On the topology of transitive and cohomogeneity one actions

Manuel Amann

October 2019

Universität
Augsburg
University

Symmetry and Shape
Santiago de Compostela

On the topology of transitive and cohomogeneity one actions

Manuel Amann

October 2019

Universität Augsburg
University

Symmetry and Shape
Santiago de Compostela

Geometry vs. Topology via Symmetry

In this talk we want to understand different aspects of the interplay of

In this talk we want to understand different aspects of the interplay of

- Geometry (mainly in the form of lower curvature bounds and Alexandrov geometry)

Geometry vs. Topology via Symmetry

In this talk we want to understand different aspects of the interplay of

- Geometry (mainly in the form of lower curvature bounds and Alexandrov geometry)
- Group Actions (via cohomogeneity one and transitive actions)

Geometry vs. Topology via Symmetry

In this talk we want to understand different aspects of the interplay of

- Geometry (mainly in the form of lower curvature bounds and Alexandrov geometry)
- Group Actions (via cohomogeneity one and transitive actions)
- Topology (as equivariant cohomology and rational ellipticity)

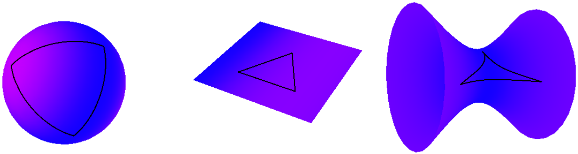
Equivariant cohomology of Cohomogeneity One Alexandrov Spaces

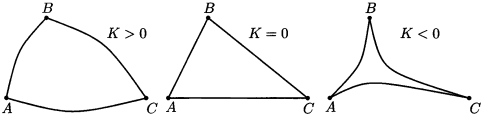
Alexandrov spaces

Toponogov's sectional curvature characterisation via fat and thin triangles can be adapted to impose a lower curvature bound on metric spaces.

Alexandrov spaces

Toponogov's sectional curvature characterisation via fat and thin triangles can be adapted to impose a lower curvature bound on metric spaces. Recall that an Alexandrov space (with lower curvature bound κ) is a geodesic length space which is basically defined by the fact that its geodesic triangles are "fatter" than the ones in the "model space" $M(\kappa)$:





Alexandrov spaces

Alexandrov spaces arise as

Alexandrov spaces

Alexandrov spaces arise as

- Gromov–Hausdorff limits of manifolds with lower sectional curvature bound, or as

Alexandrov spaces

Alexandrov spaces arise as

- Gromov–Hausdorff limits of manifolds with lower sectional curvature bound, or as
- quotients of manifolds by group actions.

Alexandrov spaces

Alexandrov spaces arise as

- Gromov–Hausdorff limits of manifolds with lower sectional curvature bound, or as
- quotients of manifolds by group actions.

The category is closed under taking products, and the category of Alexandrov spaces with curvature bounded below by 1 is closed under joins.

The (isometric) action of a compact Lie group on an Alexandrov space X is

Cohomogeneity one Alexandrov spaces

The (isometric) action of a compact Lie group on an Alexandrov space X is

- transitive if it only has one orbit. In this case X is a homogeneous manifold.

Cohomogeneity one Alexandrov spaces

The (isometric) action of a compact Lie group on an Alexandrov space X is

- transitive if it only has one orbit. In this case X is a homogeneous manifold.
- of cohomogeneity 1 if it has an orbit of codimension 1.

Cohomogeneity one Alexandrov spaces

In analogy to cohomogeneity one manifolds there is a “double mapping cylinder decomposition” of cohomogeneity one Alexandrov spaces M (with orbit space a compact interval, i.e. not being a manifold).

Cohomogeneity one Alexandrov spaces

In analogy to cohomogeneity one manifolds there is a “double mapping cylinder decomposition” of cohomogeneity one Alexandrov spaces M (with orbit space a compact interval, i.e. not being a manifold).

- Let G act by cohomogeneity one. The orbit space is a closed interval, over its interior we find the principal orbits G/H of codimension 1, over the endpoints the singular/exotic orbits G/K_0 and G/K_1.

