The transverse Jacobi equation

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Summary

1. The transverse Jacobi equation.
2. Comparison for the transverse Jacobi equation
3. Geometric applications.

The linear symplectic space of Jacobi fields

Joint work with Frederick Wilhelm (UCR).

(M^n, g) n-dimensional Riemannian manifold; $\gamma : \mathbb{R} \to M$ unit speed geodesic.

Definition

$\mathrm{Jac}(\gamma)$ *denotes the* $2(n-1)$*-dimensional vector space of normal Jacobi fields along* γ

$$\mathrm{Jac}(\gamma) := \{\, J \text{ Jacobi fields along } \gamma \,:\, J, J' \perp \gamma' \,\}.$$

There is a linear symplectic form

$$\omega : \mathrm{Jac}(\gamma) \times \mathrm{Jac}(\gamma) \to \mathbb{R}, \qquad \omega(X, Y) = \langle X', Y \rangle - \langle X, Y' \rangle$$

Lagrangian subspaces

Definition

Let $W \subset \mathrm{Jac}(\gamma)$ a linear subspace.

1. *W is isotropic if $\omega|_{W\times W} \equiv 0$;*
2. *L is Lagrangian if isotropic and maximal, i.e,* $\dim L = (n-1)$.

For a Lagrangian L, $\{J(t) : J \in L\} = \gamma'(t)^{\perp}$, except at isolated points.

Examples:

- Geodesic variations of γ leaving the initial point fixed;

$$L_0 := \{\, J \;:\; J(0) = 0 \,\}$$

Zeros: conjugate points.

- given $N \subset M$ a submanifold, γ orthogonal to N at $t = 0$,

$$L_N := \{\, J \;:\; J(0) \in T_{\gamma(0)}N, J'(0)^T + S_{\gamma'(0)}J(0) = 0 \,\}$$

Zeros: focal points of N.

Vertical/horizontal bundles

Choose $W \subset L$; at each $t \in \mathbb{R}$, let

$$W(t) := \{\, J(t) \,:\, J \in W \,\} \oplus \{\, J'(t) \,:\, J \in W, J(t) = 0 \,\}$$

$t \to W(t)$ is a smooth bundle, inducing a smooth splitting

$$\gamma'(t)^{\perp} = W(t) \oplus W(t)^{\perp} = W(t) \oplus H(t).$$

There is a covariant derivative of sections

$$\frac{D^{\perp}}{dt} : \Gamma(H) \to \Gamma(H), \qquad \frac{D^{\perp} Y}{dt} := Y'^{H}$$

Wilking's O'Neill's operators

Definition

Whenever possible,

- *choose* $J \in W$ *with* $J(t) = v$*, and define* $A_t : W(t) \to H(t)$ *as*

$$A_t(v) := J'^H(t),$$

- *and* $A_t^* : H(t) \to W(t)$ *as*

$$A_t^*(v) := J'^W(t).$$

A_t, A_t^* admit smooth extensions to all t.

The transverse Jacobi equation

Theorem (Wilking, 2007)

Let $L \subset \operatorname{Jac}(\gamma)$ Lagrangian, and $W \subset L$. Then for any $J \in L$, we have that for every $t \in \mathbb{R}$,

$$\frac{{D^{\perp}}^2 Y}{dt^2} + \{R(t)Y\}^H + 3A_t A_t^* Y = 0$$

where $Y = J^H$.

Example: Riemannian submersions

$\pi : M^{n+k} \to B^n$ Riemannian submersion.

$\gamma : \mathbb{R} \to M$ horizontal geodesic.

Projectable Jacobi fields $\mathcal{P}$: those arising from horizontally lifting to γ geodesic variations of $\bar{\gamma} := \pi \circ \gamma$ in B.

The holonomy vector fields are obtained lifting horizontally $\pi \circ \gamma$ to M:

$$W = \{\, J \in \mathcal{P} \ :\ J \text{ vertical } \,\}; \qquad \dim W = k$$

Any Lagrangian $L \subset \mathcal{P}$ contains W; H are the horizontal parts.

