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Lagrangian submanifolds of the complex quadric

Joeri Van der Veken

SYMMETRY AND SHAPE
celebrating the 60th birthday of Prof. J. Berndt

Santiago de Compostela – 30/10/2019

0 – Outline

1. How we started research on Q^n
2. The complex quadric Q^n
3. The Gauss map of a hypersurface of a sphere
4. Study of Lagrangian submanifolds of Q^n
5. Question

1 – How we started research on Q^n

The homogeneous nearly Kähler $(S^3 \times S^3, g)$

1 – How we started research on Q^n

The homogeneous nearly Kähler $(S^3 \times S^3, g)$ has

- an almost complex structure J
- and an almost product struture P,
- which anti-commute,
- and the curvature tensor is given by

$$\begin{aligned} R(X,Y)Z &= \frac{5}{12}\left(g(Y,Z)X - g(X,Z)Y\right) \\ &\quad + \frac{1}{12}\left(g(X,JZ)JY - g(Y,JZ)JX + 2g(X,JY)JZ\right) \\ &\quad + \frac{1}{3}\left(g(PY,Z)PX - g(PX,Z)PY\right. \\ &\qquad \left. + g(JPY,Z)JPX - g(JPX,Z)JPY\right). \end{aligned}$$

2 – The complex quadric Q^n

Definition

$Q^n := \{[(z_0, \ldots, z_{n+1})] \in \mathbb{C}P^{n+1}(4) \mid z_0^2 + \ldots + z_{n+1}^2 = 0\}.$

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Lemma

$$\pi^{-1}Q^n = \left\{ u + iv \;\middle|\; u, v \in \mathbb{R}^{n+2},\ \langle u, u\rangle = \langle v, v\rangle = \frac{1}{2},\ \langle u, v\rangle = 0 \right\} \subseteq S^{2n+3}(1),$$

where $\langle \cdot, \cdot \rangle$ *is the Euclidean metric on* $\mathbb{R}^{n+2}$.

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Remark. Alternative descriptions:

- Q^n is the Grassmannian of oriented 2-planes in $\mathbb{R}^{n+2}$
- $Q^n = \dfrac{\mathrm{SO}(n+2)}{\mathrm{SO}(n) \times \mathrm{SO}(2)}$

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$\mathrm{Re}(z_0^2 + \ldots + z_{n+1}^2) = 0 \quad \Rightarrow \bar{z} \perp z \quad \Rightarrow \bar{z}$ is tangent to $S^{2n+3}(1)$

$\mathrm{Im}(z_0^2 + \ldots + z_{n+1}^2) = 0 \quad \Rightarrow \bar{z} \perp iz \quad \Rightarrow \bar{z}$ is horizontal

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Lemma

Any shape operator A of Q^n in $\mathbb{C}P^{n+1}(4)$, associated to a unit normal vector field, has the following properties:

1. $A^2 = \mathrm{id}$,
2. $g(AX, AY) = g(X, Y)$,
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Let $\mathcal{A}$ be the set of these operators. Choose $A_0 \in \mathcal{A}$, then

$$\mathcal{A} = \{\cos\varphi\, A_0 + \sin\varphi\, JA_0 \mid \varphi : Q^n \to \mathbb{R}\}.$$

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Lemma

For all $A \in \mathcal{A}$, there exists a non-zero one-form s such that $\nabla_X A = s(X)JA$.

2 – The complex quadric Q^n

From the equation of Gauss:

$$\begin{aligned} R^{Q^n}(X,Y)Z &= g(Y,Z)X - g(X,Z)Y \\ &\quad + g(X,JZ)JY - g(Y,JZ)JX + 2g(X,JY)JZ \\ &\quad + g(AY,Z)AX - g(AX,Z)AY \\ &\quad + g(JAY,Z)JAX - g(JAX,Z)JAY \end{aligned}$$

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Theorem (Jensen)

A Riemannian homogeneous Einstein four-manifold is symmetric and hence locally isometric to either a real space form $\mathbb{R}^4$, $S^4(c)$ or $H^4(c)$; a complex space form $\mathbb{C}P^2(4c)$ or $\mathbb{C}H^2(4c)$; or a product of surfaces $S^2(c) \times S^2(c)$ or $H^2(c) \times H^2(c)$.

3 – Outline

3 – The Gauss map of a hypersurface of a sphere

Gauss map of a hypersurface of $\mathbb{R}^{n+1}$

$a : M^n \to \mathbb{R}^{n+1}$ hypersurface with unit normal b

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Definition

The map $G : M^n \to S^n(1) : p \mapsto b(p)$ is the Gauss map of a.

