# Some recent results on polar actions

## Symmetry and Shape

Celebrating the 60th birthday of Prof. J. Berndt

28 - 31 October 2019, Santiago de Compostela

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October 30, 2019

## Definition

An isometric Lie group action on a Riemannian manifold is called *polar*, if there is a *section* i.e. a submanifold $\Sigma$ which intersects all orbits orthogonally. The action is called *hyperpolar* if $\Sigma$ is flat.

## Examples

- Polar coordinates on $\mathbb{R}^2$.
- The action of $\mathrm{SO}(n)$ on the real symmetric $n \times n$-matrices given by $A \cdot X = AXA^{-1}$.
- The action of a compact Lie group $G$ on itself by conjugation $g \cdot x = gxg^{-1}$.
- Actions of *cohomogeneity 1*.

# Polar Actions and Riemannian symmetric spaces

**Definition**

A Riemannian manifold $M$ is called *symmetric space* if for any point $p \in M$ there is an isometry of $M$ which leaves $p$ fixed and whose differential at $p$ is minus the identity on $T_pM$.

Polar actions are connected *in several ways* to the theory of symmetric spaces:

**Polar actions arising from symmetric spaces**

- Isotropy *actions* $K \curvearrowright G/K$ of symmetric spaces are hyperpolar.
- Isotropy *representations* of symmetric spaces are polar
- ...and they induce polar actions on spheres and projective spaces.
- *Hermann actions*: If $G/H$ and $G/K$ are symmetric, then $H \curvearrowright G/K$ and $H \times K \curvearrowright G$ are hyperpolar.

# Classification results I

- Cohomogeneity-one actions on:
  $S^n$ Hsiang-Lawson (1971), $\mathbb{C}P^n$ Takagi (1973), $\mathbb{H}P^n$ D'Atri (1979), $\mathbb{O}P^2$ Iwata (1981)
- Polar representations / actions on $S^n$ Dadok (1985)
- Hyperpolar actions on irreducible compact Riemannian symmetric spaces $G/K$ K. (1998)
  – they are Hermann actions or of cohomogeneity 1 –
- Polar actions on $S^n, \mathbb{C}P^n, \mathbb{H}P^n, \mathbb{O}P^2$
  Podestà-Thorbergsson (1998)
- Polar actions with fixed point on $G/K$ irreducible of *higher rank* (i.e. $\mathrm{rk}(G/K) \geq 2$) are hyperpolar. Brück (1998)
- Polar actions on irreducible compact Hermitian symmetric spaces of higher rank. Podestà-Thorbergsson, Biliotti-Gori (2002-2005) – they are hyperpolar –

# Classification results II

- K. (2007) Polar actions on symmetric spaces $G/K$ of higher rank with $G$ compact, simple are hyperpolar.
- K. (2009) Polar actions on the exceptional compact Lie groups $\mathrm{G}_2, \mathrm{F}_4, \mathrm{E}_6, \mathrm{E}_7, \mathrm{E}_8$ are hyperpolar.

**Theorem (Lytchak 2012)**

*Polar singular foliations of irreducible compact Riemannian symmetric spaces of higher rank are hyperpolar if the codimension of the leaves is at least 3*

**Theorem (K. and Lytchak 2012)**

*Proof of Biliotti's conjecture (2005): Polar actions on compact irreducible symmetric spaces of higher rank are hyperpolar.*

# Hyperpolar actions on reducible symmetric spaces I

**Definition**

An isometric Lie group action on a Riemannian manifold is called *indecomposable* if its orbits do not agree with those of a product action.

