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Hopf Real Hypersurfaces in the Indefinite Complex Projective Space

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This talk is based on the following joint work with Makoto Kimura (Ibaraki University, Japan)

M. Kimura, —, *Hopf Real Hypersurfaces in the Indefinite Complex Projective Space*, Mediterr. J. Math. (2019) 16: 27.
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Summary

Introduction

The theory of real hypersurfaces in complex space forms is very well-developed.

J. Berdnt,
T. Cecil, G. Kaimakamis, M. Kimura, S. Maeda,
Y. Maeda, S. Montiel, K. Panagiotidou, Juan de Dios Pérez,
P. Ryan, Y. J. Suh, R. Takagi...

R. Takagi, *On homogeneous real hypersurfaces in a complex projective space*. Osaka J. Math. **10** (1973), 495–506

The classification of (extrinsically) homogeneous real hypersurfaces in $\mathbb{C}P^n$, $n \geq 2$: Six types of tubes of certain radii over some complex submanifolds [A_0, A_1, B, C, D, E].

N: a unit normal vector field to M in $\mathbb{C}P^n$,
J: the complex structure. $\xi = -JN$; A: shape operator.
All these examples satisfy $A\xi = \mu\xi$.

J. Berndt, *Real hypersurfaces with constant principal curvatures in complex hyperbolic space*, J. Reine Angew. Math. **395** (1989), 132–141.

Theorem A

Let M be a real hypersurface in $\mathbb{C}H^n$, $n \geq 2$, such that ξ is principal, and M has constant principal curvatures. Then, M is an open subset of one of the following:

(A) *A tube of radius $r > 0$ over a totally geodesic $\mathbb{C}H^k$, $k = 0, \ldots, n-1$;*

(B) *a tube of radius $r > 0$ over a totally geodesic $\mathbb{R}H^n$;*

(C) *a horosphere.*

Introduction

Hundreds of works about real hypersurfaces in non-flat complex space forms have appeared, also in

- the quaternionic space forms,
- the Grassmanian of 2-complex planes, and
- the complex quadric.

T. E. Cecil and P. J. Ryan, *Geometry of Hypersurfaces*, Springer Monographs in Mathematics, Springer, New York, NY (2015) DOI 10.1007/978-1-4939-3246-7

Introduction

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T. E. Cecil and P. J. Ryan, *Geometry of Hypersurfaces*, Springer Monographs in Mathematics, Springer, New York, NY (2015) DOI 10.1007/978-1-4939-3246-7

Next, we move to real hypersurfaces
in the indefinite complex projective space $\mathbb{C}P_p^n$.

A. Bejancu, K. L. Duggal, *Real hypersurfaces of indefinite Kaehler manifolds*, Internat. J. Math. Math. Sci. **16** (1993), no. 3, 545–556.

H. Anciaux, K. Panagiotidou, *Hopf Hypersurfaces in pseudo-Riemannian complex and para-complex space forms*, Diff. Geom. Appl. **42** (2015) 1-14 DOI: 10.1016/j.difgeo.2015.05.004

Introduction

- We allow the normal vector to have its own causal character, without changing the metric.
- We recover the almost contact metric structure (g, ξ, η, ϕ).
- Examples:
  1. Families of non-degenerate real hypersurfaces whose shape operator is diagonalisable,
  2. An example with degenerate metric and non-diagonalisable *shape operator*.
- A rigidity result.
- $AX = aX + b\eta(X)\xi$, $\forall X \in TM$.
- $A\phi = \phi A$.

Summary

Preliminaries

See [2] (Barros-Romero) for more details.
$\mathbb{C}_p^{n+1}$ the Euclidean complex space endowed with the following pseudo-Riemannian metric of index $2p$:
$z = (z_1, \ldots, z_{n+1})$, $w = (w_1, \ldots, w_{n+1}) \in \mathbb{C}^{n+1}$,

$$g(z,w) = \mathrm{Re}\left(-\sum_{j=1}^{p} z_j \bar{w}_j + \sum_{j=p+1}^{n+1} z_j \bar{w}_j \right),$$

where $\bar{w}$ is the complex conjugate of $w \in \mathbb{C}$.

