Bisectors and foliations in the complex hyperbolic space

Maciej Czarnecki

Uniwersytet Łódzki, Łódź, Poland

Symmetry and shape
Universidade de Santiago de Compostela, Spain
October 29, 2019

Summary

Summary

1. Bisectors in complex hyperbolic spaces
2. Complex cross–ratio and Goldman invariant
3. Separating bisectors
4. Representation in de Sitter space

Summary

1. Bisectors in complex hyperbolic spaces
2. Complex cross–ratio and Goldman invariant
3. Separating bisectors
4. Representation in de Sitter space

Summary

1. Bisectors in complex hyperbolic spaces
2. Complex cross–ratio and Goldman invariant
3. Separating bisectors
4. Representation in de Sitter space

Summary

1. Bisectors in complex hyperbolic spaces
2. Complex cross–ratio and Goldman invariant
3. Separating bisectors
4. Representation in de Sitter space

Complex hyperbolic distance

Definition

For the Hermitian form $\langle X|Y\rangle = X_1\overline{Y_1} + \ldots + X_n\overline{Y_n} - X_{n+1}\overline{Y_{n+1}}$ in $\mathbb{C}^{n+1}$ we define *n–dimensional complex hyperbolic space* as projectivization of negative vectors i.e.

$$\mathbb{C}H^n = \left\{X \in \mathbb{C}^{n+1} \mid \langle X|X\rangle < 0\right\}/\mathbb{C}^*$$

and its ideal boundary $\mathbb{C}H^n(\infty)$ as projectivization of null vectors.

The Bergman metric makes $\mathbb{C}H^n$ an Hadamard manifold of sectional curvature between $-1/4$ and -1 and the distance given by

$$\cosh^2\frac{d(x,y)}{2} = \frac{\langle X|Y\rangle\langle Y|X\rangle}{\langle X|X\rangle\langle Y|Y\rangle}.$$

Complex hyperbolic distance

Definition

For the Hermitian form $\langle X|Y\rangle = X_1\overline{Y_1} + \ldots + X_n\overline{Y_n} - X_{n+1}\overline{Y_{n+1}}$ in $\mathbb{C}^{n+1}$ we define *n–dimensional complex hyperbolic space* as projectivization of negative vectors i.e.

$$\mathbb{C}H^n = \left\{X \in \mathbb{C}^{n+1} \mid \langle X|X\rangle < 0\right\}/\mathbb{C}^*$$

and its ideal boundary $\mathbb{C}H^n(\infty)$ as projectivization of null vectors.

The Bergman metric makes $\mathbb{C}H^n$ an Hadamard manifold of sectional curvature between $-1/4$ and -1 and the distance given by

$$\cosh^2\frac{d(x,y)}{2} = \frac{\langle X|Y\rangle\langle Y|X\rangle}{\langle X|X\rangle\langle Y|Y\rangle}.$$

Complex geodesics and complex hyperplanes

A *complex geodesic* is the projectivization of a vector space in $\mathbb{C}^{n+1}$ spanned by two linearly indpent negative vectors. It is isometric to real hyperbolic plane $\mathbb{R}H^2$.

A *complex hyperplane* is the projectivization of a vector space in $\mathbb{C}^{n+1}$ spanned by n linearly indpent negative vectors. It is isometric to $\mathbb{C}H^{n-1}$ and orthogonal to a unit positive vector (its *polar vector*)

Proposition

Let H_1 and H_2 be complex hyperplanes in $\mathbb{C}H^n$ with polar vectors C_1 and C_2. Then

1. $H_1 \cap H_2 = \emptyset$ *iff* $|\langle C_1 | C_2 \rangle| > 1$.
2. $\angle(H_1, H_2) = \alpha$ *iff* $|\langle C_1 | C_2 \rangle| = \cos \alpha$.

Complex geodesics and complex hyperplanes

A *complex geodesic* is the projectivization of a vector space in $\mathbb{C}^{n+1}$ spanned by two linearly indpent negative vectors. It is isometric to real hyperbolic plane $\mathbb{R}H^2$.