Cohomogeneity one Alexandrov spaces

In analogy to cohomogeneity one manifolds there is a “double mapping cylinder decomposition” of cohomogeneity one Alexandrov spaces M (with orbit space a compact interval, i.e. not being a manifold).

- Let G act by cohomogeneity one. The orbit space is a closed interval, over its interior we find the principal orbits G/H of codimension 1, over the endpoints the singular/exotic orbits G/K_0 and G/K_1.
- Due to the slice theorem the normal cones (corresponding to normal disc bundles in the manifold setting) over G/K_0 and G/K_1 have common boundary G/H.

Cohomogeneity one Alexandrov spaces

In analogy to cohomogeneity one manifolds there is a "double mapping cylinder decomposition" of cohomogeneity one Alexandrov spaces M (with orbit space a compact interval, i.e. not being a manifold).

- Let G act by cohomogeneity one. The orbit space is a closed interval, over its interior we find the principal orbits G/H of codimension 1, over the endpoints the singular/exotic orbits G/K_0 and G/K_1.
- Due to the slice theorem the normal cones (corresponding to normal disc bundles in the manifold setting) over G/K_0 and G/K_1 have common boundary G/H.
- We glue them along this boundary to obtain M. We obtain bundles

$$K_i/H \hookrightarrow G/H \to G/K_i$$

Cohomogeneity one Alexandrov spaces

In analogy to cohomogeneity one manifolds there is a "double mapping cylinder decomposition" of cohomogeneity one Alexandrov spaces M (with orbit space a compact interval, i.e. not being a manifold).

- Let G act by cohomogeneity one. The orbit space is a closed interval, over its interior we find the principal orbits G/H of codimension 1, over the endpoints the singular/exotic orbits G/K_0 and G/K_1.
- Due to the slice theorem the normal cones (corresponding to normal disc bundles in the manifold setting) over G/K_0 and G/K_1 have common boundary G/H.
- We glue them along this boundary to obtain M. We obtain bundles

$$K_i/H \hookrightarrow G/H \to G/K_i$$

- In the manifold case K_i/H is a unit sphere, in the Alexandrov case it is a positively curved homogeneous space.

Cohomogeneity one Alexandrov spaces

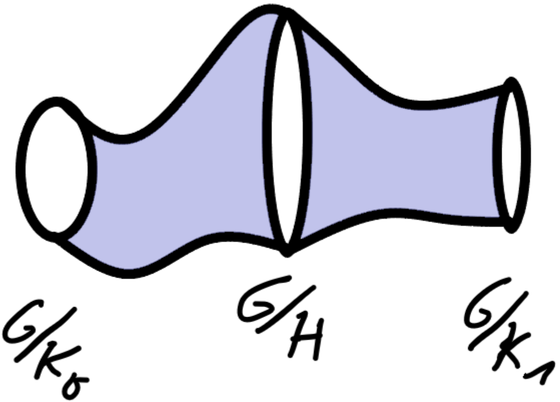
In analogy to cohomogeneity one manifolds there is a “double mapping cylinder decomposition” of cohomogeneity one Alexandrov spaces M (with orbit space a compact interval, i.e. not being a manifold).

- Let G act by cohomogeneity one. The orbit space is a closed interval, over its interior we find the principal orbits G/H of codimension 1, over the endpoints the singular/exotic orbits G/K_0 and G/K_1.
- Due to the slice theorem the normal cones (corresponding to normal disc bundles in the manifold setting) over G/K_0 and G/K_1 have common boundary G/H.
- We glue them along this boundary to obtain M. We obtain bundles

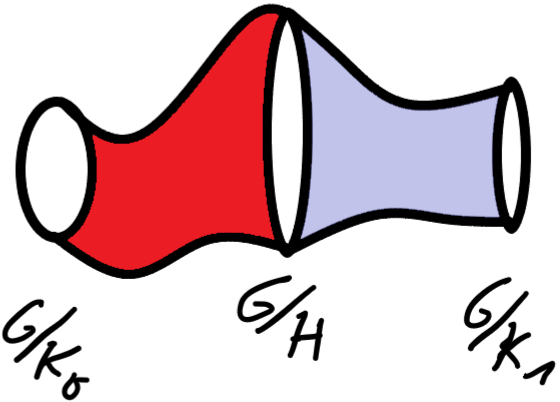
$$K_i/H \hookrightarrow G/H \to G/K_i$$

- In the manifold case K_i/H is a unit sphere, in the Alexandrov case it is a positively curved homogeneous space. These are classified, but provide a much richer setting than just spheres in the manifold case!