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Remark:

- Any *parallel hypersurface* to a, given by

$$a_t : M^n \to \mathbb{R}^{n+1} : p \mapsto a(p) + t\, b(p)$$

for some $t \in \mathbb{R}$, has the same Gauss map.

Gauss map of a hypersurface of $S^{n+1}(1)$

$a : M^n \to S^{n+1}(1) \subseteq \mathbb{R}^{n+2}$ hypersurface with unit normal b.

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Remark:

- $\langle \frac{a(p)}{\sqrt{2}}, \frac{a(p)}{\sqrt{2}} \rangle = \langle \frac{b(p)}{\sqrt{2}}, \frac{b(p)}{\sqrt{2}} \rangle = \frac{1}{2}, \langle \frac{a(p)}{\sqrt{2}}, \frac{b(p)}{\sqrt{2}} \rangle = 0 \Rightarrow \frac{a(p)}{\sqrt{2}} + i\frac{b(p)}{\sqrt{2}} \in \pi^{-1}Q^n$

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- A *parallel hypersurface* to a is now given by

  $$a_t : M^n \to S^{n+1}(1) \subseteq \mathbb{R}^{n+2} : p \mapsto \cos t\, a(p) + \sin t\, b(p)$$

  for some $t \in \mathbb{R}$. Since $b_t = \cos t\, b - \sin t\, a$ is a unit normal to a_t, $a_t + ib_t = e^{-it}(a + ib)$ and a_t has the same Gauss map as a.

Proposition

The Gauss map $G : M^n \to Q^n$ of a hypersurface $a : M^n \to S^{n+1}(1)$ is a Lagrangian immersion.

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Proof. Diagonalize the shape operator S of a: $Se_j = \lambda_j e_j$.

For the horizontal lift $\hat{G} : M^n \to \pi^{-1}Q^n : p \mapsto \frac{1}{\sqrt{2}}(a(p) + ib(p))$, one has

$$(d\hat{G})e_j = \frac{1}{\sqrt{2}}(e_j - iSe_j) = \frac{1}{\sqrt{2}}(1 - i\lambda_j)e_j.$$

$(d\hat{G})e_1, \ldots, (d\hat{G})e_n$ are linearly independent $\Rightarrow$ G is an immersion.

$\forall j, k \in \{1, \ldots, n\} : \langle (d\hat{G})e_j, i(d\hat{G})e_k \rangle = 0$ $\Rightarrow$ G is Lagrangian.

$\square$

3 – The Gauss map of a hypersurface of a sphere

Proposition

If the principal curvatures of a hypersurface $a : M^n \to S^{n+1}(1)$ are constant, then its Gauss map $G : M^n \to Q^n$ is a minimal Lagrangian immersion.

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Proposition

If the principal curvatures of a hypersurface $a : M^n \to S^{n+1}(1)$ are constant, then its Gauss map $G : M^n \to Q^n$ is a minimal Lagrangian immersion.

The statement follows from the following formula by Palmer:

$$g(JH, \cdot) = -\frac{1}{n}\, d \left(\mathrm{Im} \left(\log \prod_{j=1}^{n} (1 + i\lambda_j) \right) \right).$$

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Remark. Originally defined as level sets of *isoparametric functions* F on the ambient space: functions for which $\|\nabla F\| = \phi_1(F)$, $\Delta F = \phi_2(F)$.

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Full classification of isoparametric hypersurfaces of $\mathbb{R}^{n+1}$.

Theorem (Somigliana, Levi-Civita, Segre)

An isoparametric hypersurface of $\mathbb{R}^{n+1}$ *is an op part of a hyperplane* $\mathbb{R}^n$*, of a hypersphere* $S^n(r)$ *or of a product immersion* $S^k(r) \times \mathbb{R}^{n-k}$*.*

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  $$a_2 : S^k(r_1) \times S^{n-k}(r_2) \to S^{n+1}(1) : (p_1, p_2) \mapsto (p_1, p_2)$$

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- A tube around the Veronese surface in $S^4(1)$ (*Cartan's example*)

$$a_3 : \mathbb{R}P^2 \times S^1(\varepsilon) \to S^4(1)$$

3 – The Gauss map of a hypersurface of a sphere

Principal curvatures of these examples:

- $a_1 : \quad \lambda_1 = \ldots = \lambda_n = \frac{\sqrt{1-r^2}}{r}$
- $a_2 : \quad \lambda_1 = \ldots = \lambda_k = \frac{r_2}{r_1},\ \lambda_{k+1} = \ldots = \lambda_n = -\frac{r_1}{r_2}$
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Theorem (Münzner, 1981)

Let g be the number of distinct constant principal curvatures of an isoparametric hypersurface of $S^{n+1}(1)$, then $g \in \{1, 2, 3, 4, 6\}$.