A *complex hyperplane* is the projectivization of a vector space in $\mathbb{C}^{n+1}$ spanned by n linearly indpent negative vectors. It is isometric to $\mathbb{C}H^{n-1}$ and orthogonal to a unit positive vector (its *polar vector*)

Proposition

Let H_1 and H_2 be complex hyperplanes in $\mathbb{C}H^n$ with polar vectors C_1 and C_2. Then

1. $H_1 \cap H_2 = \emptyset$ *iff* $|\langle C_1 | C_2 \rangle| > 1$.
2. $\angle(H_1, H_2) = \alpha$ *iff* $|\langle C_1 | C_2 \rangle| = \cos \alpha$.

Complex geodesics and complex hyperplanes

A *complex geodesic* is the projectivization of a vector space in $\mathbb{C}^{n+1}$ spanned by two linearly indpent negative vectors. It is isometric to real hyperbolic plane $\mathbb{R}H^2$.

A *complex hyperplane* is the projectivization of a vector space in $\mathbb{C}^{n+1}$ spanned by n linearly indpent negative vectors. It is isometric to $\mathbb{C}H^{n-1}$ and orthogonal to a unit positive vector (its *polar vector*)

Proposition

Let H_1 and H_2 be complex hyperplanes in $\mathbb{C}H^n$ with polar vectors C_1 and C_2. Then

1. $H_1 \cap H_2 = \emptyset$ *iff* $|\langle C_1 | C_2 \rangle| > 1$.
2. $\angle(H_1, H_2) = \alpha$ *iff* $|\langle C_1 | C_2 \rangle| = \cos\alpha$.

Bisectors

Definition

For $z_1, z_2 \in \mathbb{C}H^n$ we define a *bisector* as an equidistant from z_1 and z_2

$$\mathfrak{E}(z_1, z_2) = \{z \mid d(z, z_1) = d(z, z_2)\}.$$

Bisectors are in one-to-one correspondence with pairs of points on the ideal boundary $\mathbb{C}H^n(\infty)$. These points (called *vertices* of bisector) are ends of the unique geodesic line through z_1 and z_2.

For the bisector $\mathfrak{E}$ of vertices p and q we call the geodesic line σ a *spine* while the complex geodesic

$$\Sigma = \mathrm{span}_{\mathbb{C}}(p, q) \cap \mathbb{C}H^n \simeq \mathbb{C}H^1 \simeq \mathbb{R}H^2$$

a *complex spine*. Observe that $\mathfrak{E} \cap \Sigma = \sigma$.

Bisectors

Definition

For $z_1, z_2 \in \mathbb{C}H^n$ we define a *bisector* as an equidistant from z_1 and z_2

$$\mathfrak{E}(z_1, z_2) = \{z \mid d(z, z_1) = d(z, z_2)\}.$$

Bisectors are in one-to-one correspondence with pairs of points on the ideal boundary $\mathbb{C}H^n(\infty)$. These points (called *vertices* of bisector) are ends of the unique geodesic line through z_1 and z_2.

For the bisector $\mathfrak{E}$ of vertices p and q we call the geodesic line σ a *spine* while the complex geodesic

$$\Sigma = \operatorname{span}_{\mathbb{C}}(p, q) \cap \mathbb{C}H^n \simeq \mathbb{C}H^1 \simeq \mathbb{R}H^2$$

a *complex spine*. Observe that $\mathfrak{E} \cap \Sigma = \sigma$.

Bisectors

Definition

For $z_1, z_2 \in \mathbb{C}H^n$ we define a *bisector* as an equidistant from z_1 and z_2

$$\mathfrak{E}(z_1, z_2) = \{z \mid d(z, z_1) = d(z, z_2)\}.$$

Bisectors are in one-to-one correspondence with pairs of points on the ideal boundary $\mathbb{C}H^n(\infty)$. These points (called *vertices* of bisector) are ends of the unique geodesic line through z_1 and z_2.

For the bisector $\mathfrak{E}$ of vertices p and q we call the geodesic line σ a *spine* while the complex geodesic

$$\Sigma = \mathrm{span}_{\mathbb{C}}(p, q) \cap \mathbb{C}H^n \simeq \mathbb{C}H^1 \simeq \mathbb{R}H^2$$

a *complex spine*. Observe that $\mathfrak{E} \cap \Sigma = \sigma$.