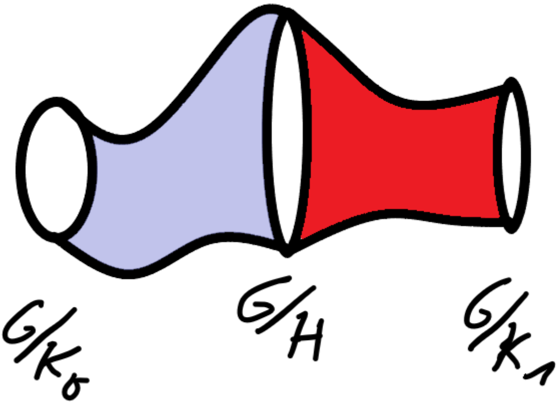
Cohomogeneity one Alexandrov spaces



Cohomogeneity one Alexandrov spaces



Cohomogeneity one Alexandrov spaces

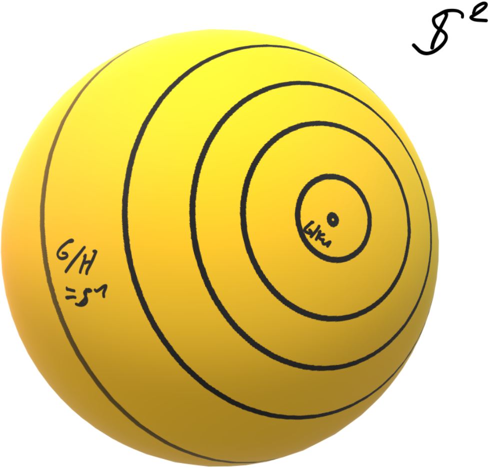


Cohomogeneity one Alexandrov spaces

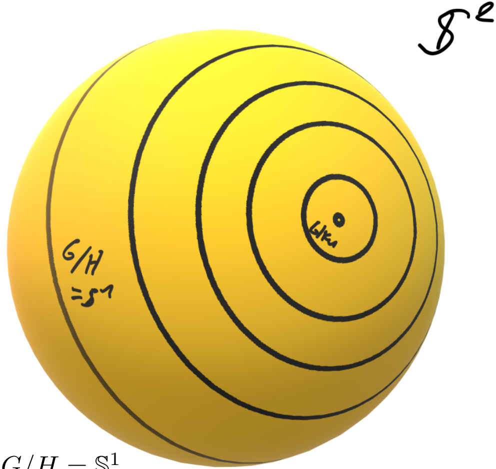
$G = \mathbb{S}^1$, $K_0 = K_1 = \mathbb{S}^1$, $H = \{e\}$

Cohomogeneity one Alexandrov spaces

$G = \mathbb{S}^1,\ K_0 = K_1 = \mathbb{S}^1,\ H = \{e\}$



$G = \mathbb{S}^1,\ K_0 = K_1 = \mathbb{S}^1,\ H = \{e\}$



principal orbit: $G/H = \mathbb{S}^1$,
singular orbit: $G/K_i = \{e\}$,
normal fibre: $K_i/H = \mathbb{S}^1$

Equivariant Formality

Let us bring in topology to this setting. Recall the definition of equivariant cohomology for $G \curvearrowright M$ as the cohomology

$$H^*_G(M) := H^*(M_G)$$

of the Borel construction $M_G = M \times_G \mathbf{E}G$

Equivariant Formality

Let us bring in topology to this setting. Recall the definition of equivariant cohomology for $G \curvearrowright M$ as the cohomology

$$H^*_G(M) := H^*(M_G)$$

of the Borel construction $M_G = M \times_G \mathbf{E}G$
with *Borel fibration*

$$M \hookrightarrow M_G \to \mathbf{B}G = \mathbf{E}G/G$$

Definition

The action $G \curvearrowright M$ is called equivariantly formal if there is a module isomorphism $H^*(M_G) \cong H^*(M) \otimes H^*(\mathbf{B}G)$.