E.g. the action of $G \times G$ on $G \times G$, given by

$$(a, b) \cdot (x, y) = (axb^{-1}, ayb^{-1})$$

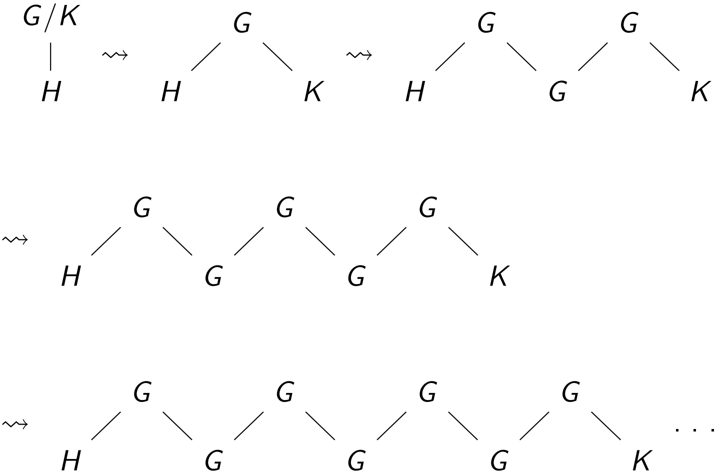
is hyperpolar and indecomposable.

Motivated by: $H \curvearrowright G/K$ hyperpolar $\iff$ $H \times K \curvearrowright G$ hyperpolar we define

**Definition ("Expanding factors")**

$$H \curvearrowright M_1 \times G_2/K_2 \quad \rightsquigarrow \quad H \times K_2 \curvearrowright M_1 \times G_2.$$

# Hyperpolar actions on reducible symmetric spaces II




Question: Do we obtain all indecomposable hyperpolar actions by this construction?

Antwort: Yes, if they are of cohomogeneity $\geq 2$.

# Hyperpolar actions on reducible symmetric spaces III

**Theorem (K. 2016, Transformation Groups)**

*A indecomposable hyperpolar action of cohomogeneity $\geq 2$ on a Riemannian symmetric space is orbit equivalent to a Hermann action, i.e. has the same orbits as a Hermann action.*

For cohomogeneity-one actions, an analogous statement does not hold:

**Counterexample**

The Spin(9)-action on $S^8 \times S^{15}$ is of cohomogeneity one and indecomposable.

## Other examples of cohomogeneity-one actions on products

Consider the 3 inequivalent irreducible representations

$$\begin{aligned} \mathrm{Spin}(8) &\to \mathrm{SO}(8) \times \mathrm{SO}(8) \times \mathrm{SO}(8), \\ g &\mapsto (\varrho_0(g), \varrho_1(g), \varrho_2(g)). \end{aligned}$$

This defines

- $\mathrm{Spin}(8) \curvearrowright S^7 \times S^7 \times S^7$ and
- $\mathrm{Spin}(8) \curvearrowright S^7 \times S^7 \times G_2(\mathbb{R}^8)$,

both indecomposable cohomogeneity-one actions.

# Polar isotropy: a characterization of symmetric spaces

**Theorem (Díaz-Ramos, Domínguez-Vázquez, K. 2018)**

*Let $G$ be a simply connected semisimple compact Lie group and let $H$ be a closed subgroup. Assume the homogeneous space $M = G/H$ is equipped with a $G$-invariant Riemannian metric $\mu$. Then the isotropy action of $H$ on $G/H$ is polar with respect to $\mu$ if and only if the Riemannian manifold $(G/H, \mu)$ is a symmetric space.*

This result had been proved before for the case $G$ simple:

A. Kollross, F. Podestà: *Homogeneous spaces with polar isotropy.* manuscripta math. 110 (4), 487– 503 (2003)

# Polar isotropy: non-compact case

**Theorem (Díaz-Ramos, Domínguez-Vázquez, K. 2018/K., Samiou 2000/Di Scala 2006)**

*Let $M$ be a homogeneous Riemannian manifold on which a Lie group $H$ acts polarly with a fixed point. Then $M$ is locally symmetric if one of the following holds:*

1. *The $H$-action is of cohomogeneity $\leq 2$.*
2. *$\Sigma$ is a compact, locally symmetric space.*

**Theorem (Díaz-Ramos, Domínguez-Vázquez, K. 2018)**

*The isotropy actions of* $\mathrm{SU}(n,1)/\mathrm{SU}(n)$, $\mathrm{Sp}(n,1)/\mathrm{Sp}(n)\times\mathrm{U}(1)$ *and* $\mathrm{Spin}(8,1)/\mathrm{Spin}(7)$ *are non-polar.*
*Generalized Heisenberg groups and non-symmetric Damek-Ricci spaces have non-polar isotropy actions.*

