HOMOGENEOUS CR-SUBMANIFOLDS IN COMPLEX HYPERBOLIC SPACES

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CR-submanifolds

$\bar{M}$ Hermitian manifold, J complex structure

Definition

$M \subset \bar{M}$ CR-submanifold if

$$T_pM = \Delta_p \oplus \Delta_p^\perp, \ \forall p \in M,$$

where:

- *Δ_p complex subspace of T_pM $(J\Delta_p \subset \Delta_p)$*
- *$\Delta_p^\perp$ totally real subspace of T_pM $(\Delta_p^\perp \perp J\Delta_p^\perp)$*

Examples

- Complex submanifolds
- Totally real submanifolds
- Real hypersurfaces

Motivation

Problem

To classify homogeneous CR-submanifolds of $\mathbb{C}H^n$

- Homogeneous real hypersurfaces (Berndt-Tamaru)
- Homogeneous complex submanifolds (Di Scala-Ishi-Loi)
- Homogeneous Lagrangian submanifolds
  - $H < AN$ (Hashinaga-Kajigaya)

Our problem

To classify those CR-submanifolds of $\mathbb{C}H^n$ given by the action of $H < AN$.

The isometry group of $\mathbb{C}H^n$

- $\mathbb{C}H^n$ symmetric space $\Rightarrow \mathbb{C}H^n = G/K$
  - $G = \mathrm{Isom}^0(\mathbb{C}H^n) = SU(1,n)$
  - $K = G_o = S(U(1)U(n))$, $o \in \mathbb{C}H^n$
- $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ Cartan decomposition

- $\mathfrak{a} \subset \mathfrak{p}$ maximal abelian subspace, $\mathfrak{a} \simeq \mathbb{R}$
- For $\lambda \in \mathfrak{a}^*$, $\mathfrak{g}_\lambda = \{X \in \mathfrak{g} : [A, X] = \lambda(A)X, \forall A \in \mathfrak{a}\}$ root spaces
- $\Sigma = \{\pm\alpha, \pm 2\alpha\}$ set of roots
- $\mathfrak{g} = \mathfrak{g}_{-2\alpha} \oplus \mathfrak{g}_{-\alpha} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$ root space decomposition

- $\Sigma^+ = \{\alpha, 2\alpha\}$ positive roots
- $\mathfrak{n} = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$ nilpotent Lie algebra
- $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ Iwasawa decomposition $\rightarrow G = KAN$
- $\mathfrak{a} \oplus \mathfrak{n}$ solvable part: $J\mathfrak{a} = \mathfrak{g}_{2\alpha}$, $J\mathfrak{g}_\alpha = \mathfrak{g}_\alpha$, $\mathfrak{g}_\alpha \simeq \mathbb{C}^{n-1}$

Lemma

$G = \mathrm{Isom}^0(\mathbb{C}H^n)$, $o \in \mathbb{C}H^n$, $H < G$ *connected subgroup,* $Lie(H) = \mathfrak{h}$.

$$M = H \cdot o \;\; \textit{CR-submanifold} \;\Leftrightarrow\; \mathfrak{h}_{\mathfrak{a}\oplus\mathfrak{n}} = \mathfrak{c} \oplus \mathfrak{r}.$$

Example

- $G = KAN$, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$, where $\mathfrak{n} = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$
- $\mathfrak{w}^\perp \subset \mathfrak{g}_\alpha$ totally real vector subspace of dimension k
- $\mathfrak{w} = \mathfrak{g}_\alpha \ominus \mathfrak{w}^\perp$
- $\mathfrak{s} = \mathfrak{a} \oplus \mathfrak{w} \oplus \mathfrak{g}_{2\alpha}$ Lie subalgebra with associated Lie group S

$W^{2n-k} = S \cdot o$ Berndt-Brück submanifold of $\mathbb{C}H^n$

W^{2n-k} is a CR-submanifold

$$\mathfrak{s} \;=\; \underbrace{(\mathfrak{a} \oplus \mathfrak{c} \oplus \mathfrak{g}_{2\alpha})}_{\text{complex}} \oplus \underbrace{J\mathfrak{w}^\perp}_{\text{totally real}}, \quad \text{where} \quad \mathfrak{c} = \mathfrak{w} \ominus J\mathfrak{w}^\perp$$

Lemma

$G = \mathrm{Isom}^0(\mathbb{C}H^n)$, $o \in \mathbb{C}H^n$, $H < G$ *connected subgroup,* $Lie(H) = \mathfrak{h}$.

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$W^{2n-k} = S \cdot o$ homogeneous, minimal, ruled by $\mathbb{C}H^{n-k}$

W^{2n-k} is a CR-submanifold

$$\mathfrak{s} \; = \underbrace{(\mathfrak{a} \oplus \mathfrak{c} \oplus \mathfrak{g}_{2\alpha})}_{\text{complex}} \oplus \underbrace{J\mathfrak{w}^\perp}_{\text{totally real}}, \quad \text{where} \quad \mathfrak{c} = \mathfrak{w} \ominus J\mathfrak{w}^\perp$$

Main result

Our problem

To classify those CR-submanifolds of $\mathbb{C}H^n$ given by the action of $H < AN$.