Equivariant Formality

Let us bring in topology to this setting. Recall the definition of equivariant cohomology for $G \curvearrowright M$ as the cohomology

$$H^*_G(M) := H^*(M_G)$$

of the Borel construction $M_G = M \times_G \mathbf{E}G$
with *Borel fibration*

$$M \hookrightarrow M_G \to \mathbf{B}G = \mathbf{E}G/G$$

Definition

The action $G \curvearrowright M$ is called equivariantly formal if there is a module isomorphism $H^*(M_G) \cong H^*(M) \otimes H^*(\mathbf{B}G)$.

Remark

This is a highly prominent condition allowing for many different examples like torus actions on simply-connected Kähler manifolds or Hamiltonian torus actions.

Cohen–Macaulay actions

Let us generalise this:

Cohen–Macaulay actions

Let us generalise this: An action $G \curvearrowright X$ (then applied to the G-cohomogeneity one-Alexandrov space X) is called Cohen–Macaulay if

$$\dim_{H^*(\mathbf{B}G)} H^*_G(X) = \operatorname{depth} H^*_G(X)$$

Cohen–Macaulay actions

Let us generalise this: An action $G \curvearrowright X$ (then applied to the G-cohomogeneity one-Alexandrov space X) is called Cohen–Macaulay if

$$\dim_{H^*(\mathbf{B}G)} H^*_G(X) = \operatorname{depth} H^*_G(X)$$

That is, the Krull dimension of $H^*(\mathbf{B}G)/\operatorname{Ann}(H^*_G(X))$ equals the length of a maximal regular sequence of $H^*_G(X)$.

Let us generalise this: An action $G \curvearrowright X$ (then applied to the G-cohomogeneity one-Alexandrov space X) is called Cohen–Macaulay if

$$\dim_{H^*(\mathbf{B}G)} H^*_G(X) = \operatorname{depth} H^*_G(X)$$

That is, the Krull dimension of $H^*(\mathbf{B}G)/\operatorname{Ann}(H^*_G(X))$ equals the length of a maximal regular sequence of $H^*_G(X)$.

"Forgetting the free part, we act with fixed-points."

Cohen–Macaulay actions

Remark

- *It is easy to see that an equivariantly formal action is Cohen–Macaulay.*

Cohen–Macaulay actions

Remark

- *It is easy to see that an equivariantly formal action is Cohen–Macaulay.*
- *The G-action on a cohomogeneity one manifold is known to be Cohen–Macaulay.*

Cohen–Macaulay actions

Remark

- *It is easy to see that an equivariantly formal action is Cohen–Macaulay.*
- *The G-action on a cohomogeneity one manifold is known to be Cohen–Macaulay.*
- *Together with Leopold Zoller we recently suggested two further variants of equivariant formality: $\mathcal{MOD}$-formality and actions of formal core (and prove the toral rank conjecture and a version of the maximal symmetry rank conjecture in non-negative curvature for them).*

Inclusions are denoted by $\iota_i \colon H \to K_i$.

Theorem (A., Zarei)

Let X be a closed simply-connected Alexandrov space and G be a compact connected Lie group which acts on X by cohomogeneity one with a group diagram (G, H, K_0, K_1), where the classifying spaces of the isotropy groups H, K_0, and K_1 are Sullivan spaces. Then $H^*_G(X;\mathbb{Q})$ is a Cohen–Macaulay $H^*(\mathbf{B}G;\mathbb{Q})$-module if and only if one of the following statements holds.

