$, $(0, \mu)$-spaces can be realized as homogeneous real hypersurfaces in $Q^{n*}(c)$ with some min. sec. curv. $-c$.

- In [Cho–Kubo-Taketomi-Tamaru-H.] we first realized $(0, 4)$-space, by investigating Lie structure of Boeckx's Lie algebra $\mathfrak{g}_{\alpha,\beta}$

Theorem (Cho-Kubo-Taketomi-Tamaru-H. 2018)

- $(0, 4)$-space can be realized as a horosphere (whose center at infinity is the equivalence class of an $\mathcal{A}$-principal) in $Q^{n*}(-8)$
- The Boeckx invariant of $(0, 4)$-space is $I = -1$

Remark

- In order to calculate I, we define (κ, μ)-space as a contact metric mfd($\Phi = d\eta$) satifying (κ, μ) condition.
- min. sec. curv. $c = -8 \Longleftrightarrow (0, 4)$-space satisfies $\Phi = d\eta$

Main results

- $Q^{n*}(c)$: noncompact complex quadric with min. sec. curv. $-c$

Main results(Cho-Kubo-Taketomi-Tamaru-H., $\cdots$):
$\forall \mu \in \mathbb{R}$, $(0,\mu)$-spaces can be realized as homogeneous real hypersurfaces in $Q^{n*}(c)$ with some min. sec. curv. $-c$.

- In [Cho–Kubo-Taketomi-Tamaru-H.] we first realized $(0,4)$-space, by investigating Lie structure of Boeckx's Lie algebra $\mathfrak{g}_{\alpha,\beta}$

Theorem (Cho-Kubo-Taketomi-Tamaru-H. 2018)

- $(0,4)$-space can be realized as a horosphere (whose center at infinity is the equivalence class of an $\mathcal{A}$-principal) in $Q^{n*}(-8)$
- The Boeckx invariant of $(0,4)$-space is $I=-1$

Remark

- In order to calculate I, we define (κ,μ)-space as a contact metric mfd($\Phi = d\eta$) satifying (κ,μ) condition.
- min. sec. curv. $c=-8 \Longleftrightarrow (0,4)$-space satisfies $\Phi = d\eta$

Main results

- $Q^{n*}(c)$: noncompact complex quadric with min. sec. curv. $-c$

Main results(Cho-Kubo-Taketomi-Tamaru-H., $\cdots$):
$\forall \mu \in \mathbb{R}$, $(0, \mu)$-spaces can be realized as homogeneous real hypersurfaces in $Q^{n*}(c)$ with some min. sec. curv. $-c$.

- In [Cho–Kubo-Taketomi-Tamaru-H.] we first realized $(0, 4)$-space, by investigating Lie structure of Boeckx's Lie algebra $\mathfrak{g}_{\alpha,\beta}$

Theorem (Cho-Kubo-Taketomi-Tamaru-H. 2018)

- $(0, 4)$-space can be realized as a horosphere (whose center at infinity is the equivalence class of an $\mathcal{A}$-principal) in $Q^{n*}(-8)$
- The Boeckx invariant of $(0, 4)$-space is $I = -1$

Remark

- In order to calculate I, we define (κ, μ)-space as a contact metric mfd($\Phi = d\eta$) satifying (κ, μ) condition.
- min. sec. curv. $c = -8 \iff (0, 4)$-space satisfies $\Phi = d\eta$

Main results

Theorem (Kubo-Taketomi-Tamaru-H.)

- $\forall c < -8$, $(0, -\frac{c}{2})$-space can be realized as a tube of radius $r = r(c)$ around Q^{n-1*} in $Q^{n*}(c)$.
- The Boeckx invariants of $(0, -\frac{c}{2})$-spaces are $I < -1$

Remark

- radius $r = r(c)$ is decided s.t. $(0, -\frac{c}{2})$-space satisfies $\Phi = d\eta$

- Recently, Cho proved that $(0, -\frac{c}{2})$-space $(c > -8)$ can be realized as a tube of radius $r = r(c)$ around $\mathbb{R}H^n$ in $Q^{n*}(c)$.
- Klein-Suh calculated principal curvatures of contact real hypersurfaces with constant mean curvature in Q^{n*}
- By using their results, Cho checked (κ, μ) condition directly

Main results

Theorem (Kubo-Taketomi-Tamaru-H.)

- $\forall c < -8$, $(0, -\frac{c}{2})$-space can be realized as a tube of radius $r = r(c)$ around Q^{n-1*} in $Q^{n*}(c)$.
- The Boeckx invariants of $(0, -\frac{c}{2})$-spaces are $I < -1$

Remark

- radius $r = r(c)$ is decided s.t. $(0, -\frac{c}{2})$-space satisfies $\Phi = d\eta$

- Recently, Cho proved that $(0, -\frac{c}{2})$-space ($c > -8$) can be realized as a tube of radius $r = r(c)$ around $\mathbb{R}H^n$ in $Q^{n*}(c)$.
- Klein-Suh calculated principal curvatures of contact real hypersurfaces with constant mean curvature in Q^{n*}
- By using their results, Cho checked (κ, μ) condition directly

Summary

(κ, μ)	real hypersurface	Boeckx Inv.
$(0, -\frac{c}{2})$ $(c > -8)$	a tube around $\mathbb{R}H^n$ in $Q^{n*}(c)$	$-1 < I < 1$
$(0, 4)$	a horoshpere in $Q^{n*}(-8)$	$I = -1$
$(0, -\frac{c}{2})$ $(c < -8)$	a tube around Q^{n-1*} in $Q^{n*}(c)$	$I < -1$

Thank you for your kind attention!

Summary

(κ, μ)	real hypersurface	Boeckx Inv.
$(0, -\frac{c}{2})$ $(c > -8)$	a tube around $\mathbb{R}H^n$ in $Q^{n*}(c)$	$-1 < I < 1$
$(0, 4)$	a horoshpere in $Q^{n*}(-8)$	$I = -1$
$(0, -\frac{c}{2})$ $(c < -8)$	a tube around Q^{n-1*} in $Q^{n*}(c)$	$I < -1$

Thank you for your kind attention!