$$\mathbb{S}^1 = \{a \in \mathbb{C} : a\bar{a} = 1\} = \{\mathrm{e}^{i\theta} : \theta \in \mathbb{R}\}.$$

$$\mathbb{S}_{2p}^{2n+1} = \{z \in \mathbb{C}_p^{n+1} : g(z,z) = 1\}.$$

$$x, y \in \mathbb{S}_{2p}^{2n+1}, \quad x \sim y \Leftrightarrow \exists a \in \mathbb{S}^1 : x = a\, y.$$

$$\pi : \mathbb{S}_{2p}^{2n+1} \to \mathbb{S}_{2p}^{2n+1} / \sim =: \mathbb{C}P_p^n.$$

The manifold $\mathbb{C}P_p^n$ is called the *Indefinite Complex Projective Space*.

Let g be the metric on $\mathbb{C}P_p^n$ such that π becomes a semi-Riemannian submersion.

Let $\bar{\nabla}$ be its Levi-Civita connection.

$\mathbb{C}P_p^n$ admits a complex structure J induced by π.

M: a connected, orientable, immersed real hypersurface in $\mathbb{C}P^n_p$.
N : a unit normal vector field such that $\varepsilon = g(N,N) = \pm 1$.
$\xi = -JN$: The *structure* vector field on M. Clearly, $g(\xi,\xi) = \varepsilon$.
Given $X \in TM$, we decompose JX in its tangent and normal parts, namely

$$JX = \phi X + \varepsilon\,\eta(X)N,$$

where ϕX is the tangential part, and η is the 1-form on M. Given $X, Y \in TM$,

$$\eta(X) = g(X,\xi), \quad \phi\xi = 0, \quad \eta(\xi) = \varepsilon,$$
$$\phi^2 X = -X + \varepsilon\eta(X)\xi, \quad \eta(\phi X) = 0,$$
$$g(\phi X, \phi Y) = g(X,Y) - \varepsilon\eta(X)\eta(Y), \quad g(\phi X, Y) + g(X, \phi Y) = 0.$$

(g, ϕ, η, ξ) is called an *almost contact metric structure* on M.

Next, if ∇ is the Levi-Civita connection of M, we have the Gauss and Weingarten formulae:

$$\bar{\nabla}_X Y = \nabla_X Y + \varepsilon g(AX, Y)N, \quad \bar{\nabla}_X N = -AX,$$

for any $X, Y \in TM$, where A is the shape operator associated with

Definition 1

Let M be a real hypersurface in $\mathbb{C}P^n_p$. We will say that M is $Hopf$ when its structure vector field ξ is everywhere principal, i. e., it is an eigenvector of A.

Its associated principal curvature $\mu = \varepsilon g(A\xi, \xi)$ will be called the *Hopf curvature*: $A\xi = \mu\xi$.

Summary

Recall the projection $\pi : \mathbb{S}^{2n+1}_{2p} \to \mathbb{C}P^n_p$.

$$\begin{array}{ccc} \tilde{M}^{2n} & \longrightarrow & \mathbb{S}^{2n+1}_{2p} \\ \downarrow & & \downarrow \\ M^{2n-1} & \longrightarrow & \mathbb{C}P^n_p \end{array}$$

Given $0 \leq q \leq p \leq m \leq n+2$, $m > q+1$, the case $q = 0$ and $m = n+2$ is not considered.

We define the following map $\mathbf{pr} : \mathbb{C}_p^{n+1} \to \mathbb{C}_p^{n+1}$:

- if $1 \leq q$ and $m \leq n+1$,
  $\mathbf{pr}(z) = (z_1, \ldots, z_q, 0, \ldots, 0, z_m, \ldots, z_{n+1})$,
- if $q = 0$ and $m \leq n+1$,
  $\mathbf{pr}(z) = (0, \ldots, 0, z_m, \ldots, z_{n+1})$,
- if $1 \leq q$ and $m = n+2$,
  $\mathbf{pr}(z) = (z_1, \ldots, z_q, 0, \ldots, 0)$.