# Polar isotropy: open questions

- Can the main result in the compact case (i.e. the characterization of symmetric spaces by polar isotropy) be generalized to the non-compact case?
- For example, can $\mathrm{Sp}(n,1)/\mathrm{Sp}(n)$ be endowed with an invariant Riemannian metric such that its isotropy action is polar?
- More generally, study the question in generality without assuming $G$ is semisimple. (We are not aware of any irreducible non-symmetric Riemannian homogeneous space with a non-trivial polar isotropy action.)
- Is there a conceptual or geometric proof linking polar isotropy with symmetry?

# Asystatic actions I

## Definition

A homogeneous space is called *asystatic* if its isotropy representation has no non-zero fixed vectors, or, equivalently, if the isotropy subgroups at sufficiently close points are different.
A proper Lie group action is called *asystatic* if all of its principal orbits are asystatic.

## Original definition by Lie

Sophus Lie
(1842-1899)

*Ist eine Gruppe des Raumes $x_1 \cdots x_s$ so beschaffen, dass alle ihre Transformationen, welche einen Punkt von allgemeiner Lage invariant lassen, gleichzeitig alle Punkte einer continuirlichen, durch diesen Punkt gehenden Mannigfaltigkeit festhalten, so rechnen wir die Gruppe der einen Classe zu und nennen sie systatisch. Alle übrigen Gruppen aber, also die, welche nicht systatisch sind, rechnen wir zur andern Classe und nennen sie asystatisch.*)*

Sophus Lie: *Theorie der Transformationsgruppen*, 1888, vol. 1, p. 501

# Asystatic actions I

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## Original definition by Lie (translated)

Sophus Lie
(1842-1899)

*If a group of the space $x_1, \dots, x_s$ is constituted in such a way that all its transformations which leave invariant a point in general position do simultaneously fix all points of a continuous manifold passing through this point, then we reckon this group as belonging to the one class and we call them systatic. But we reckon all the remaining groups, hence those which are not systatic, as belonging to the other class, and we call them asystatic.*

S. Lie: *Theory of transformation groups. I.* Translated by Jol Merker. Springer, 2015.

# Asystatic actions II

## Examples

- The action of $\mathrm{SO}(m)$ on $\mathbb{R}^m$ is *asystatic*. The isotropy group $\mathrm{SO}(m-1)$ acts on $T_pS^{m-1}$, leaving only the zero vector fixed.
- The action of $\mathrm{U}(n)$ on $\mathbb{R}^{2n}$ is *not asystatic*. The isotropy group $\mathrm{U}(n-1)$ leaves a non-zero vector in $T_pS^{2n-1}$ fixed.
- For $m = 2n$ the two actions are orbit equivalent.

## Lemma

*Asystatic actions are polar (w.r.t. any invariant Riemannian metric).*

The example above shows that the converse does not hold.

# Asystatic actions III

**Theorem (Gorodski, K. 2015)**

*A polar action on a compact Riemannian symmetric space of rank one is orbit equivalent to an asystatic action.*

**Open question**

Is any polar Action on a compact Riemannian symmetric space orbit equivalent to an asystatic action?

**Theorem (Faye Ried 2019, Master's thesis)**

*Hermann actions on classical compact symmetric spaces are orbit equivalent to asystatic actions.*

# Infinitesimally polar actions

**Definition**

An isometric Lie group action is called *infinitesimally polar* if all of its slice representations are polar.

