Cheeger deformations and positive Ricci and scalar curvatures

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Positive Curvatures

1. Examples of manifolds with positive sectional curvature are sparse:
   1. $\mathbb{S}^n$, $\mathbb{K}P^n, \mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}, Ca$;
   2. $W^6 = SU(3)/T^2$, $W^{12} = Sp(3)/Sp(1)^3, W^{24} = F_4/\mathrm{Spin}(8)$;
   3. $B^7 = SO(5)/SO(3)$, $B^{13} = SU(5)/Sp(2) \cdot S^1$;
   4. $W^7_{p,q} = SU(3)/\mathrm{diag}(z^p, z^q, \bar{z}^{p+q})$;
   5. Some biquotients.
2. In every known example so far, some hypothesis of symmetry is assumed.
3. In this short talk we will always consider a closed and connected Riemannian manifold (M, g) with an isometric action by a compact and connected Lie group G with bi–invariant metric Q.

The Lawson–Yau theorem

In contrast with the shortage of examples of manifolds with positive sectional curvature, manifolds with positive scalar curvature are much more abundant. In fact, Lawson and Yau proved the following:

Theorem (Lawson–Yau [2])

Let (M, g) be a closed and connected Riemannian manifold with an effective isometric action by a compact connected and non–abelian Lie Group G. Then, M admits a G–invariant metric of positive scalar curvature.

Riemannian manifolds with isometric actions

On the context of Riemannian manifolds with isometric actions one observes that:

- Via the Gray–O'Neill formula, positive sectional curvature on (M, g) implies positive sectional curvature on M^{reg}/G (this is not necessarily true for positive Ricci curvature);
- Generally, one can not lift positive sectional curvature from the orbit space. Example: $SO(3) \curvearrowright \mathbb{R}P^2 \times \mathbb{R}P^2$.

Question: can one lift positive Ricci curvature from the orbit space?

The Searle–Wilhelm theorem

Searle and Wilhelm proved that:

Theorem (Searle-Wilhelm [3])

Let G be a compact connected Lie group G acting effectively on a closed Riemannian manifold (M, g). Assume that:

- *A principal orbit has finite fundamental group;*
- *On the orbital distance metric, M^{reg}/G satisfies $\mathrm{Ric}_{M^{reg}/G} \geq 1$.*

Then, M admits a G-invariant metric of positive Ricci curvature after a finite Cheeger deformation.

1. The hypothesis of a principal orbit with finite fundamental group implies that the orbits have sectional curvature bounded from below on the normal homogeneous space metric;
2. The metric on the quotient is required to be a Riemannian submersion metric.

The Searle and Wilhelm theorem

The proof of Searle and Wilhelm theorem is divided in two steps:

- To make a G–invariant conformal transformation of the initial metric on a neighbourhood of the singular strata;
- To use Cheeger deformations on the conformally changed metric to obtain positive Ricci curvature on any compact subset of M^{reg}.

This work was motivated by the question:

Question: can one simplify their proof by avoiding the step 1 and use only Cheeger deformations to prove their theorem?

Why Cheeger deformations?

1. Cheeger deformations are obtained via introducing a parameter $\frac{1}{t}$, $t > 0$ in $\pi : (M \times G, g \times \frac{1}{t}Q) \to (M, g_t)$, where one considers the action
$$r \cdot (p, g) := (r \cdot p, gr^{-1})$$
and consider the Riemannian submersion $\pi(p, g) := g \cdot p$.
2. There are appropriate reparametrizations of 2-planes $\{\overline{X}, \overline{Y}\}$ such that the expression for the sectional curvature of g_t is given by:

Theorem

$$\kappa_t(\overline{X}, \overline{Y}) = K_g(\overline{X}, \overline{Y}) + \frac{t^3}{4}\|[PU, PV]\|_Q^2 + z_t(\overline{X}, \overline{Y}),$$

where z_t is a non–negative term.

Cheeger deformations do not decrease sectional curvature of reparametrized planes.

Theorem (Cavenaghi–Sperança [1])

Let (M^n, g) be a closed and connected Riemannian manifold with an isometric action by a compact connected Lie group G. Denote by $\mathcal{H}_p$ the horizontal space of the G-action at p. Assume that:

1. *A principal orbit of G on M has finite fundamental group,*
2. $\mathrm{Ric}_{M^{reg}/G} \geq 1$.

If g has directions of negative Ricci curvature after each finite Cheeger deformation, then

(a) *There exists a singular point $p \in M$ with a non–zero vector $X \in \mathcal{H}_p$ that is fixed by the isotropy representation $\rho : G_p \to O(\mathcal{H}_p)$,*

(b) *The resctriction ρ a $X^\perp \cap \mathcal{H}_p$ is reducible and if $X^\perp \cap \mathcal{H}_p$ has exactly two ρ–irreducible components, then*

$$\dim \mathcal{H}_1 - \dim G_p Y_1 > \frac{(k-1)\dim \mathcal{H}_1}{\dim \mathcal{H}_p - 1},$$

where k is the dimension of a principal orbit.

Corollary (Cavenaghi–Sperança [1])

Let (M, g) be a closed and connected Riemannian manifold with an isometric G–action by a connected and compact Lie group. Assume that

1. *A principal orbit of G on M has finite fundamental group,*
2. $\mathrm{Ric}_{M^{reg}/G} \geq 1$,

Then, g develops positive Ricci curvature after a finite Cheeger deformation if one of the following (equivalent) assumptions hold

(a) *The singular strata is composed by isolated orbits;*

(b) *For every singular point p, the isotropy representation is irreducible;*

(c) *The only fixed vector by the linear isotropy representation on a singular point is the zero vector;*

(d) *For every singular point p, the G–action induced on the unitary tangent space T^1_pM has no fixed points.*

Theorem (Cavenaghi–Sperança [1])

Let (M, g) be a closed and connected Riemannian manifold with an isometric action by a compact connected and non–abelian Lie group G. Then, g develops positive scalar curvature after a finite Cheeger deformation.

This improvement to Lawson–Yau theorem leads to the interesting open problem:

Problem 1

Is the space of those metrics contractible?

Corollary

A closed Riemannian manifold (M, g), with an isometric action by a compact and connected Lie group G and satisfying the hypothesis of Searle and Wilhelm, admits a metric of positive Ricci curvature after a finite Cheeger deformation if, and only if,

$$\mathrm{Ric}^{\mathcal{H}}(X) > 0,$$

for every horizontal singular point p and every vector $X \in \mathcal{H}_p$ such that $\mathfrak{g}_X = \mathfrak{g}_p$.

Remark

To give a simplified proof for Searle–Wilhelm theorem one only needs to make a conformal change to obtain positive horizontal Ricci cuvature on the directions X such that $\mathfrak{g}_X = \mathfrak{g}_p$.

Theorem (Cavenaghi–Sperança [1])

Let $\mathbb{S}^n$ the unitary sphere on $\mathbb{R}^{n+1}$ with the $SO(n-2)$–action that fixes its first 3 entries. Then, for each $n \geq 5$, there exists a $SO(n-2)$-invariant metric g on $\mathbb{S}^n$ that satisfies:

1. $\mathrm{Ric}_{M^{reg}/G} \geq 1$
2. *There is a non–zero vector $X \in \mathcal{H}_p$, where p is a singular point, such that $\mathrm{Ric}_{g_t}(X) < 0$ for every Cheeger deformation g_t.*

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