Type A

Consider $t \in \mathbb{R}$, $t \neq 0, 1$, *and* $0 \leq q \leq p \leq m \leq n+2$, $m > q+1$. *With this notation, we define*

$$\begin{aligned}
\tilde{\mathbf{M}}_q^m(t) &= \left\{ z = (z_1, \dots, z_n) \in \mathbb{S}_{2p}^{2n+1} \;:\; g(\mathbf{pr}(z), \mathbf{pr}(z)) = t \right\} \\
\mathbf{M}_q^m(t) &= \pi(\tilde{\mathbf{M}}_q^m(t)) \subset \mathbb{C}P_p^n \\
A\xi &= \mu\xi
\end{aligned}$$

For a suitable $r > 0$,

(A_+) $\varepsilon = +1$, $0 < t = \cos^2(r) < 1$,
$\mu = 2\cot(2r)$, $\lambda_1 = -\tan(r)$, $\lambda_2 = \cot(r)$.

(A_-) $\varepsilon = -1$, $1 < t = \cosh^2(r)$,
$\mu = 2\coth(2r)$, $\lambda_1 = -\tanh(r)$, $\lambda_2 = \coth(r)$.

$$\dim V_{\lambda_1} = 2(m-q-2), \quad \dim V_{\lambda_2} = 2(n+q-m+1).$$

Examples

Type B

Given $t > 0$, $t \neq 1$, $Q(z) = -\sum_{j=1}^{p} z_j^2 + \sum_{j=p+1}^{n+1} z_j^2$,

$$\tilde{\mathbf{M}}_t = \left\{ z = (z_1, \ldots, z_{n+1}) \in \mathbb{S}_{2p}^{2n+1} \ : \ Q(z)\overline{Q(z)} = t \right\}, \ \mathbf{M}_t = \pi(\tilde{\mathbf{M}}_t).$$

$$\varepsilon = \operatorname{sign}(t(1-t)) = \pm 1, \quad A\xi = \mu\xi, \quad g(\xi,\xi) = \varepsilon.$$

(B_+) $\varepsilon = +1$, $0 < t = \sin^2(2r) < 1$, $\mu = 2\cot(2r)$, $\lambda_1 = \cot(r)$, $m_1 = n-1$, $\lambda_2 = \tan(r)$, $m_2 = n-1$, $\phi V_{\lambda_1} = V_{\lambda_2}$.

(B_0) $\varepsilon = -1$, $\mu = \sqrt{3}$, $\lambda = 1/\sqrt{3}$, $\dim V_\mu = n$, $\dim V_\lambda = n-1$, $\phi V_\mu = V_\lambda$, $\xi \in V_\mu$.

(B_-) $\varepsilon = -1$, $1 < t = \cosh^2(2r)$, $\mu = 2\tanh(2r)$, $\lambda_1 = \coth(r)$, $m_1 = n-1$, $\lambda_2 = \tanh(r)$, $m_2 = n-1$, $\phi V_{\lambda_1} = V_{\lambda_2}$.

A degenerate example

Recall $Q(z) = -\sum_{j=1}^{p} z_j^2 + \sum_{j=p+1}^{n+1} z_j^2$.

$$\tilde{\mathbf{M}}_1 = \left\{ z = (z_1, \ldots, z_{n+1}) \in \mathbb{S}_{2p}^{2n+1} \; : \; Q(z)\overline{Q(z)} = 1, \; z \neq Q(z)\bar{z} \right\}.$$

$\mathbf{M}_1 = \pi(\tilde{\mathbf{M}}_1)$ is a real hypersurface in $\mathbb{C}P_p^n$ such that:

1. The normal vector N is lightlike, so that $N \in T\mathbf{M}_1$.
2. The induced metric g is degenerate, with $\{N, \xi\}$ spanning its radical.
3. If $AX = -\bar{\nabla}_X N$, for any $X \in TM$, then M is Hopf: $A\xi = 0$.
4. The shape operator is not diagonalisable:
   $\mathbb{D} = T\mathbf{M}_1 \cap JT\mathbf{M}_1 = V_0 \oplus V_2$. $\lambda_1 = 0$, $\lambda_2 = 2$,
   $\dim V_0 = \dim V_2 = n - 1$. $\xi \in V_0$. But
   For $V \notin \mathbb{D}$ s.t. $T\mathbf{M}_1 = \mathbb{D} \oplus \mathrm{Span}\{V\} \Rightarrow 0 \neq AV \in \mathbb{D}$.
5. It is the tube of radius $s = \pi/4$ over a totally complex submanifold.