Properties of bisectors

1. A bisector is a real analytic fibration over its spine with respect to the orthogonal projection onto the complex spine $\mathfrak{E} = \bigcup_{z \in \sigma} \Pi_{\Sigma}^{-1}(z)$ (*slice decomposition*).
2. For $z \in \mathbb{C}H^n$ the bisector $\mathfrak{E}$ is equidistant from z iff $z \in \Sigma \setminus \sigma$.
3. A bisector is a real hypersurface which is Hadamard and even in $\mathbb{C}H^2$ it has 3 distinct principal curvatures: -1, $-1/4$ and some between $-1/2$ and $-1/4$.
4. Every two bisectors are congruent

Observe that in case of $\mathbb{R}H^n$ all these properties trivialize — bisectors are totally geodesic.

Properties of bisectors

1. A bisector is a real analytic fibration over its spine with respect to the orthogonal projection onto the complex spine $\mathfrak{E} = \bigcup_{z\in\sigma} \Pi_\Sigma^{-1}(z)$ (*slice decomposition*).
2. For $z \in \mathbb{C}H^n$ the bisector $\mathfrak{E}$ is equidistant from z iff $z \in \Sigma \setminus \sigma$.
3. A bisector is a real hypersurface which is Hadamard and even in $\mathbb{C}H^2$ it has 3 distinct principal curvatures: -1, $-1/4$ and some between $-1/2$ and $-1/4$.
4. Every two bisectors are congruent

Observe that in case of $\mathbb{R}H^n$ all these properties trivialize — bisectors are totally geodesic.

Properties of bisectors

1. A bisector is a real analytic fibration over its spine with respect to the orthogonal projection onto the complex spine $\mathfrak{E} = \bigcup_{z\in\sigma} \Pi_{\Sigma}^{-1}(z)$ (*slice decomposition*).
2. For $z \in \mathbb{C}H^n$ the bisector $\mathfrak{E}$ is equidistant from z iff $z \in \Sigma \setminus \sigma$.
3. A bisector is a real hypersurface which is Hadamard and even in $\mathbb{C}H^2$ it has 3 distinct principal curvatures: -1, $-1/4$ and some between $-1/2$ and $-1/4$.
4. Every two bisectors are congruent

Observe that in case of $\mathbb{R}H^n$ all these properties trivialize — bisectors are totally geodesic.

Properties of bisectors

1. A bisector is a real analytic fibration over its spine with respect to the orthogonal projection onto the complex spine $\mathfrak{E} = \bigcup_{z\in\sigma} \Pi_{\Sigma}^{-1}(z)$ (*slice decomposition*).
2. For $z \in \mathbb{C}H^n$ the bisector $\mathfrak{E}$ is equidistant from z iff $z \in \Sigma \setminus \sigma$.
3. A bisector is a real hypersurface which is Hadamard and even in $\mathbb{C}H^2$ it has 3 distinct principal curvatures: -1, $-1/4$ and some between $-1/2$ and $-1/4$.
4. Every two bisectors are congruent

Observe that in case of $\mathbb{R}H^n$ all these properties trivialize — bisectors are totally geodesic.

Properties of bisectors

1. A bisector is a real analytic fibration over its spine with respect to the orthogonal projection onto the complex spine $\mathfrak{E} = \bigcup_{z\in\sigma} \Pi_{\Sigma}^{-1}(z)$ (*slice decomposition*).
2. For $z \in \mathbb{C}H^n$ the bisector $\mathfrak{E}$ is equidistant from z iff $z \in \Sigma \setminus \sigma$.
3. A bisector is a real hypersurface which is Hadamard and even in $\mathbb{C}H^2$ it has 3 distinct principal curvatures: -1, $-1/4$ and some between $-1/2$ and $-1/4$.
4. Every two bisectors are congruent

Observe that in case of $\mathbb{R}H^n$ all these properties trivialize — bisectors are totally geodesic.