Wilking's transverse equation for L/W $\Leftrightarrow$ standard Jacobi equation for $\bar{\gamma}$ in B and A_t is the O'Neill tensor.

Comparison

Some overlap with results from Verdiani-Ziller.

Definition (The Riccati operator)

Define $\hat{S}(t) : H(t) \to H(t)$ *as*

$$\hat{S}(t)(v) := J'(t)^H,$$

where $J \in L$ *with* $J(t) = v$*.*

Important: Well defined whenever any $J \in L$ with $J(t) = 0$ lies in W.

Definition

W is of full index in an interval $I \subset \mathbb{R}$ if the above happens at every point of I.

$$(\hat{S}_t J^H)' = \hat{S}_t{}' J^H + \hat{S}_t J^{H\prime} = (\hat{S}_t{}' + \hat{S}_t{}^2) J^H$$

Wilking's equation:

$$\hat{S}_t{}' + \hat{S}_t{}^2 + \{R(t)\}^H + 3A_tA_t^* = 0$$

Taking traces leaves

$$\hat{s}_t{}' + \hat{s}_t{}^2 + \hat{r}_t = 0,$$

where, if $k = \dim H$, $\hat{S}_t^0$ the trace free part of $\hat{S}_t$,

- $\hat{s}_t = \frac{1}{k} \operatorname{Trace} \hat{S}_t$;
- $\hat{r}_t = \frac{1}{k}(|\hat{S}_t^0|^2 + \operatorname{Trace}\left[R(t)^H + 3A_tA_t^*\right]) \geq \frac{1}{k} \operatorname{Trace} R(t)^H$

Intermediate Ricci curvature appears naturally:

Definition

$\mathrm{Ric}_k \geq \ell$ *if for any* $v \in T_pM$, *and any* $(k+1)$*-orthogonal frame* $\{v, e_1, \dots, e_k\}$ *we have*

$$\sum_{i=1}^{k} \sec(v, e_i) \geq \ell.$$

Riccati comparison, dimension one

$$s' + s^2 + r(t) = 0. \tag{1}$$

Denote by s_a solutions of $s' + s^2 + a = 0$, with $a = 1, 0, -1$.

Lemma

Suppose $r \geq a$, and let s be a solution of (1) defined in $[t_0, t_{max}]$, with $s(t_0) \leq s_a(t_0)$. then

- $s(t) \leq s_a(t)$,
- *if there is some t_1 with $s(t_1) = s_a(t_1)$, then $s \equiv s_a$ and $r \equiv a$ in $[t_0, t_1]$.*

Corollary

If $r \geq 1$, $[t_0, t_{max}] \subset [0, \pi]$, and $\alpha \in [0, \pi - t_0)$, then the only solution of the Riccati equation with $s(t_0) \leq \cot(t_0 + \alpha)$ that is defined in $[t_0, \pi - \alpha)$ is $s(t) = \cot(t + \alpha)$, and $r \equiv 1$.

Positive intermediate Ricci comparison

Theorem (F.Wilhelm, LG)

$\mathrm{Ric}_k \geq k$, *and* L *a Lagrangian along* γ *with Riccati operator* S_t.
If there is a k-dimensional subspace $\mathcal{H}_0 \perp \gamma'(0)$ *such that the Ricatti operator for* L *satisfies*

$$\mathrm{Trace}\, S_0|_{\mathcal{H}_0} \leq 0,$$

then:

1. *There is some nonzero* $J \in L$, $J(0) \in \mathcal{H}_0$, *such that* $J(t_1) = 0$ *for some* $t_1 \in (0, \pi/2]$.
2. *If no* $J \in L$ *vanishes before time* $\pi/2$, *then there are subspaces* W, H *in* L, *with* $H_0 = \mathcal{H}_0$, *such that* L *splits as* $L = W \oplus H$ *orthogonally for every* $t \in [0, \pi/2]$. *Moreover, every field in* H *is of the form*

$$\sin(t + \frac{\pi}{2}) \cdot E(t),$$

where E *is a parallel vector field.*

Existence of focal points for Ric_k.