**Theorem (Lytchak-Thorbergsson 2010)**

*The orbit space of a proper isometric Lie group action on a Riemannian manifold is a Riemannian orbifold if and only if the action is infinitesimally polar.*

- Gorodski-Lytchak 2014: Classification of infinitesimally polar actions on spheres.
- Gorodski-K. 2015: Classification of infinitesimally polar actions on $\mathbb{C}P^n$, $\mathbb{H}P^n$, $\mathbb{O}P^2$.

$$\mathrm{SO}(3) \cdot \mathrm{G}_2 \curvearrowright \mathbb{O}P^2$$

- Berndt—Brück—Díaz-Ramos—Tamaru 1998–2007: Classification results on cohomogeneity-one actions and hyperpolar foliations on non-compact symmetric spaces
- Wu 1992: Polar Actions on real hyperbolic space
- Berndt, Díaz-Ramos 2012: homogeneous polar foliations on $\mathbb{C}H^n$
- Díaz-Ramos, Domínguez-Vázquez, K. 2017: polar actions on $\mathbb{C}H^n$

**Theorem (K. 2018)**

*Classification of polar actions on* $\mathbb{O}H^2 = \mathrm{F}_4/\mathrm{Spin}(9)$ *which leave a totally geodesic submanifold invariant:* $\mathrm{G}_2 \cdot \mathrm{SO}_0(1,2)$, $\mathrm{SU}(3) \cdot \mathrm{SU}(1,2)$, $\mathrm{Sp}(1) \cdot \mathrm{Sp}(1,2)$, $\mathrm{Spin}(7) \cdot \mathrm{SO}_0(1,1)$, $\mathrm{Spin}(6) \cdot \mathrm{Spin}(1,2)$, $\mathrm{Spin}(3) \cdot \mathrm{Spin}(1,5)$, $\mathrm{SO}(2) \cdot \mathrm{Spin}(1,6)$, $\mathrm{Spin}(1,7)$, $\mathrm{Spin}(1,8)$

# A model for $\mathfrak{f}_4$

**Theorem (K. 2018)**

*Define the following binary operation on* $\mathfrak{so}(8) \times \mathbb{O} \times \mathbb{O} \times \mathbb{O}$*:*
$[(A, u, v, w), (B, x, y, z)] = (C, r, s, t)$, *where*

$$\begin{aligned} C &= AB - BA - 4u \wedge x - 4\lambda^2(v \wedge y) - 4\lambda(w \wedge z), \\ r &= Ax - Bu + \overline{vz} - \overline{yw}, \\ s &= \lambda(A)y - \lambda(B)v + \overline{wx} - \overline{zu}, \\ t &= \lambda^2(A)z - \lambda^2(B)w + \overline{uy} - \overline{xv}, \end{aligned}$$

*where* $\lambda$ *is the triality automorphism of* $\mathfrak{so}(8)$ *and*
$x \wedge y := xy^t - yx^t \in \mathfrak{so}(8)$ *for* $x, y \in \mathbb{R}^8 = \mathbb{O}$*.*
*The 52-dimensional real algebra thus defined is the Lie algebra of exceptional compact simple Lie group* $\mathrm{F}_4$*.*

# Generalization: A model for $\mathfrak{e}_8$

## Theorem (K. 2018)

*Define the following binary operation on* $(\mathfrak{so}(8) \oplus \mathfrak{so}(8)) \times (\mathbb{O} \otimes \mathbb{O}) \times (\mathbb{O} \otimes \mathbb{O}) \times (\mathbb{O} \otimes \mathbb{O})$*:*
*Let* $[(A, u, v, w), (B, x, y, z)] = (C, r, s, t)$*, where*

$$\begin{aligned} C &= [A, B] - 4u \curlywedge x - 4\Lambda^2(v \curlywedge y) - 4\Lambda(w \curlywedge z), \\ r &= A.x - B.u + \overline{vz} - \overline{yw}, \\ s &= \Lambda(A).y - \Lambda(B).v + \overline{wx} - \overline{zu}, \\ t &= \Lambda^2(A).z - \Lambda^2(B).w + \overline{uy} - \overline{xv}, \end{aligned}$$

*where* $\Lambda := \lambda \otimes \lambda$ *and* $X \curlywedge Y := (XY^t - YX^t, X^tY - Y^tX)$ *for* $X, Y \in X, Y \in \mathbb{R}^{8\times 8} = \mathbb{O} \otimes \mathbb{O}$.
*The 248-dimensional real algebra thus defined is the Lie algebra of exceptional compact simple Lie group* $\mathrm{E}_8$.