Spine and polar vectors of slices

Assume that a bisector $\mathfrak{E}$ has vertices p and q represented by such null vectors that $\langle P|Q\rangle = -2$. Then

1. its spine σ is parametrized by arc–length as
$$\gamma(t) = \frac{1}{2}\left(e^{-\frac{t}{2}}P + e^{\frac{t}{2}}Q\right)$$
2. a polar vector to a slice of $\mathfrak{E}$ at $\gamma(t)$ is
$$C(t) = \frac{1}{2}\left(e^{-\frac{t}{2}}P - e^{\frac{t}{2}}Q\right)$$

Spine and polar vectors of slices

Assume that a bisector $\mathfrak{E}$ has vertices p and q represented by such null vectors that $\langle P|Q\rangle = -2$. Then

1. its spine σ is parametrized by arc–length as

$$\gamma(t) = \frac{1}{2}\left(e^{-\frac{t}{2}}P + e^{\frac{t}{2}}Q\right)$$

2. a polar vector to a slice of $\mathfrak{E}$ at $\gamma(t)$ is

$$C(t) = \frac{1}{2}\left(e^{-\frac{t}{2}}P - e^{\frac{t}{2}}Q\right)$$

Spine and polar vectors of slices

Assume that a bisector $\mathfrak{E}$ has vertices p and q represented by such null vectors that $\langle P|Q\rangle = -2$. Then

1. its spine σ is parametrized by arc–length as

$$\gamma(t) = \frac{1}{2}\left(e^{-\frac{t}{2}}P + e^{\frac{t}{2}}Q\right)$$

2. a polar vector to a slice of $\mathfrak{E}$ at $\gamma(t)$ is

$$C(t) = \frac{1}{2}\left(e^{-\frac{t}{2}}P - e^{\frac{t}{2}}Q\right)$$

Bisector foliations

Definition

A *bisector foliation* in $\mathbb{C}H^n$ is a foliation of all the leaves being bisectors.

By the slice decomposition every bisector foliation decomposes in a (real) codimension 2 totally geodesic foliation of $\mathbb{C}H^n$.

Theorem (Cz, P. Walczak 2006, based on Ferus 1973)

Every cospinal (i.e. having one common complex spine Σ of leaves) bisector foliations in $\mathbb{C}H^n$ is that of bisectors of (real) spines in $\Sigma \simeq \mathbb{R}H^2$ orthogonal to a curve of geodesic curvature ≤ 1.

Bisector foliations

Definition

A *bisector foliation* in $\mathbb{C}H^n$ is a foliation of all the leaves being bisectors.

By the slice decomposition every bisector foliation decomposes in a (real) codimension 2 totally geodesic foliation of $\mathbb{C}H^n$.

Theorem (Cz, P. Walczak 2006, based on Ferus 1973)

Every cospinal (i.e. having one common complex spine Σ of leaves) bisector foliations in $\mathbb{C}H^n$ is that of bisectors of (real) spines in $\Sigma \simeq \mathbb{R}H^2$ orthogonal to a curve of geodesic curvature ≤ 1.

Bisector foliations

Definition

A *bisector foliation* in $\mathbb{C}H^n$ is a foliation of all the leaves being bisectors.

By the slice decomposition every bisector foliation decomposes in a (real) codimension 2 totally geodesic foliation of $\mathbb{C}H^n$.

Theorem (Cz, P. Walczak 2006, based on Ferus 1973)

Every cospinal (i.e. having one common complex spine Σ of leaves) bisector foliations in $\mathbb{C}H^n$ is that of bisectors of (real) spines in $\Sigma \simeq \mathbb{R}H^2$ orthogonal to a curve of geodesic curvature ≤ 1.

Complex cross–ratio and Goldman invariant

Definition

A *Korányi–Reimann complex cross–ratio* assigns to a quadruple of points $x_1, x_2, x_3, x_4 \in \mathbb{C}H^n(\infty)$ a number

$$[x_1, x_2, x_3, x_4] = \frac{\langle X_3|X_1\rangle\langle X_4|X_2\rangle}{\langle X_4|X_1\rangle\langle X_3|X_2\rangle}$$

Definition

For a bisector $\mathfrak{E}$ of vertices p and q and a complex hyperplane H with polar vector C we define a *Goldman invariant* by

$$\eta(\mathfrak{E}, H) = \eta(p, q, c) = \frac{\langle P|C\rangle\langle C|Q\rangle}{\langle P|Q\rangle\langle C|C\rangle}$$