The horosphere

Type C

Given $t > 0$,

$$\tilde{\mathbf{H}}(t) = \left\{z = (z_1, \ldots, z_n) \in \mathbb{S}^{2n+1}_{2p} : (z_1 - z_{n+1})(\bar{z}_1 - \bar{z}_{n+1}) = t\right\},$$
$$\mathbf{H}(t) = \pi(\tilde{\mathbf{H}}(t)) \subset \mathbb{C}P^n_p.$$

The unit normal vector N on $\mathbf{H}(t)$ is time-like.

$$A\xi = 2\xi, \quad AX = X, \quad \forall X \in T\mathbf{H}(t),\ X \perp \xi.$$

Summary

A Ridigity Result

Theorem 1

Let $f_i : M_q^{2n-1} \to \mathbb{C}P_p^n$, $i = 1, 2$ *two isometric immersions of the same connected manifold in* $\mathbb{C}P_p^n$*, with Weingarten endomorphisms* A_1 *and* A_2*. If for each point* $p \in M$, $A_1(p) = A_2(p)$*, there exists an isometry* $\Phi : \mathbb{C}P_p^n \to \mathbb{C}P_p^n$ *such that* $f_2 = \Phi \circ f_1$*.*

We are strongly using that $\mathbb{S}_{2p}^{2n+1}$ is a space of constant sectional curvature.

Summary

The shape operator vs the almost contact metric structure

Theorem 2

Let M be a connected, non-degenerate, oriented real hypersurface in $\mathbb{C}P_p^n$, $n \geq 2$, such that $AX = \lambda X + \rho\eta(X)\xi$ for any $X \in TM$, for some functions $\lambda, \rho \in C^\infty(M)$. Then, M is locally congruent to one of the following real hypersurfaces:

1. *A real hypersurface of type A_+, with $m = q + 2$, $q \leq p \leq m = q + 2$, $\mu = 2\cot(2r)$ and $\lambda = \cot(r)$, $r \in (0, \pi/2)$;*
2. *A real hypersurface of type A_+, with $m = n + q + 1$, $0 \leq q \leq 1$, $\mu = 2\cot(2r)$ and $\lambda = -\tan(r)$, $r \in (0, \pi/2)$;*
3. *A real hypersurface of type A_-, with $m = q + 2$, $q \leq p \leq m = q + 2$, $\mu = 2\coth(2r)$, $r > 0$ and $\lambda = \coth(r)$;*
4. *A real hypersurface of type A_-, with $m = q + 2$, $q \leq p \leq m = q + 2$, $\mu = 2\coth(2r)$, $r > 0$ and $\lambda = \tanh(r)$;*
5. *A horosphere.*

Corollary 1

Let M be a non-degenerate real hypersurface in $\mathbb{C}P^n_p$ such that its Weingarten endomorphism is diagonalisable. The following are equivalent:

1. ξ *is a Killing vector field;*
2. $A\phi = \phi A$;
3. M *is an open subset of one of the following:*
   - a. *A real hypersurface of type* A_+, *with* $m = q + 2$, $q \leq p \leq m = q + 2$, $\mu = 2\cot(2r)$ *and* $\lambda = \cot(r)$, $r \in (0, \pi/2)$;
   - b. *A real hypersurface of type* A_+, *with* $m = n + q + 1$, $0 \leq q \leq 1$, $\mu = 2\cot(2r)$ *and* $\lambda = -\tan(r)$, $r \in (0, \pi/2)$;
   - c. *A real hypersurface of type* A_-, *with* $m = q + 2$, $q \leq p \leq m = q + 2$, $\mu = 2\coth(2r)$, $r > 0$ *and* $\lambda = \coth(r)$;
   - d. *A real hypersurface of type* A_-, *with* $m = q + 2$, $q \leq p \leq m = q + 2$, $\mu = 2\coth(2r)$, $r > 0$ *and* $\lambda = \tanh(r)$;
   - e. *A horosphere.*

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H. Anciaux, K. Panagiotidou, *Hopf Hypersurfaces in pseudo-Riemannian complex and para-complex space forms*, Diff. Geom. Appl. **42** (2015) 1-14 DOI: 10.1016/j.difgeo.2015.05.004

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Thank you very much for your kind attention!