Theorem (F.Wilhelm, LG)

Let M be a complete manifold with $\mathrm{Ric}_k \geq k$, and $N \subset M$ a submanifold (possibly not embedded, not complete) with $\dim N \geq k$. Then

1. *for any geodesic $\gamma : \mathbb{R} \to M$ with $\gamma(0) \in N$, $\gamma'(0) \perp N$, there are at least*

   $$\dim N - k + 1$$

   focal points to N in the interval $[-\pi/2, \pi/2]$;

2. *if for every geodesic as above, the first focal point is at time $\pi/2$ or $-\pi/2$, then N is totally geodesic*

Diameter rigidity vs. focal rigidity

Theorem (Gromoll-Grove's diameter rigidity)

Let M be a compact Riemannian manifold with $\sec \geq 1$ and $\operatorname{diam} = \pi/2$. Then M is homeomorphic to a sphere, or isometric to a compact projective space.

Definition

The focal radius of a submanifold N is the smallest time t_0 such that there is a focal point to N along a geodesic $\gamma : \mathbb{R} \to M$ with $\gamma(0) \in N$, $\gamma'(0) \perp N$.

Theorem (Focal rigidity, F. Wilhelm, LG)

Let M be a compact Riemannian manifold with $\mathrm{Ric}_k \geq k$; if M contains an embedded submanifold N with $\dim N \geq k$, and with focal radius $\pi/2$, then the universal cover of M is isometric to a round sphere, or to a compact projective space with N totally geodesic in M.

Sphere Theorem for Ric_k

Definition

Let $\gamma : [0, \ell] \to M$ a unit geodesic. The index of γ is the number (with multiplicity) of conjugate points to $\gamma(0)$ along γ.

Lemma (Index of "long" geodesics, F.Wilhelm, LG)

(M^n, g) with $\mathrm{Ric}_k \geq k$. Then any unit geodesic $\gamma : [0, b] \to M$ with $b \geq \pi$ satisfies

$$\mathrm{index}(\gamma) \geq n - k.$$

Case of sec: David González-Álvaro, LG.

Theorem (Sphere Theorem for Ric_k)

(M^n, g) with $\mathrm{Ric}_k \geq k$. Suppose there is some $p \in M$ with $\mathrm{conj}_p > \pi/2$. Then

- *the universal cover of M is $(n-k)$-connected;*
- *if $k \leq n/2$, then the universal cover of M is homeomorphic to a sphere.*

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Proof of the Sphere Theorem

We can assume $\pi_1 M = 0$, $n - k \geq 2$.

Ω_p: Loops based at p (or a finite dimensional manifold approximation).

Enough to show that Ω_p is $(n-k-1)$-connected .

Lemma

Let $f : P \to \mathbb{R}$ be a proper, smooth function, and $a < b$ such that every critical point in $f^{-1}[a,b]$ has index $\geq m$. Then

$$f^{-1}(-\infty, a] \hookrightarrow f^{-1}(-\infty, b]$$

is $(m-1)$-connected.

$E : \Omega_p \to [0, \infty)$ the energy function

$$E(\alpha) = \frac{1}{2} \int_0^{\ell} |\alpha'(t)|^2 \, dt$$

Its critical points are geodesic loops based at p.

Proof of the Sphere Theorem (cont.)