Complex cross–ratio and Goldman invariant

Definition

A *Korányi–Reimann complex cross–ratio* assigns to a quadruple of points $x_1, x_2, x_3, x_4 \in \mathbb{C}H^n(\infty)$ a number

$$[x_1, x_2, x_3, x_4] = \frac{\langle X_3|X_1\rangle\langle X_4|X_2\rangle}{\langle X_4|X_1\rangle\langle X_3|X_2\rangle}$$

Definition

For a bisector $\mathfrak{E}$ of vertices p and q and a complex hyperplane H with polar vector C we define a *Goldman invariant* by

$$\eta(\mathfrak{E}, H) = \eta(p, q, c) = \frac{\langle P|C\rangle\langle C|Q\rangle}{\langle P|Q\rangle\langle C|C\rangle}$$

Metric properties of $[.,.,.,.]$ and η

Theorem (Goldman, Mostow)

Let η be a Goldman invariant for a bisector $\mathfrak{E}$ and a complex hyperplane H. Then $\mathfrak{E} \cap H = \emptyset$ iff $(\operatorname{Im}\eta)^2 + 2\operatorname{Re}\eta \geq 1$.

Thus a condition for separating bisectors as functions of their ends?

No, because we obtain an equation of degree 8 involving cross–ratios of ends. Even in case case of distance of geodesics it could be unsolvable (M. Sandler example).

If we restrict to $n = 2$ the following formula would be useful

Theorem (Parker)

Let σ_1 and σ_2 be geodesic lines in $\mathbb{C}H^2$ of ends p_1, q_1 and p_2, q_2 respectively. Then

$$d(\sigma_1, \sigma_2) \geq |[p_2, q_1, p_1, q_2]| + |[q_2, q_1, p_1, p_2]|$$

Theorem (Goldman, Mostow)

Let η be a Goldman invariant for a bisector $\mathfrak{E}$ and a complex hyperplane H. Then $\mathfrak{E} \cap H = \emptyset$ iff $(\operatorname{Im} \eta)^2 + 2 \operatorname{Re} \eta \geq 1$.

Thus a condition for separating bisectors as functions of their ends?

No, because we obtain an equation of degree 8 involving cross–ratios of ends. Even in case case of distance of geodesics it could be unsolvable (M. Sandler example).

If we restrict to $n = 2$ the following formula would be useful

Theorem (Parker)

Let σ_1 and σ_2 be geodesic lines in $\mathbb{C}H^2$ of ends p_1, q_1 and p_2, q_2 respectively. Then

$$d(\sigma_1, \sigma_2) \geq |[p_2, q_1, p_1, q_2]| + |[q_2, q_1, p_1, p_2]|$$

Metric properties of $[.,.,.,.]$ and η

Theorem (Goldman, Mostow)

Let η be a Goldman invariant for a bisector $\mathfrak{E}$ and a complex hyperplane H. Then $\mathfrak{E} \cap H = \emptyset$ iff $(\operatorname{Im}\eta)^2 + 2\operatorname{Re}\eta \geq 1$.

Thus a condition for separating bisectors as functions of their ends?

No, because we obtain an equation of degree 8 involving cross–ratios of ends. Even in case case of distance of geodesics it could be unsolvable (M. Sandler example).

If we restrict to $n = 2$ the following formula would be useful

Theorem (Parker)

Let σ_1 and σ_2 be geodesic lines in $\mathbb{C}H^2$ of ends p_1, q_1 and p_2, q_2 respectively. Then

$$d(\sigma_1, \sigma_2) \geq |[p_2, q_1, p_1, q_2]| + |[q_2, q_1, p_1, p_2]|$$

Metric properties of $[.,.,.,.]$ and η

Theorem (Goldman, Mostow)

Let η be a Goldman invariant for a bisector $\mathfrak{E}$ and a complex hyperplane H. Then $\mathfrak{E} \cap H = \emptyset$ iff $(\mathrm{Im}\,\eta)^2 + 2\,\mathrm{Re}\,\eta \geq 1$.

Thus a condition for separating bisectors as functions of their ends?

No, because we obtain an equation of degree 8 involving cross–ratios of ends. Even in case case of distance of geodesics it could be unsolvable (M. Sandler example).

