On Mean curvature flow of Singular Riemannian foliations: Non compact cases

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In honor of Professor Jürgen Berndt's 60th birthday.

[ACG19] Marcos M. Alexandrino, Leonardo F. Cavenaghi and Icaro Gonçalves, *Mean curvature flow of singular Riemannian foliations: Non compact cases*, arXiv:1909.04201 (2019)

Definition

Given a Riemannian manifold M and an immersion $\varphi : L_0 \to M$, a smooth family of immersions $\varphi_t : L_0 \to M$, $t \in [0, T)$ is called a solution of the ***mean curvature flow*** *(MCF for short) if φ_t satisfies the evolution equation*

$$\frac{d}{dt}\varphi_t(x) = \vec{H}(t, x),$$

where $\vec{H}(t, x)$ is the mean curvature of $L(t) := \varphi_t(L_0)$.

Definition

A submanifold L of a space form $M(k)$ is called ***isoparametric*** *if its normal bundle is flat and the principal curvatures along any parallel normal vector field are constant.*

An ***isoparametric foliation*** *$\mathcal{F}$ on $M(k)$ is a partition of $M(k)$ by submanifolds parallel to a given isoparametric submanifold L.*

Jurgen Berndt, Sergio Console, Carlos Enrique Olmos *Submanifolds and Holonomy* Chapman & Hall/CRC Monographs and Research Notes in Mathematics(2003)

G. Thorbergsson, *Singular Riemannian Foliations and Isoparametric Submanifolds* Milan J. Math. Vol. 78 (2010) 355–370

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A singular foliation $\mathcal{F} = \{L\}$ *is called a* ***generalized isoparametric*** *if*

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Examples:

1. $\mathcal{F} = \{G(x)\}_{x \in M}$, where G is Lie subgroup of $Iso(M)$
2. isoparametric foliations,
3. Singular Riemannian foliations with compact leaves on $\mathbb{R}^n$, $\mathbb{S}^n$ and projective spaces (see Clifford foliations for non homogenous examples).

Example (Holonomy foliations)

- *L is a Riemannian manifold ,*
- *E is a **Euclidean** vector bundle over L (i.e., with an inner product $\langle\, ,\, \rangle_p$ on each fiber E_p)*
- *∇^E is a metric connection on E, i.e.*

$$X\langle \xi, \eta\rangle = \langle \nabla^E_X \xi, \eta\rangle + \langle \xi, \nabla^E_X \eta\rangle.$$

- *the **connection (Sasaki) metric** g^E on E*

*Define the **holonomy foliation** $\mathcal{F}^h$ on E, by declaring two vectors $\xi, \eta \in E$ in the same leaf if they can be connected to one another via a composition of parallel transports (with respect to ∇^E).*

Example (Model)

Consider a Euclidean vector bundle $\mathbb{R}^n \to E \to L$, with a metric connection ∇^E and a the Sasaki metric g^E. Let $\mathcal{F}^0_p = \{L^0_\xi\}_{\xi \in E_p}$ be a SRF with compact leaves on the fiber E_p. Assume $\mathcal{F}^0$ is invariant by the the holonomy group H_p at p i.e., the group sends leaves to leaves.

- *$\mathcal{F} = \{L_\xi\}_{\xi \in E_p}$ with leaves $L_\xi = H(L^0_\xi)$ where H is the holonomy groupoid associate to the connection ∇^E.*

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ACG19 + Alexandrino, Inagaki, Struchiner(18) imply

Lemma (Semi-local Model)

Let $\mathcal{F}$ be a SRF with closed leaves. Then $\mathcal{F}|_{\mathrm{Tub}_\epsilon(L_q)}$ is foliated diffeomorphic to the foliation defined in Model. Therefore $\mathrm{Tub}_\epsilon(L_q)$ admits a metric so that $\mathcal{F}|_{\mathrm{Tub}_\epsilon(L_q)}$ is a generalized isoparametric foliation.

Theorem A (ACG19)

Let $\mathcal{F} := \{L\}$ be a generalized isoparametric foliation with closed leaves on a complete manifold M so that $M/\mathcal{F}$ is compact. Let $L_0 \in \mathcal{F}$ be a regular leaf of M and let $L(t)$ denote the MCF evolution of L_0. Assume that $T < \infty$. Then:

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(a) *$L(t)$ converges to a singular leaf L_T of $\mathcal{F}$.*

(b) *If the curvature of M is bounded and the shape operator along each leaf is bounded, then $\varphi_t(p)$ converges to a point of L_T, for each $p \in L(0)$. In addition the singularity is of type I, i.e.,*

$$\limsup_{t \to T^-} \|A_t\|_\infty^2 (T - t) < \infty,$$

where $\|A_t\|_\infty$ is the sup norm of the second fundamental form of $L(t)$.

Lemma (basins of attraction)

Let L_q be a singular leaf. Then there exists an $\epsilon = \epsilon(L_q)$ such that if $L(t_0)$ lies in $\mathrm{Tub}_\epsilon(L_q)$ we have:

(a) *For any $t > t_0$ the distance $r(t) = \mathrm{dist}(L(t), L_q)$ satisfies*

$$C_1^2(t - t_0) \leq r^2(t_0) - r^2(t) \leq C_2^2(t - t_0)$$

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(b) *$T < \infty$ and $L(t) \subset \mathrm{Tub}_\epsilon(L_q)$ for all $t \in (t_0, T)$.*

(c) *If $L(t)$ converges to L_q at time T then for any $t \in (t_0, T]$,*

$$C_1\sqrt{T - t} \leq r(t) \leq C_2\sqrt{T - t}.$$

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Proof of item (a) of Theorem A the fact that $M/\mathcal{F}$ is compact, $T < \infty$ and the lemma imply $L(t) \to L_T$ **Q.E.D**

Proposition (ACG19)

Let M be a compact Riemannian manifold and $\mathcal{F}$ be a generalized isoparametric foliation on M, with possible non-closed leaves. Assume that the MCF $t \to L(t)$ of a regular leaf $L(0)$ as initial datum has $T < \infty$.

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Let M be a compact Riemannian manifold and $\mathcal{F}$ be a generalized isoparametric foliation on M, with possible non-closed leaves. Assume that the MCF $t \to L(t)$ of a regular leaf $L(0)$ as initial datum has $T < \infty$. Then $t \to L(t)$ must converge to the closure of a singular leaf. In other words, $\pi(L(t))$ converges to a singular point of $M/\overline{\mathcal{F}}$, where $\pi : M \to M/\overline{\mathcal{F}}$ is the canonical projection.

Lemma Under bounded curvature conditions, if $N \subset \partial \mathrm{Tub}_{\epsilon}(L)$ then $-\frac{C}{r(x)} - c_1 \leq \mathrm{tr}(A_{\nabla r}) \leq -\frac{C}{r(x)} + c_1$, where dim $N >$ dim L.

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Then $N(t)$ converges to a singular leaf L.

Sketch of proof: Type I convergence

Let $f^0(t)$ be the distance between L_x and its focal set with respect with g^0. From Radeschi and Alexandrino (2015) $f^0(t) \geq C\sqrt{T-t}$. We also have that $\|A^0(t)\|_0 = \frac{1}{f^0(t)}$

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(1), (2), (5) imply:

$$\|A_x(t)\|\sqrt{T-t} \leq C_5$$

and hence **type I convergence**.

Thank you very much!

Moitíssimas grazas!

Vielen Dank!