Choose some $\pi < \ell < 2\,\mathrm{conj}_p$

1. The long geodesic lemma $\equiv$ geodesics longer than ℓ have index $\geq n - k$,

$$\Rightarrow E^{-1}(-\infty, b/2] \hookrightarrow \Omega_p \text{ is } (n-k-1)\text{-connected.}$$

2. $E^{-1}(-\infty, b/2]$ has no critical points (contradiction).
   - $n - k \geq 2$ and Ω_p connected $\Rightarrow E^{-1}(-\infty, b/2]$ is connected.
   - Any geodesic loop in $E^{-1}(-\infty, b/2]$ is not connected to $\{p\}$, because

Lemma (Long homotopy lemma, Abresch-Meyer)

$\gamma : [0,\ell] \to M$ a unit geodesic loop based at p. If $\ell < 2\,\mathrm{conj}_p$, and $\{\gamma_s\}$ is a homotopy from γ to $\{p\}$, then there is some $s_0 \in (0,1)$ such that

$$\mathrm{length}(\gamma_{s_0}) > 2\,\mathrm{conj}_p\,.$$

Thanks!

Bounds on the second fundamental form

Theorem

Let M be a complete Riemannian manifold with $\sec \geq a$ (where a can take the values $\{1, 0, -1\}$.

Then the second fundamental form of N satisfies

- $|\,\mathrm{II}_N\,| \leq \cot(\mathrm{foc}\, N)$ *if* $a = 1$;
- $|\,\mathrm{II}_N\,| \leq \frac{1}{\mathrm{foc}\, N}$ *if* $a = 0$;
- $|\,\mathrm{II}_N\,| \leq \coth(\mathrm{foc}\, N)$ *if* $a = -1$.

There is a similar theorem for k-submanifolds and Ric_k.

Theorem

Let M be a compact Riemannian manifold. Given $D, r > 0$ the class $\mathcal{S}$ of closed Riemannian manifolds that can be isometrically embedded into M with focal radius $\geq r$ and intrinsic diameter $\leq D$ is precompact in the $C^{1,\alpha}$–topology. In particular, $\mathcal{S}$ contains only finitely many diffeomorphism types.

A soft connectivity principle

Theorem

Let M be a simply connected, complete Riemannian n–manifold with $\mathrm{Ric}_k M \geq k$, and let $N \subset M$ be a compact, connected, embedded, ℓ–dimensional submanifold. If for some $r \in \left[0, \frac{\pi}{2}\right)$,

$$\mathrm{foc}_N > r, \tag{2}$$

and for all unit vectors v normal to N and all k–dimensional subspaces $W \subset TN$,

$$|\mathrm{Trace}\,(S_v|_W)| \leq k \cot\left(\frac{\pi}{2} - r\right), \tag{3}$$

then the inclusion $N \hookrightarrow M$ is $(2\ell - n - k + 2)$–connected.

Soft: hypothesis are satisfied for an C^2-open set in the space of metrics and immersions.

Infinite focal radius in sec ≥ 0.

Theorem (Soul theorem)

M a complete, noncompact manifold with $\sec \geq 0$*. Then there is a totally geodesic, totally convex compact submanifold* $N \subset M$ *such that M is diffeomorphic to the normal bundle of S.*

The metric relation of how N sits inside M was described by Perelman.

Theorem (F.Wilhelm, LG)

M a complete, noncompact manifold with $\sec \geq 0$*. If N is a closed submanifold with infinite focal radius, then N is a soul of M.*

Infinite focal radius in $\mathrm{Ric}_k \geq 0$.

Theorem (Soul theorem for nonnegative Ric_k curvature)

Let M be a complete Riemannian n–manifold with $Ric_k \geq 0$, and let N be any closed submanifold of M with $\dim(N) \geq k$ and infinite focal radius.
Then

- *N is totally geodesic;*
- *the normal bundle $\nu(N)$ with the pull back metric $\left(\exp_N^{\perp}\right)^*(g)$ is a complete manifold with $Ric_k \geq 0$, that covers M;*
- *N lies in $\nu(N)$ as in the description of Perelman's rigidity theorem (vertizontal flats, Riemannian submersion into N, etc.*