If we restrict to $n = 2$ the following formula would be useful

Theorem (Parker)

Let σ_1 and σ_2 be geodesic lines in $\mathbb{C}H^2$ of ends p_1, q_1 and p_2, q_2 respectively. Then

$$d(\sigma_1, \sigma_2) \geq |[p_2, q_1, p_1, q_2]| + |[q_2, q_1, p_1, p_2]|$$

Local separation of bisectors

In $\mathbb{C}H^2$ every complex geodesic is a complex hyperplane. For given bisectors $\mathfrak{E}_j$ of vertices p_j, q_j we define their spines σ_j, complex spines Σ_j, and polar vectors C_j, $j = 1, 2$.

1. Taking such representatives of p's and q's that $\langle P_j | Q_j \rangle = -2$ we have $C_j = \frac{1}{4} P_j \boxtimes Q_j$ where $\boxtimes$ denotes Hermitian cross–product in $\mathbb{C}^3$.
2. Assume that complex hyperbolic reflection along $C_1 - C_2$ sends σ_2 onto geodesic disjoint with σ_1.
3. Then using complex hyperbolic trigonometry we find such $k = k(\angle(C_1, C_2))$ that $d(\mathfrak{E}_1, \mathfrak{E}_2) \geq kd(\sigma_1, \sigma_2)$ for the angle small enough.
4. Thus in terms of vertices of bisectors only (Parker's formula) we expressed separations of close bisectors. This is in fact enough for local condition on bisector foliation.

Local separation of bisectors

In $\mathbb{C}H^2$ every complex geodesic is a complex hyperplane. For given bisectors $\mathfrak{E}_j$ of vertices p_j, q_j we define their spines σ_j, complex spines Σ_j, and polar vectors C_j, $j = 1, 2$.

1. Taking such representatives of p's and q's that $\langle P_j | Q_j \rangle = -2$ we have $C_j = \frac{1}{4} P_j \boxtimes Q_j$ where $\boxtimes$ denotes Hermitian cross–product in $\mathbb{C}^3$.
2. Assume that complex hyperbolic reflection along $C_1 - C_2$ sends σ_2 onto geodesic disjoint with σ_1.
3. Then using complex hyperbolic trigonometry we find such $k = k(\angle(C_1, C_2))$ that $d(\mathfrak{E}_1, \mathfrak{E}_2) \geq k d(\sigma_1, \sigma_2)$ for the angle small enough.
4. Thus in terms of vertices of bisectors only (Parker's formula) we expressed separations of close bisectors. This is in fact enough for local condition on bisector foliation.

Local separation of bisectors

In $\mathbb{C}H^2$ every complex geodesic is a complex hyperplane. For given bisectors $\mathfrak{E}_j$ of vertices p_j, q_j we define their spines σ_j, complex spines Σ_j, and polar vectors C_j, $j = 1, 2$.

1. Taking such representatives of p's and q's that $\langle P_j | Q_j \rangle = -2$ we have $C_j = \frac{1}{4} P_j \boxtimes Q_j$ where $\boxtimes$ denotes Hermitian cross–product in $\mathbb{C}^3$.
2. Assume that complex hyperbolic reflection along $C_1 - C_2$ sends σ_2 onto geodesic disjoint with σ_1.
3. Then using complex hyperbolic trigonometry we find such $k = k(\angle(C_1, C_2))$ that $d(\mathfrak{E}_1, \mathfrak{E}_2) \geq kd(\sigma_1, \sigma_2)$ for the angle small enough.
4. Thus in terms of vertices of bisectors only (Parker's formula) we expressed separations of close bisectors. This is in fact enough for local condition on bisector foliation.

Local separation of bisectors

In $\mathbb{C}H^2$ every complex geodesic is a complex hyperplane. For given bisectors $\mathfrak{E}_j$ of vertices p_j, q_j we define their spines σ_j, complex spines Σ_j, and polar vectors C_j, $j = 1, 2$.