Theorem

Let $H < AN$ be a connected subgroup acting on $\mathbb{C}H^n$ in such a way that the orbit $M = H \cdot o$ is a CR-submanifold. Then, its Lie algebra $\mathfrak{h}$ is conjugated to one of the following

1. $\mathfrak{h} = \mathfrak{r}$, *or*
2. $\mathfrak{h} = \mathfrak{a} \oplus \mathfrak{r}$, *or*
3. $\mathfrak{h} = \mathfrak{c} \oplus \mathfrak{r} \oplus \mathfrak{g}_{2\alpha}$, *or*
4. $\mathfrak{h} = \mathfrak{a} \oplus \mathfrak{c} \oplus \mathfrak{r} \oplus \mathfrak{g}_{2\alpha}$,

where $\mathfrak{r}, \mathfrak{c} \subset \mathfrak{g}_{\alpha}$, $\mathfrak{r}$ is a totally real subspace and $\mathfrak{c}$ is a complex one.

Idea of the proof

1. Lemma: $\mathfrak{h}$ = complex $\oplus$ totally real
2. $\mathrm{pr}\colon \mathfrak{g} \to \mathfrak{a} \oplus \mathfrak{g}_{2\alpha}$ projection onto $\mathfrak{a} \oplus \mathfrak{g}_{2\alpha}$
3. $\mathrm{pr}(\mathfrak{h}) \in \{0,\ \mathfrak{a},\ \mathfrak{g}_{2\alpha},\ \mathbb{R}(aB + bZ),\ \mathfrak{a} \oplus \mathfrak{g}_{2\alpha}\}$, where $B \in \mathfrak{a}$, $Z \in \mathfrak{g}_{2\alpha}$

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$\mathrm{pr}(\mathfrak{h}) = \mathfrak{a}$: $\mathfrak{h} = \mathbb{R}(B + X) \oplus \mathfrak{w}$, for $\mathfrak{w} \subset \mathfrak{g}_\alpha$, $X \in \mathfrak{g}_\alpha \ominus \mathfrak{w}$. Let $U, V \in \mathfrak{w}$.

- Due to the properties of the Lie bracket of $\mathfrak{a} \oplus \mathfrak{n}$,

$$2[U, V] = \langle JU, V\rangle Z \Rightarrow \langle JU, V\rangle = 0 \Rightarrow \mathfrak{w} \text{ totally real subspace,}$$
$$2[B + X, U] = U + \langle JX, U\rangle Z \Rightarrow \langle JX, U\rangle = 0 \Rightarrow X \perp \mathbb{C}\mathfrak{w}.$$

- $\langle J(B + X), U\rangle = \langle Z + JX, U\rangle = 0 \Rightarrow \mathfrak{h}$ totally real subspace.
- $\mathrm{Ad}(\mathrm{Exp}(2X))\mathfrak{h} = e^{2\,\mathrm{ad}(X)}\mathfrak{h} = \mathfrak{a} \oplus \mathfrak{w}$.

Main result

Theorem

Let $H < AN$ be a connected subgroup acting on $\mathbb{C}H^n$ in such a way that the orbit $M = H \cdot o$ is a CR-submanifold. Then, its Lie algebra $\mathfrak{h}$ is conjugated to one of the following

1. $\mathfrak{h} = \mathfrak{r} \rightsquigarrow H \cdot o$ *is a horosphere in a totally geodesic* $\mathbb{R}H^k$,
2. $\mathfrak{h} = \mathfrak{a} \oplus \mathfrak{r} \rightsquigarrow H \cdot o$ *is a totally geodesic* $\mathbb{R}H^k$,
3. $\mathfrak{h} = \mathfrak{r} \oplus \mathfrak{c} \oplus \mathfrak{g}_{2\alpha} \rightsquigarrow H \cdot o$ *is a direct product of a horosphere in a totally geodesic* $\mathbb{C}H^k$ *and a horosphere in a totally geodesic* $\mathbb{R}H^\ell$,
4. $\mathfrak{h} = \mathfrak{a} \oplus \mathfrak{r} \oplus \mathfrak{c} \oplus \mathfrak{g}_{2\alpha} \rightsquigarrow H \cdot o$ *is the Berndt-Brück submanifold* W^{2m-k} *in a totally geodesic* $\mathbb{C}H^m$,

where $\mathfrak{r}, \mathfrak{c} \subset \mathfrak{g}_\alpha$, $\mathfrak{r}$ *is a totally real subspace and* $\mathfrak{c}$ *is a complex one.*

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