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Until today, the classification of isoparametric hypersurfaces of $S^{n+1}(1)$ is still not completely understood.

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Let $f : M^n \to Q^n$ be a Lagrangian immersion. Assume $A \in \mathcal{A}$ is fixed.

If X is tangent to M^n, the decomposition

$$AX = BX - JCX$$

into a tangent and a normal part, defines two $(1,1)$-tensor fields on M^n.

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Lemma

The $(1,1)$-tensor fields B and C on M^n satisfy

1. *B and C are symmetric,*
2. *$B^2 + C^2 = \mathrm{id}$,*
3. *$[B, C] = 0$.*

Hence, for every $p \in M^n$, there exists an ONB $\{e_1, \ldots, e_n\}$ of T_pM^n and $\theta_1, \ldots, \theta_n \in \mathbb{R}$, determined up to an integer multiple of π, such that

$$Ae_j = \cos(2\theta_j)e_j - \sin(2\theta_j)Je_j.$$

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Lemma

Let $f : M^n \to Q^n$ be a Lagrangian immersion and $A_0, A \in \mathcal{A}$. Then there exists a function $\varphi : M^n \to \mathbb{R}$ such that $A = \cos\varphi\, A_0 + \sin\varphi\, JA_0$ along M^n and the local angle functions $\theta_1, \ldots, \theta_n$ associated to A are related to the local angle functions $\theta_1^0, \ldots, \theta_n^0$ associated to A_0 by

$$\theta_j = \theta_j^0 - \frac{\varphi}{2}.$$

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Example. One can choose $A \in \mathcal{A}$ such that

$$\theta_1 + \ldots + \theta_n = 0 \mod \pi.$$

Equation of Gauss:

$$\begin{aligned} g(R(X,Y)Z,W) = {} & g(Y,Z)g(X,W) - g(X,Z)g(Y,W) \\ & + g(BY,Z)g(BX,W) - g(BX,Z)g(BY,W) \\ & + g(CY,Z)g(CX,W) - g(CX,Z)g(CY,W) \\ & + g(h(Y,Z),h(X,W)) - g(h(X,Z),h(Y,W)) \end{aligned}$$

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Equation of Codazzi:

$$\begin{aligned} (\overline{\nabla}h)(X,Y,Z) - (\overline{\nabla}h)(Y,X,Z) = {} & g(CY,Z)JBX - g(CX,Z)JBY \\ & - g(BY,Z)JCX + g(BX,Z)JCY \end{aligned}$$

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Idea:

Take a horizontal lift $\hat{f} : M^n \to \pi^{-1}Q^n$ and put

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Remark:

- We have to work locally to guarantee that a is an immersion
- We expect a relation between the angle functions of f and the principal curvatures of a.

Theorem (VdV, Wijffels)

PART I

Let $a : M^n \to S^{n+1}(1)$ be a hypersurface with unit normal b and denote by $G : M^n \to Q^n : p \mapsto [a(p) + ib(p)]$ its Gauss map. After a suitable choice of $A \in \mathcal{A}$, the relation between the principal curvatures a and the angle functions of G is

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4 – Study of Lagrangian submanifolds of Q^n – main results

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Remark. The choice of A comes down to choosing

$$\overline{\hat{G}(p)} = \frac{1}{\sqrt{2}}(a(p) - ib(p))$$

as a unit normal to $\pi^{-1}Q^n$ in $S^{2n+3}(1)$ along $\hat{G}$.

Theorem (VdV, Wijffels)

PART II

Conversely, if $f : M^n \to Q^n$ is a Lagrangian immersion, then for every point of M^n there exist an open neighborhood U of that point in M^n and an immersion $a : U \to S^{n+1}(1)$ with Gauss map $f|_U$. This immersion is not unique, nor are its principal curvature functions. However, for any choice of the hypersurface a and of the almost product structure $A \in \mathcal{A}$, the principal curvature functions of a are related to the corresponding angle functions of f by

$$\cot(\theta_j - \theta_k) = \pm\frac{\lambda_j\lambda_k + 1}{\lambda_j - \lambda_k}$$

for $j, k = 1, \ldots, n$ in points where $\lambda_j \neq \lambda_k$.