1. Taking such representatives of p's and q's that $\langle P_j | Q_j \rangle = -2$ we have $C_j = \frac{1}{4} P_j \boxtimes Q_j$ where $\boxtimes$ denotes Hermitian cross–product in $\mathbb{C}^3$.
2. Assume that complex hyperbolic reflection along $C_1 - C_2$ sends σ_2 onto geodesic disjoint with σ_1.
3. Then using complex hyperbolic trigonometry we find such $k = k(\angle(C_1, C_2))$ that $d(\mathfrak{E}_1, \mathfrak{E}_2) \geq kd(\sigma_1, \sigma_2)$ for the angle small enough.
4. Thus in terms of vertices of bisectors only (Parker's formula) we expressed separations of close bisectors. This is in fact enough for local condition on bisector foliation.

Local separation of bisectors

In $\mathbb{C}H^2$ every complex geodesic is a complex hyperplane. For given bisectors $\mathfrak{E}_j$ of vertices p_j, q_j we define their spines σ_j, complex spines Σ_j, and polar vectors C_j, $j = 1, 2$.

1. Taking such representatives of p's and q's that $\langle P_j | Q_j \rangle = -2$ we have $C_j = \frac{1}{4} P_j \boxtimes Q_j$ where $\boxtimes$ denotes Hermitian cross–product in $\mathbb{C}^3$.
2. Assume that complex hyperbolic reflection along $C_1 - C_2$ sends σ_2 onto geodesic disjoint with σ_1.
3. Then using complex hyperbolic trigonometry we find such $k = k(\angle(C_1, C_2))$ that $d(\mathfrak{E}_1, \mathfrak{E}_2) \geq k d(\sigma_1, \sigma_2)$ for the angle small enough.
4. Thus in terms of vertices of bisectors only (Parker's formula) we expressed separations of close bisectors. This is in fact enough for local condition on bisector foliation.

Representation in de Sitter space

(Real) de Sitter n–space Λ^n is a set of unit vectors in $\mathbb{R}^{n+1}$ with respect to the standard Lorentz form. Every oriented totally geodesic hypersurface in $\mathbb{R}H^n$ is represented by a unique point on Λ^n.

Theorem (Cz, Langevin 2013)

A continuous and unbounded curve Γ in Λ^n represents a totally geodesic codimension 1 foliation of $\mathbb{R}H^n$ iff at every point the tangent vector to Γ is time–like or light–like.

Description of bisector foliation in complex de Sitter space $\mathbb{C}\Lambda^n$ is much more complicated because every bisector is represented by a hyperbola. Thus we could follow conformal methods of studying Dupin foliation by Langevin and P. Walczak.

Representation in de Sitter space

(Real) de Sitter n–space Λ^n is a set of unit vectors in $\mathbb{R}^{n+1}$ with respect to the standard Lorentz form. Every oriented totally geodesic hypersurface in $\mathbb{R}H^n$ is represented by a unique point on Λ^n.

Theorem (Cz, Langevin 2013)

A continuous and unbounded curve Γ in Λ^n represents a totally geodesic codimension 1 foliation of $\mathbb{R}H^n$ iff at every point the tangent vector to Γ is time–like or light–like.

Description of bisector foliation in complex de Sitter space $\mathbb{C}\Lambda^n$ is much more complicated because every bisector is represented by a hyperbola. Thus we could follow conformal methods of studying Dupin foliation by Langevin and P. Walczak.

Representation in de Sitter space

(Real) de Sitter n–space Λ^n is a set of unit vectors in $\mathbb{R}^{n+1}$ with respect to the standard Lorentz form. Every oriented totally geodesic hypersurface in $\mathbb{R}H^n$ is represented by a unique point on Λ^n.

Theorem (Cz, Langevin 2013)

A continuous and unbounded curve Γ in Λ^n represents a totally geodesic codimension 1 foliation of $\mathbb{R}H^n$ iff at every point the tangent vector to Γ is time–like or light–like.

Description of bisector foliation in complex de Sitter space $\mathbb{C}\Lambda^n$ is much more complicated because every bisector is represented by a hyperbola. Thus we could follow conformal methods of studying Dupin foliation by Langevin and P. Walczak.

¡Moitas grazas!
Thank you!
¡Muchas gracias!

Spanish–Polish Mathematical Meeting

Łódź, September 6–10, 2021

RSME, SEMA, CSM
+ PTM (=Sociedad Matematica Polaca)