1. $\operatorname{rk} H = \operatorname{rk} K_0 = \operatorname{rk} K_1$.
2. $\operatorname{rk} H < \max\{\operatorname{rk} K_0, \operatorname{rk} K_1\}$ *and*

$$\operatorname{im} H^*(\mathbf{B}\iota_0) + \operatorname{im} H^*(\mathbf{B}\iota_1) = H^*(\mathbf{B}H;\mathbb{Q})$$

Cohen–Macaulay cohomogeneity one Alexandrov spaces

Remark

- *The theorem comprises the orbifold case!*

Remark

- *The theorem comprises the orbifold case!*
- *We extended the known rational homotopy theory of homogeneous spaces in order to incorporate non-connected stabiliser groups (leading to the condition of "Sullivan spaces").*

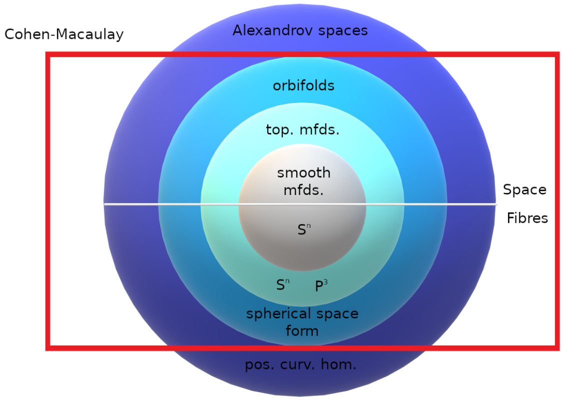
Remark

- *The theorem comprises the orbifold case!*
- *We extended the known rational homotopy theory of homogeneous spaces in order to incorporate non-connected stabiliser groups (leading to the condition of "Sullivan spaces").*
- *We prove that if X is a cohomogeneity one Alexandrov space of $\mathrm{curv} \geq 1$, then X is Cohen–Macaulay if and only if it is equivariantly formal provided that $\chi(X) \neq 0$ in the case when $\dim X$ is odd.*

Remark

- *The theorem comprises the orbifold case!*
- *We extended the known rational homotopy theory of homogeneous spaces in order to incorporate non-connected stabiliser groups (leading to the condition of "Sullivan spaces").*
- *We prove that if X is a cohomogeneity one Alexandrov space of $\mathrm{curv} \geq 1$, then X is Cohen–Macaulay if and only if it is equivariantly formal provided that $\chi(X) \neq 0$ in the case when $\dim X$ is odd.*
- *Using the join construction we can provide several examples of non-Cohen–Macaulay Alexandrov spaces.*

Cohen–Macaulay cohomogeneity one Alexandrov spaces



Rational Ellipticity of Cohomogeneity One Alexandrov Spaces

Rational ellipticity

Definition

A nilpotent space X *is rationally elliptic if* $\dim \pi_*(X) \otimes \mathbb{Q} < \infty$.

Rational ellipticity

Definition

A nilpotent space X is rationally elliptic if $\dim \pi_*(X) \otimes \mathbb{Q} < \infty$.

Theorem (Grove–Halperin)

Cohomogeneity one manifolds are rationally elliptic.

Rational ellipticity

Definition

A nilpotent space X is rationally elliptic if $\dim \pi_*(X) \otimes \mathbb{Q} < \infty$.

Theorem (Grove–Halperin)

Cohomogeneity one manifolds are rationally elliptic.

Remark

If $K_0/H = K_1/H = \mathbb{S}^1$, the cohomogeneity one manifold is known to admit non-negative sectional curvature.

Rational ellipticity

Bott–Grove–Halperin speculated:

Conjecture

Non-negatively curved manifolds are rationally elliptic.

Rational ellipticity

Bott–Grove–Halperin speculated:

Conjecture

Non-negatively curved manifolds are rationally elliptic.

Remark

Hence, this is true for cohomogeneity one manifolds.

Rational ellipticity

Bott–Grove–Halperin speculated:

Conjecture

Non-negatively curved manifolds are rationally elliptic.

Remark

Hence, this is true for cohomogeneity one manifolds.
It is obviously wrong for cohomogeneity one Alexandrov spaces, since, for example,

$$H^*(\Sigma\mathbb{C}\mathbf{P}^2) = \Lambda\langle x, y\rangle/_{xy=0}$$

($\deg x = 3, \deg x = 5$) *and the Euler characteristic of the suspension* $\Sigma\mathbb{C}\mathbf{P}^2$ *of* $\mathbb{C}\mathbf{P}^2$ *is negative.*

Rational ellipticity

Bott–Grove–Halperin speculated:

Conjecture

Non-negatively curved manifolds are rationally elliptic.