Some classification theorems for minimal Lagrangian immersions into Q^n

4 – Study of Lagrangian submanifolds of Q^n – main results

Some classification theorems for minimal Lagrangian immersions into Q^n

Theorem (Li, Ma, VdV, Vrancken, Wang)

Let $f : M^n \to Q^n$, $n \geq 2$, be a minimal Lagrangian immersion with constant local angle functions. If g is the number of different constant local angle functions modulo π, then $g \in \{1, 2, 3, 4, 6\}$. Moreover,

- *if $g = 1$, then f is the Gauss map of a part of* $a_1 : S^n(r) \to S^{n+1}(1)$;
- *if $g = 2$, then f is the Gauss map of a part of* $a_2 : S^k(r_1) \times S^{n-k}(r_2) \to S^{n+1}(1)$;
- *if $g = 3$, then f is the Gauss map of a part of* $a_3 : \mathbb{R}P^2 \times S^1(\varepsilon) \to S^4(1)$ *or of tubes around standard embeddings* $\mathbb{C}P^2 \to S^7(1)$, $\mathbb{H}P^2 \to S^{13}(1)$ *or* $\mathbb{O}P^2 \to S^{25}(1)$.

4 – Study of Lagrangian submanifolds of Q^n – main results

Theorem (Li, Ma, VdV, Vrancken, Wang)

Let $f : M^n \to Q^n$, $n \geq 2$, be a totally geodesic Lagrangian immersion. Then f is the Gauss map of a part of $a_1 : S^n(r) \to S^{n+1}(1)$ or of a part of $a_2 : S^k(r_1) \times S^{n-k}(r_2) \to S^{n+1}(1)$.

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Theorem (Li, Ma, VdV, Vrancken, Wang)

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- *$n = 2$ and f is the Gauss map of a part of* $a_2 : S^1(r_1) \times S^1(r_2) \to S^3(1)$; $\quad c = 0$
- *$n = 3$ and f is the Gauss map of a part of* $a_3 : \mathbb{R}P^2 \times S^1(\varepsilon) \to S^4(1)$. $\quad c = \frac{1}{8}$

4 – Study of Lagrangian submanifolds of Q^n – main results

Some steps in the proof of the last theorem

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Lemma

Let $f : M^n \to Q^n$, $n \geq 2$, be a Lagrangian immersion, such that M^n has constant sectional curvature c. Then

$$\begin{aligned}&\sin(\theta_i-\theta_j)\sin(\theta_i+\theta_j-2\theta_k)(\delta_{k\ell}h(e_i,e_j)+h_{ij}^{\ell}Je_k)\\&+\sin(\theta_j-\theta_k)\sin(\theta_j+\theta_k-2\theta_i)(\delta_{i\ell}h(e_j,e_k)+h_{jk}^{\ell}Je_i)\\&+\sin(\theta_k-\theta_i)\sin(\theta_k+\theta_i-2\theta_j)(\delta_{j\ell}h(e_i,e_k)+h_{ik}^{\ell}Je_j)=0\end{aligned}$$

for all i, j, k, ℓ. In particular, if i, j, k are mutually different, then

$$h_{ii}^k\sin(\theta_i-\theta_k)\sin(\theta_i+\theta_k-2\theta_j)=h_{jj}^k\sin(\theta_j-\theta_k)\sin(\theta_j+\theta_k-2\theta_i),$$
$$h_{ij}^k\sin(\theta_i-\theta_j)\sin(\theta_i+\theta_j-2\theta_k)=0$$

and if i, j, k, ℓ are mutually different, then

$$h_{ij}^k\sin(\theta_i-\theta_j)\sin(\theta_i+\theta_j-2\theta_\ell)=0.$$

Proposition

Let $f : M^n \to Q^n$, $n \geq 2$, be a minimal Lagrangian immersion such that M^n has constant sectional curvature and choose $A \in \mathcal{A}$ such that $\theta_1 + \ldots + \theta_n = 0 \mod \pi$. Then either

- *all local angle functions are the same modulo π, or*
- *all local angle functions are mutually different modulo π.*

In the former case, the immersion is the Gauss map of a part of $a_1 : S^n(r) \to S^{n+1}(1)$.

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The conclusion follows by algebraic computations using all the obtained relations and the equations of Gauss and Codazzi.

Proposition

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Remark. For $M^2 \to Q^2 \cong S^2(\frac{1}{2}) \times S^2(\frac{1}{2})$, the classification was already obtained by Castro and Urbano.

5 – Question

Question:

Are there other Riemannian manifolds (M,g) with anti-commuting almost complex structure J and almost product structure P such that

$$R(X,Y)Z = a\,(g(Y,Z)X - g(X,Z)Y)$$
$$+\, b\,(g(X,JZ)JY - g(Y,JZ)JX + 2g(X,JY)JZ)$$
$$+\, c\,(g(PY,Z)PX - g(PX,Z)PY + g(JPY,Z)JPX - g(JPX,Z)JPY)\,?$$

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Only examples that I know of so far:

- real space forms (no J, no P), complex space forms (no P)
- the homogeneous nearly Kähler $S^3 \times S^3$
- the complex quadric, the hyperbolic complex quadric

Remark. All such manifolds will be Einstein.

5 – References

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Thank you for your attention!