Remark

Hence, this is true for cohomogeneity one manifolds.
It is obviously wrong for cohomogeneity one Alexandrov spaces, since, for example,

$$H^*(\Sigma\mathbb{C}\mathbf{P}^2) = \Lambda\langle x, y\rangle/_{xy=0}$$

($\deg x = 3, \deg x = 5$) and the Euler characteristic of the suspension $\Sigma\mathbb{C}\mathbf{P}^2$ of $\mathbb{C}\mathbf{P}^2$ is negative.
This is an Alexandrov space of positive curvature!

Rational ellipticity

Theorem (A., Galaz-García, Zarei)

Let (G, K_0, K_1, H) be a group diagram of connected Lie groups of the cohomogeneity one Alexandrov space X. Then X is nilpotent, and it is rationally elliptic if and only if, without restriction, either

- *X is a smooth manifold, or*
- *K_0/H rationally is an odd-dimensional sphere (and actually a sphere out of dimension 7).*

Equivariant formality of $\mathbb{Z}_2 \oplus \mathbb{Z}_2$-symmetric spaces

Equivariant formality of the isotropy action

Let us finally provide a result for equivariant formality on certain homogeneous manifolds.

Equivariant formality of the isotropy action

Let us finally provide a result for equivariant formality on certain homogeneous manifolds.

Conjecture

Let G be a compact connected Lie group and let σ be an abelian Lie group of automorphisms of G. Then the isotropy action on G/G_0^σ, where G_0^σ denotes the identity component of the fixed point set of σ, is equivariantly formal.

Equivariant formality of the isotropy action

Let us finally provide a result for equivariant formality on certain homogeneous manifolds.

Conjecture

Let G be a compact connected Lie group and let σ be an abelian Lie group of automorphisms of G. Then the isotropy action on G/G_0^σ, where G_0^σ denotes the identity component of the fixed point set of σ, is equivariantly formal.

We extend our result to the following which was independently discovered and proved by totally different techniques by Noshari.

Equivariant formality of the isotropy action

Let us finally provide a result for equivariant formality on certain homogeneous manifolds.

Conjecture

Let G be a compact connected Lie group and let σ be an abelian Lie group of automorphisms of G. Then the isotropy action on G/G_0^σ, where G_0^σ denotes the identity component of the fixed point set of σ, is equivariantly formal.

We extend our result to the following which was independently discovered and proved by totally different techniques by Noshari.

Theorem (A.–Kollross, Noshari)

The conjecture holds whenever $|\sigma| \leq 7$.

Equivariant formality of the isotropy action

Let us finally provide a result for equivariant formality on certain homogeneous manifolds.

Conjecture

Let G be a compact connected Lie group and let σ be an abelian Lie group of automorphisms of G. Then the isotropy action on G/G_0^σ, where G_0^σ denotes the identity component of the fixed point set of σ, is equivariantly formal.

We extend our result to the following which was independently discovered and proved by totally different techniques by Noshari.

Theorem (A.–Kollross, Noshari)

The conjecture holds whenever $|\sigma| \leq 7$.

In particular, in this situation equivariant formality of the isotropy action implies formality of G/G_0^σ.

Conclusion

Remark

Despite all three setting/results/proofs being rather different, we can provide one common underlying tool which, in the field of equivariant cohomology, provides an effective new approach to the area:

Conclusion

Remark

Despite all three setting/results/proofs being rather different, we can provide one common underlying tool which, in the field of equivariant cohomology, provides an effective new approach to the area:

In each case the construction of a (respective/distinct) rational model allows for concrete computations and sometimes endows you with a...

Conclusion

Remark

Despite all three setting/results/proofs being rather different, we can provide one common underlying tool which, in the field of equivariant cohomology, provides an effective new approach to the area:

In each case the construction of a (respective/distinct) rational model allows for concrete computations and sometimes endows you with a...

"better grasp on buried maths"

Thank you very much